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Finite-time Consensus for Second-order Multi-agent Networks with Inherent Nonlinear Dynamics Under an Undirected Fixed Graph

Yongcan Cao and Wei Ren

Abstract— **This paper studies finite-time consensus for networked multi-agent systems with second-order dynamics in the presence of inherent nonlinear dynamics under an undirected fixed interaction graph. We propose a nonlinear distributed consensus algorithm and then present sufficient conditions such that finite-time consensus can be achieved. By employing a similar stability analysis, it is expected that finite-time consensus can be achieved with/without inherent nonlinear dynamics when the initial undirected interaction graph is connected and the difference between the Laplacian matrix for any $t > 0$ and the Laplacian matrix at $t = 0$ is small enough.**

I. Introduction

Distributed control of networked multi-agent systems has been investigated extensively in the systems and controls society. The main research problem in distributed control of networked multi-agent systems is to have a group of mobile agents achieve desired group behaviors through local information exchange. Compared with the traditional centralized control, distributed control has a number of benefits, such as easy implementation, low complexity, high robustness, and good scalability. Although benefits of distributed control are foreseen, the design and analysis of distributed control are more complicated and challenging than those of the traditional centralized control.

Consensus is one fundamental problem in distributed control of network multi-agent systems. The main objective of consensus to design local control algorithms such that a group of agents reaches a common state, such as positions, phases, and velocities. In the systems and controls society, the pioneer work is given in [1] where an asynchronous agreement problem is studied for distributed decision making problems. Subsequently, the authors in [2]–[5] study consensus for first-order kinematics under various information flow constraints. For more results about consensus, the readers are referred to [6]–[9] and references therein.

Finite-time consensus, one interesting research problem in consensus, refers to the agreement of a group of agents on a common state in finite time. Finite-time consensus is first studied in [10] where a nonsmooth consensus algorithm is proposed and the finite-time convergence analysis of the closed-loop system is presented under an undirected fixed/switching interaction graph. Then a continuous nonlinear consensus algorithm is proposed in [11] to guarantee the finite-time stability under an undirected fixed interaction graph. The proposed algorithm in [11] is shown in [12] to be able to guarantee finite-time consensus under an undirected switching interaction and a directed fixed interaction graph when each strongly connected component of the topology is detail-balanced. Another continuous nonlinear consensus algorithm is proposed in [13] to guarantee the finite-time stability under a directed fixed interaction graph. Different from [10]–[13] where the final state of all agents is in general not controllable, several nonlinear consensus algorithms are proposed in [14] to guarantee the finite-time χ-consensus, where the final equilibrium state can be controlled by designing the χ function. In contrast to [10]–[14] where finite-time consensus is studied for first-order kinematics, a nonlinear algorithm is proposed in [15] to solve the finite-time consensus for double-integrator dynamics under an undirected fixed interaction graph.

The existing research on consensus focuses mainly on the case when no inherent dynamics is considered for the agents. However, in many practical systems, inherent (nonlinear) dynamics often exists for the agents. For example, in the synchronization of complex dynamical networks [16]–[19], to name a few, the dynamics of each node is normally described by the sum of a continuously differentiable function describing the inherent dynamics associated with the node and the coupling item identifying the corresponding connection between the node and the other nodes. More comprehensive details on the study of the synchronization of complex dynamical networks can be found in [20], [21]. Recently, inherent nonlinear dynamics has also been considered in the consensus problem [22]–[25]. The authors in [22] study first-order consensus of multi-agent systems in the presence of inherent nonlinear dynamics. Sufficient conditions are given to guarantee first-order consensus under a directed fixed interaction graph. The authors in [23] study finite-time consensus for first-order kinematics with inherent nonlinear dynamics. Sufficient conditions are given to guarantee finite-time consensus under an undirected switching interaction graph. The authors in [24] study second-order consensus of multi-agent systems with inherent nonlinear dynamics. Sufficient conditions are derived to guarantee second-order consensus under a directed fixed interaction graph. In [25], the authors propose a connectivity-preserving second-order consensus algorithm for multi-agent systems with inherent nonlinear dynamics when there exists a virtual leader. It can be observed that finite-time consensus for second-order dynamics with inherent nonlinear dynamics has not been considered in the existing literature.

In this paper, we study finite-time consensus for second-order dynamics with inherent nonlinear dynamics under

The authors are with the Department of Electrical Engineering, University of California, Riverside, CA, 92521. This work was supported by NSF CAREER Award ECCS–0748287.

an undirected fixed interaction graph. Compared with the study of finite-time consensus for second-order dynamics and consensus for second-order dynamics with inherent nonlinear dynamics, finite-time consensus for second-order dynamics with inherent nonlinear dynamics is more challenging because these two problems are considered simultaneously. It is also worth emphasizing that the proposed techniques in the study of finite-time consensus for second-order dynamics and consensus for second-order dynamics with inherent nonlinear dynamics are not applicable in the study of finite-time consensus for second-order dynamics with inherent nonlinear dynamics. To solve the problem, we first propose a nonlinear distributed consensus algorithm and then present sufficient conditions such that finite-time consensus can be achieved. By employing a similar stability analysis, it is expected that finite-time consensus can be achieved with/without inherent nonlinear dynamics when the initial undirected interaction graph is connected and the difference between the Laplacian matrix for any $t > 0$ and the Laplacian matrix at $t = 0$ is small enough.

II. Preliminaries

A. Graph Theory Notions

For a team of n agents, the interaction among all agents can be modeled by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$, where $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$ and $\mathcal{W} \subseteq \mathcal{V}^2$ represent, respectively, the agent set and the edge set. An edge in an undirected graph denoted as (v_i, v_j) means that agents i and j can obtain information from each other. Accordingly, agent i is a neighbor of agent j and vice versa. An undirected graph is connected if there is an undirected path between every pair of distinct agents.

There are two commonly used matrices used to represent the interaction graph: the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ with $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{W}$ and $a_{ij} = 0$ otherwise, and the Laplacian matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ with $\ell_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ and $\ell_{ij} = -a_{ij}$, $i \neq j$. In particular, we let that $a_{ii} = 0$, $i = 1, \cdots, n$, (*i.e.*, agent i is not a neighbor of itself), $a_{ij} = a_{ji}$ (*i.e.*, $\mathcal{A}$ and $\mathcal{L}$ are symmetric). It is straightforward to verify that $\mathcal{L}$ is symmetric semi-definite and $\mathcal{L}$ has at least one eigenvalue equal to 0 with a corresponding left eigenvector $\mathbf{1}_n^T$ and a corresponding right eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n$ is an $n \times 1$ all-one column vector.

B. Notations

We use $\mathbb{R}$ to denote the set of real number. $\mathbf{0}_n \in \mathbb{R}^n$ is used to denote the $n \times 1$ all-zero column vector and $\mathbf{0}_{m\times n} \in \mathbb{R}^{m\times n}$ is used to denote the $m \times n$ all-zero matrix. $\text{diag}(\kappa_1, \cdots, \kappa_n)$ is used to denote the $n\times n$ diagonal matrix with the ith diagonal entry given by κ_i. $I_n \in \mathbb{R}^{n\times n}$ is used to denote the identity matrix. $\otimes$ is used to denote the Kronecker product of matrices. We use $\lambda_2(\cdot)$ and $\lambda_{\max}(\cdot)$ to denote, respectively, the smallest nonzero eigenvalue and the largest eigenvalue of a symmetric Laplacian matrix corresponding to an undirected connected interaction graph. $\|\cdot\|$ is used to denote the $2-$norm. We use $\text{sgn}(\cdot)$ to denote the signum function. Define $\text{sig}(x)^\alpha \triangleq \text{sgn}(x)\,\|x\|^\alpha$. Note that $\text{sig}(x)^\alpha$ is continuous with respect to x when $\alpha > 0$. Let $f : [0, \infty) \mapsto J \subseteq \mathbb{R}^n$ be a continuous function. The upper right-hand derivative of $f(t)$ is given by $D^+ f(t) = \limsup_{h\to 0^+} \frac{1}{h}[f(t+h) - f(t)]$. Given two real vectors $x \triangleq [x_1, \ldots, x_p]^T \in \mathbb{R}^p$ and $y \triangleq [y_1, \ldots, y_p]^T \in \mathbb{R}^p$, we use $x \leq y$ to denote that $x_i \leq y_i, \forall i = 1, \ldots, p$.

Define

$$M \triangleq \frac{1}{n}\left[nI_n - \mathbf{1}_n\mathbf{1}_n^T\right]. \tag{1}$$

Let $\varrho_i, i = 1, \cdots, n$, be the ith eigenvalue of M satisfying that $\varrho_1 \leq \varrho_2 \leq \cdots \leq \varrho_n$. Because M is a Laplacian matrix corresponding to an undirected connected graph, $\varrho_1 = 0$ and $\varrho_i > 0, i = 2, \cdots, n$. Let Γ be the unique orthonormal matrix such that $M = \Gamma^T \text{diag}(\varrho_1, \cdots, \varrho_n)\Gamma$. It is worth noting that $M^2 = M$.

C. Problem Statement

Consider a group of n agents given by

$$\dot{r}_i = v_i, \quad \dot{v}_i = f(t, r_i, v_i) + u_i, \quad i = 1, \cdots, n, \tag{2}$$

where $r_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^m$ are, respectively, the position and the velocity of the ith agent, $f(t, r_i, v_i) : \mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m \mapsto \mathbb{R}^m$ is the unknown inherent nonlinear dynamics for the ith agent, and $u_i \in \mathbb{R}^m$ is the control input for the ith agent. Here we assume that

$$\|f(t, r_i, v_i) - f(t, r_j, v_j)\| \leq \gamma(\|r_i - r_j\| + \|v_i - v_j\|), \tag{3}$$

where γ is a known positive constant. The objective here is to design u_i such that $\|r_i(t) - r_j(t)\| \to 0$ and $\|v_i(t) - v_j(t)\| \to 0$ in finite time for all $i, j = 1, \cdots, n$. That is, all agents' states reach consensus in finite time. Due to the existence of the nonlinear term $f(t, r_i, v_i)$, $\dot{v}_i(t)$ in general does not approach $\mathbf{0}_m$, which is different from the traditional case where $\dot{v}_i(t) \to \mathbf{0}_m$ as $t \to \infty$.

Remark 2.1: Because $f(t, r_i, v_i)$ in (2) is unknown, (2) can not be converted to the well-studied second-order system by letting $u_i = -f(t, r_i) + \phi_i$ with ϕ_i being the new control input to be designed. Moreover, due to the existence of the unknown nonlinear term $f(t, r_i, v_i)$, the consensus problem is more challenging than that without the nonlinear term $f(t, r_i, v_i)$.

III. Finite-time Convergence Under an Undirected Fixed Interaction

In this section, we consider an undirected fixed interaction graph. We use $\mathcal{A}$, $\mathcal{L}$, and $\mathcal{G}$ to denote, respectively, the adjacency matrix, the Laplacian matrix, and the undirected graph associated with the n agents.

We propose the following nonlinear finite-time consensus algorithm for (2) as

$$\begin{aligned} u_i = &-\sum_{j=1}^{n} a_{ij}[\text{sig}(r_i - r_j)^{\alpha_1} + \text{sig}(v_i - v_j)^{\alpha_2}] \\ &- \beta\sum_{j=1}^{n} a_{ij}[(r_i - r_j) + (v_i - v_j)], \end{aligned} \tag{4}$$

where β is a positive constant, a_{ij} is the (i, j)th entry of the adjacency matrix $\mathcal{A}$ characterizing the interaction among the n agents, and $\alpha_1 \in (0, 1)$ and $\alpha_2 \in (0, 1)$ are two constant scalars. The objective of the first term in (4) is to guarantee the finite-time convergence while the objective of the second term in (4) is to guarantee the stability when the nonlinear term $f(t, r_i, v_i)$ exists in (2). It is worthwhile to mention that the stability analysis is, in general, difficult to analyze due to the existence of the unknown nonlinear term $f(t, r_i, v_i)$.

Definition 3.1: A function $f : \mathbb{R}^d \mapsto \mathbb{R}^m$ is locally Lipschitz of order χ at $x \in \mathbb{R}^d$ if there exists L_x and $\epsilon \in (0, \infty)$ such that $\|f(y) - f(y')\| \leq L_x \|y - y'\|^\chi$ for all $y, y' \in B(x, \epsilon)$, where $\chi > 0$ and $B(x, \epsilon)$ denotes the ball centered at x with radius ϵ.

Lemma 3.1: Suppose that $F(t, z) : [t_0, T) \times J \subseteq \mathbb{R}^p \mapsto \mathbb{R}^q$, is a continuous function satisfying that $D^+F = f(t, z)$, where $z \in \mathbb{R}^p$, and $f(t, z)$ is piecewise continuous in t and is locally χ-Lipschitz in z when $f(t, z)$ is continuous at t. Let $G(t, \omega) : [t_0, T) \times J \subseteq \mathbb{R}^p \mapsto \mathbb{R}^q$ be a continuous function whose upper right-hand derivative D^+G satisfies the differential inequality $D^+G \leq f(t, \omega)$ with $G[t_0, \omega(t_0)] \leq F[t_0, z(t_0)]$. Then $G(t) \leq F(t)$ for all $t \in [t_0, T)$.

Proof: The proof is similar to that of Lemma 3.3 in [23] based on the two lemmas presented in the Appendix. ■

Lemma 3.2: [26] Let $A \in \mathbb{R}^{p\times p}$ have eigenvalues β_i with associated eigenvectors $f_i, i = 1, \ldots, p$, and let $B \in \mathbb{R}^{q\times q}$ have eigenvalues ρ_j with associated eigenvectors $g_j, j = 1, \ldots, q$. Then the pq eigenvalues of $A \otimes B$ are $\beta_i\rho_j$ with associated eigenvectors $f_i \otimes g_j, i = 1, \ldots, p, j = 1, \ldots, q$.

Lemma 3.3: Let $\mathcal{L}$ be the Laplacian matrix associated with an undirected connected graph. Let

$$P \triangleq \begin{bmatrix} \beta\mathcal{L} & M \\ M & M \end{bmatrix} \tag{5}$$

and

$$Q \triangleq \begin{bmatrix} \beta\mathcal{L} - B_1 & \frac{\beta\mathcal{L}-B_1-B_2}{2} \\ \frac{\beta\mathcal{L}-B_1-B_2}{2} & \beta\mathcal{L} - B_2 \end{bmatrix}, \tag{6}$$

where B_1 and B_2 are two (time-varying) matrices satisfying the following two conditions:

1. Each row sum of $B_i, i = 1, 2,$ is equal to zero;
2. Each off-diagonal entry of $B_i, i = 1, 2,$ denoted as $\vartheta_i^j,\ i = 1, 2,\ j = 1, \ldots, n^2 - n$, satisfies that $\left|\vartheta_i^j\right| < \varphi$,

and β is a positive constant. When $\beta \geq \max\{\frac{(1+\epsilon)\lambda_{\max}(M)}{\lambda_2(\mathcal{L})}, \frac{8(n-1)\varphi}{\lambda_2(\mathcal{L})}\}$, P and Q are positive semi-definite, where ϵ is any positive constant. In particular, for any $\mu \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^n$, $\begin{bmatrix}\mu^T & \nu^T\end{bmatrix} \begin{bmatrix} p\beta\mathcal{L} & M \\ M & M \end{bmatrix} \begin{bmatrix} \mu \\ \nu \end{bmatrix}$=0 only if $\mu = \varsigma_1 \mathbf{1}_n$ and $\nu = \varsigma_2 \mathbf{1}_n$, where $p \geq 1$ is a constant scalar, ς_1 and ς_2 are two constant scalars.

Proof: We first show that P is positive semi-definite under the condition of the lemma. Note that (5) can be written as

$$P = \underbrace{\begin{bmatrix} (1+\epsilon)M & M \\ M & M \end{bmatrix}}_{P_1} + \underbrace{\begin{bmatrix} \beta\mathcal{L} - (1+\epsilon)M & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}}_{P_2}. \tag{7}$$

According to the definition of M in (1), we know that M is a Laplacian matrix corresponding to an undirected connected graph. Therefore, all eigenvalues of M are nonnegative. Note also that $P_1 = \begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1 \end{bmatrix} \otimes M$. Because the two eigenvalues of $\begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1 \end{bmatrix}$ are positive, it then follows from Lemma 3.2 that all eigenvalues of P_1 are nonnegative except two zero eigenvalues, which implies that P_1 is positive semi-definite. Therefore, P is positive semi-definite if P_2 is positive semi-definite. Note that P_2 is positive semi-definite if and only if $\beta\mathcal{L} - (1 + \epsilon)M$ is positive semi-definite. For an arbitrary vector $x \in \mathbb{R}^n$, we can rewrite x as $x = x^\parallel + x^\perp$, where $x^\parallel$ is the projection of x along the vector $\mathbf{1}_n$ and $x^\perp$ is the projection of x in the plane that is perpendicular to the vector $\mathbf{1}_n$. It follows that

$$\begin{aligned}
&x^T \left[\beta\mathcal{L} - (1+\epsilon)M\right] x \\
=&(x^\parallel + x^\perp)^T \left[\beta\mathcal{L} - (1+\epsilon)M\right] (x^\parallel + x^\perp) \\
=&(x^\parallel + x^\perp)^T \left[\beta\mathcal{L} - (1+\epsilon)M\right] x^\parallel + (x^\parallel \\
&+ x^\perp)^T \left[2\beta\mathcal{L} - (1+\epsilon)M\right] x^\perp \\
=&(x^\parallel + x^\perp)^T \left[\beta\mathcal{L} - (1+\epsilon)M\right] x^\perp \\
=&(x^\parallel)^T \left[\beta\mathcal{L} - (1+\epsilon)M\right] x^\perp \\
&+ (x^\perp)^T \left[2\beta\mathcal{L} - (1+\epsilon)M\right] x^\perp \\
=&(x^\perp)^T \left[\beta\mathcal{L} - (1+\epsilon)M\right] x^\perp \\
\geq&\beta\lambda_2(\mathcal{L})(x^\perp)^T x^\perp - (1+\epsilon)\lambda_{\max}(M)(x^\perp)^T x^\perp.
\end{aligned} \tag{8}$$

When $\beta \geq \frac{(1+\epsilon)\lambda_{\max}(M)}{\lambda_2(\mathcal{L})}$ [i.e., $\beta\lambda_2(\mathcal{L}) \geq (1+\epsilon)\lambda_{\max}(M)$], it follows that $\beta\lambda_2(\mathcal{L})(x^\perp)^T x^\perp - (1+\epsilon)\lambda_{\max}(M)(x^\perp)^T x^\perp \geq 0$, which implies that $x^T \left[\beta\mathcal{L} - (1+\epsilon)M\right] x \geq 0$. That is, $\beta\mathcal{L} - (1 + \epsilon)M$ is positive semi-definite when $\beta \geq \frac{(1+\epsilon)\lambda_{\max}(M)}{\lambda_2(\mathcal{L})}$. Therefore, P is positive semi-definite when $\beta \geq \frac{(1+\epsilon)\lambda_{\max}(M)}{\lambda_2(\mathcal{L})}$.

We next show that Q is positive semi-definite under the condition of the lemma. Note that Q can be rewritten as

$$Q = \beta \underbrace{\begin{bmatrix} \mathcal{L} & \frac{1}{2}\mathcal{L} \\ \frac{1}{2}\mathcal{L} & \mathcal{L} \end{bmatrix}}_{Q_1} + \underbrace{\begin{bmatrix} -B_1 & \frac{-B_1-B_2}{2} \\ \frac{-B_1-B_2}{2} & -B_2 \end{bmatrix}}_{Q_2}. \tag{9}$$

Similar to the previous analysis for P, when $\beta > \frac{\lambda_{\max}(Q_2)}{\lambda_2(Q_1)}$, Q is positive semi-definite. According to the Gershgorin disc theorem, $\lambda_{\max}(Q_2) \leq 4(n-1)\varphi$. Note that $Q_1 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \otimes \mathcal{L}$, it follows from Lemma 3.2 that $\lambda_2(Q_1) = \frac{1}{2}\lambda_2(\mathcal{L})$. Therefore, when $\beta > \frac{8(n-1)\varphi-1}{\lambda_2(\mathcal{L})}$, Q is positive semi-definite.

Lastly, we show that, for any $p \geq 1$, $\mu \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^n$, $\begin{bmatrix}\mu^T & \nu^T\end{bmatrix} \begin{bmatrix} p\beta\mathcal{L} & M \\ M & M \end{bmatrix} \begin{bmatrix} \mu \\ \nu \end{bmatrix}$=0 only if $\mu = \varsigma_1 \mathbf{1}_n$ and $\nu = \varsigma_2 \mathbf{1}_n$, where ς_1 and ς_2 are two constant scalars. For simplicity, we only study the case when $p = 1$. Similar analysis can be applied to the case for $p > 1$. When $p = 1$, $\begin{bmatrix} p\beta\mathcal{L} & M \\ M & M \end{bmatrix} = P$. Note that $P_1 = \begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1 \end{bmatrix} \otimes M$ and the two eigenvalues of $\begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1 \end{bmatrix}$ are positive. Without loss of generality, let the two eigenvalues of $\begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1 \end{bmatrix}$ be λ_1 and λ_2 with the corresponding right eigenvectors given by ω_1 and ω_2. It then follows from Lemma 3.2 that P_1 has only two zero eigenvalues with the associated eigenvectors given by $\omega_1 \otimes \mathbf{1}_n$ and $\omega_2 \otimes \mathbf{1}_n$. Combining with the facts that ω_1 and ω_2 are two real vectors and P_1 is positive semi-definite implies that $\begin{bmatrix} \mu^T & \nu^T \end{bmatrix} P \begin{bmatrix} \mu \\ \nu \end{bmatrix} = 0$ only if $\begin{bmatrix} \mu \\ \nu \end{bmatrix} = k_1\omega_1 \otimes \mathbf{1}_n + k_2\omega_2 \otimes \mathbf{1}_n$, where k_1 and k_2 are two constant scalars. Note also that $\begin{bmatrix} \mu^T & \nu^T \end{bmatrix} P_2 \begin{bmatrix} \mu \\ \nu \end{bmatrix} = 0$ only if $\mu = k_3\mathbf{1}_n$, where k_3 is a constant scalar. It then follows from (7) that $\begin{bmatrix} \mu^T & \nu^T \end{bmatrix} P \begin{bmatrix} \mu \\ \nu \end{bmatrix} = 0$ only if $\mu = \varsigma_1\mathbf{1}_n$ and $\nu = \varsigma_2\mathbf{1}_n$, where ς_1 and ς_2 are two constant scalars. ∎

We next present the main result for (2) using (4) under an undirected fixed interaction graph when $m = 1$ (*i.e.,* one-dimensional case). When $m > 1$ (*i.e.,* high-dimensional case), similar results can be obtained by applying the corresponding analysis to each dimension.

Theorem 3.2: Assume that the interaction graph $\mathcal{G}$ is undirected and connected. Using (4) for (2), $|r_i(t) - r_j(t)| \to 0$ and $|v_i(t) - v_j(t)| \to 0$ in finite time if $\beta \geq \max\{\frac{(1+\epsilon)\lambda_{\max}(M)}{\lambda_2(\mathcal{L})}, \frac{8(n-1)\gamma}{n\lambda_2(\mathcal{L})}\}$, where ϵ is any positive constant and γ is defined in (3).

Proof: The proof of the theorem is based on Lemma 3.1. In order to employ Lemma 3.1, the main objective here is to propose proper functions F and G such that their upper right-hand derivatives satisfy the conditions in Lemma 3.1. We will first propose the function G and analyze the corresponding upper right-hand derivative. Then we will propose the function F and analyze its upper right-hand derivative. Note that both G and F are carefully designed here.

Define $\overline{r} \triangleq \frac{1}{n}\sum_{i=1}^n r_i$, $\overline{v} \triangleq \frac{1}{n}\sum_{i=1}^n v_i$, $\tilde{r}_i \triangleq r_i - \overline{r}$, and $\tilde{v}_i \triangleq v_i - \overline{v}$. We can get

$$\dot{\tilde{r}}_i = \tilde{v}_i$$

and

$$\begin{aligned}
\dot{\tilde{v}}_i = & \frac{1}{n}\sum_{j=1}^n [f(t, r_i, v_i) - f(t, r_j, v_j)] \\
& - \sum_{j=1}^n a_{ij}[\mathrm{sig}(r_i - r_j)^{\alpha_1} + \mathrm{sig}(v_i - v_j)^{\alpha_2}] \\
& - \beta\sum_{j=1}^n a_{ij}[(r_i - r_j) + (v_i - v_j)] \\
= & \frac{1}{n}\sum_{j=1}^n [f(t, \tilde{r}_i + \overline{r}, \tilde{v}_i + \overline{r}) - f(t, \tilde{r}_j + \overline{r}, \tilde{v}_j + \overline{v})] \\
& - \sum_{j=1}^n a_{ij}[\mathrm{sig}(\tilde{r}_i - \tilde{r}_j)^{\alpha_1} + \mathrm{sig}(\tilde{v}_i - \tilde{v}_j)^{\alpha_2}] \\
& - \beta\sum_{j=1}^n a_{ij}[(\tilde{r}_i - \tilde{r}_j) + (\tilde{v}_i - \tilde{v}_j)]. \qquad (10)
\end{aligned}$$

Note that $f(t, r_i, v_i)$ satisfies (3). It follows that (10) can be written as

$$\begin{aligned}
\dot{\tilde{v}}_i = & \frac{1}{n}\sum_{j=1}^n [b_{ij}^r(t)(r_i - r_j) + b_{ij}^v(t)(v_i - v_j)] \\
& - \sum_{j=1}^n a_{ij}[\mathrm{sig}(\tilde{r}_i - \tilde{r}_j)^{\alpha_1} + \mathrm{sig}(\tilde{v}_i - \tilde{v}_j)^{\alpha_2}] \\
& - \beta\sum_{j=1}^n a_{ij}[(\tilde{r}_i - \tilde{r}_j) + (\tilde{v}_i - \tilde{v}_j)], \qquad (11)
\end{aligned}$$

where $|b_{ij}^r(t)| \leq \gamma$ and $|b_{ij}^v(t)| \leq \gamma$ due to (3). Let $B_1 = [b_{ij}^1] \in \mathbb{R}^{n\times n}$ and $B_2 = [b_{ij}^2] \in \mathbb{R}^{n\times n}$ be defined such that $b_{ij}^1 = -\frac{1}{n}b_{ij}^r(t)$ and $b_{ij}^2 = -\frac{1}{n}b_{ij}^v(t)$ for all $i \neq j$, $b_{ii}^1 = \frac{1}{n}\sum_{j=1,j\neq i}^n b_{ij}^r(t)$, and $b_{ii}^2 = \frac{1}{n}\sum_{j=1,j\neq i}^n b_{ij}^v(t)$. Define $\tilde{r} \triangleq [\tilde{r}_1, \cdots, \tilde{r}_n]^T$, $\tilde{v} \triangleq [\tilde{v}_1, \cdots, \tilde{v}_n]^T$, and $\eta \triangleq [\tilde{r}^T \ \ \tilde{v}^T]^T$. Consider the function $G(t, \eta) = \frac{1}{2}\eta^T\widetilde{P}\eta$, where

$$\widetilde{P} \triangleq \begin{bmatrix} 2\beta\mathcal{L} & M \\ M & M \end{bmatrix}. \qquad (12)$$

Note that $\widetilde{P} = P + \begin{bmatrix} \beta\mathcal{L} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}$ and $\begin{bmatrix} \beta\mathcal{L} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}$ is positive semi-definite. It follows from Lemma 3.3 that $\widetilde{P}$ is positive semi-definite under the condition of the theorem. Define

$$\chi \triangleq [\chi_1, \cdots, \chi_n]^T \qquad (13)$$

with $\chi_i = -\sum_{j=1}^n a_{ij}[\mathrm{sig}(\tilde{r}_i - \tilde{r}_j)^{\alpha_1} + \mathrm{sig}(\tilde{v}_i - \tilde{v}_j)^{\alpha_2}]$. Then the upper right-hand derivative of $G(t, \eta)$ can be derived as

$$\begin{aligned}
D^+G(t, \eta) = & \limsup_{h\to 0^+} \frac{1}{h}[G(t + h, \eta(t + h)) - G(t, \eta(t))] \\
= & 2\beta\tilde{r}^T\mathcal{L}\dot{\tilde{r}} + \tilde{r}^T M\dot{\tilde{v}} + \tilde{v}^T M\tilde{v} + \tilde{v}^T M\dot{\tilde{v}} \\
= & -\eta^T\Omega\eta + \tilde{r}^T M\chi + \tilde{v}^T M\chi,
\end{aligned}$$

where

$$\Omega = \begin{bmatrix} M(\beta\mathcal{L} - B_1) & E \\ E^T & M(\beta\mathcal{L} - B_2) - M \end{bmatrix}$$

with $E \triangleq \frac{-2\beta\mathcal{L} + M(\beta\mathcal{L} - B_1) + (\beta\mathcal{L} - B_2)M}{2}$. According to the definition of M in (1), we have that $M\mathcal{L} = \mathcal{L}$, $\mathcal{L}M = \mathcal{L}$, $MB_i = B_i$, and $B_iM = B_i$, $i = 1, 2$. Therefore, Ω can be simplified as

$$\Omega = \begin{bmatrix} \beta\mathcal{L} - B_1 & \frac{-B_1 - B_2}{2} \\ \frac{-B_1 - B_2}{2} & \beta\mathcal{L} - B_2 - M \end{bmatrix}.$$

In order to employ Lemma 3.1, it is important to propose a proper function F based on the previous function G and compute its upper right-hand derivative, which can satisfy the conditions in Lemma 3.1. For the purpose, we consider the closed-loop dynamics given by

$$\begin{aligned}
\dot{\xi}_i = & \varpi_i, \\
\dot{\varpi}_i = & -\sum_{j=1}^n a_{ij}[\mathrm{sig}(\xi_i - \xi_j)^{\alpha_1} + \mathrm{sig}(\varpi_i - \varpi_j)^{\alpha_2}], \\
& i = 1, \cdots, n, \qquad (14)
\end{aligned}$$

where $\xi_i \in \mathbb{R}$, $\varpi \in \mathbb{R}$, $\xi_i(0) = \kappa r_i(0)$, $\varpi(0) = \kappa v_i(0)$, and

$$\kappa = \begin{cases} 1, & [\eta(0)]^T \widetilde{P} \eta(0) = 0, \\ \sqrt{\frac{[\eta(0)]^T \widetilde{P} \eta(0)}{[\eta(0)]^T P \eta(0)}}, & \text{otherwise}, \end{cases} \quad (15)$$

where $\widetilde{P}$ is defined in (12), η is defined before (12), and P is defined in (5). Define $\overline{\xi} \triangleq \frac{1}{n}\sum_{j=1}^n \xi_j$, $\overline{\varpi} \triangleq \frac{1}{n}\sum_{j=1}^n \varpi_j$, $\widetilde{\xi}_i = \xi_i - \overline{\xi}$, $\widetilde{\varpi}_i = \varpi_i - \overline{\varpi}$, $\widetilde{\xi} \triangleq [\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n]^T$, and $\widetilde{\varpi} \triangleq [\widetilde{\varpi}_1, \ldots, \widetilde{\varpi}_n]^T$. Then (14) can be rewritten as

$$\dot{\widetilde{\xi}}_i = \widetilde{\varpi}_i, \quad \dot{\widetilde{\varpi}}_i = \dot{\varpi}_i - \dot{\overline{\varpi}}, \quad i = 1, \cdots, n.$$

Let $\zeta \triangleq [\widetilde{\xi}^T, \widetilde{\varpi}^T]^T$, and consider the function $F(t, \zeta) \triangleq \zeta^T P \zeta$, where P is defined in (5). Define $\widetilde{\chi} \triangleq [\widetilde{\chi}_1, \cdots, \widetilde{\chi}_n]^T$ with $\widetilde{\chi}_i = -\sum_{j=1}^n a_{ij}[\mathrm{sig}(\tilde{\xi}_i - \tilde{\xi}_j)^{\alpha_1} + \mathrm{sig}(\widetilde{\varpi}_i - \widetilde{\varpi}_j)^{\alpha_2}]$. Then the upper right-hand derivative of $F(t, \zeta)$ can be derived as

$$\begin{aligned} D^+F(t,\zeta) &= \limsup_{h \to 0^+} \frac{1}{h}[F(t+h, \zeta(t+h)) - F(t, \zeta(t))] \\ &= \beta \widetilde{\xi}^T \mathcal{L} \dot{\widetilde{\xi}} + \widetilde{\xi}^T M \dot{\widetilde{\varpi}} + \widetilde{\varpi}^T M \widetilde{\varpi} + \widetilde{\varpi}^T M \dot{\widetilde{\varpi}} \\ &= -\zeta^T \widetilde{\Omega} \zeta + \widetilde{\xi}^T M \widetilde{\chi} + \widetilde{\varpi}^T M \widetilde{\chi}, \end{aligned}$$

where $\widetilde{\Omega} = \begin{bmatrix} \mathbf{0}_{n \times n} & -\frac{1}{2}\beta\mathcal{L} \\ -\frac{1}{2}\beta\mathcal{L} & -M \end{bmatrix}$.

With the previously proposed functions F and G and the computation of D^+F and D^+G, we next check the conditions in Lemma 3.1. Note that $D^+F(t,\zeta)$ is continuous in t and is locally Lipschitz with order χ in ζ. Note also that

$$\begin{aligned} & D^+G(t,\eta) - D^+F(t,\eta) \\ =& -\eta^T \Omega \eta + \tilde{r}^T M \chi + \tilde{v}^T M \chi \\ & - (-\eta^T \widetilde{\Omega} \eta + \tilde{r}^T M \chi + \tilde{v}^T M \chi) \\ =& -\eta^T Q \eta. \end{aligned}$$

According to Lemma 3.3, Q is positive semi-definite under the condition of the theorem. Therefore, $D^+G(t,\eta) - D^+F(t,\eta) \le 0$. When $\xi_i(0) = \kappa r_i(0)$ and $\varpi(0) = \kappa v_i(0)$, it follows that $F(0, \zeta(0)) = G(0, \eta(0))$. It then follows from Lemma 3.1 that $G(t) \le F(t)$ for all $t \ge 0$. Given (14), if the undirected graph $\mathcal{G}$ is connected, it follows from Theorem 1 in [15] that $F(t) \to 0$ in finite time. Note from the definition of $G(t)$ for all $t \ge 0$ that $G(t) \ge 0$ for all $t \ge 0$. It then follows from the facts that $G(t) \ge 0$, $G(t) \le F(t)$, and $F(t) \to 0$ in finite time that $G(t) \to 0$ in finite time. From Lemma 3.3, $G(t) = 0$ only if $\eta = \begin{bmatrix} \varsigma_1 \mathbf{1}_n \\ \varsigma_2 \mathbf{1}_n \end{bmatrix}$ where ς_1 and ς_2 are two constant scalars. It then follows that $\tilde{r}(t) \to \varsigma_1 \mathbf{1}_n$ and $\tilde{v}(t) \to \varsigma_2 \mathbf{1}_n$ in finite time. Noting that $\mathbf{1}_n^T \tilde{r}(t) = 0$ and $\mathbf{1}_n^T \tilde{v}(t) = 0$ because $\mathbf{1}_n^T \tilde{r}(0) = 0$, $\mathbf{1}_n^T \tilde{v}(0) = 0$, $\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{1}_n^T \tilde{r}(t)] = 0$, and $\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{1}_n^T \tilde{v}(t)] = 0$, it follows that $\varsigma_1 = 0$ and $\varsigma_2 = 0$. That is, $\tilde{r}(t) \to \mathbf{0}_n$ and $\tilde{v}(t) \to \mathbf{0}_n$ in finite time. Because $\tilde{r}(t) = Mr(t)$, $\tilde{v}(t) = Mv(t)$, and M has only one eigenvalue equal to zero with the associated eigenvector given by $\mathbf{1}_n$, it follows that $|r_i(t) - r_j(t)| \to 0$ and $|v_i(t) - v_j(t)| \to 0$ in finite time. ■

Remark 3.3: From (11), the inherent nonlinear dynamics can be considered the disturbance introduced to the system when the inherent nonlinear dynamics does not exist. Specifically, the effect of the inherent nonlinear dynamics can be interpreted as two (time-varying) matrices added to the Laplacian matrix corresponding to the undirected fixed interaction graph. In particular, the two matrices satisfy: (1) the off-diagonal entries of them are bounded; and (2) the row sums of them are all equal to zero. Accordingly, the analysis in Theorem 3.2 can be used to analyze the finite-time consensus for second-order dynamics with/without inherent nonlinear dynamics under an undirected *switching* interaction graph. Specifically, in the presence/absence of the inherent nonlinear dynamics, finite-time consensus for second-order multi-agent systems is expected under an undirected switching interaction graph if the initial undirected interaction graph is connected and the difference between the Laplacian matrix for any $t > 0$ and the Laplacian matrix at $t = 0$ is small enough.

IV. Conclusion

In this paper, we studied finite-time consensus for distributed multi-agent systems with second-order dynamics in the presence of inherent nonlinear dynamics under an undirected fixed interaction graph. We proposed a nonlinear distributed consensus algorithm and presented sufficient conditions to guarantee finite-time consensus. By employing a similar stability analysis, it was expected that finite-time consensus can be achieved with/without inherent nonlinear dynamics when the initial undirected interaction graph is connected and the difference between the Laplacian matrix for any $t > 0$ and the Laplacian matrix at $t = 0$ is small enough.

References

[1] J. N. Tsitsiklis, D. P. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," *IEEE Transactions Automatic Control*, vol. 31, no. 9, pp. 803–812, 1986.

[2] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions Automatic Control*, vol. 48, no. 6, pp. 988–1001, June 2003.

[3] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions Automatic Control*, vol. 49, no. 9, pp. 1520–1533, September 2004.

[4] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions Automatic Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[5] M. Cao, A. S. Morse, and B. D. O. Anderson, "Agreeing asynchronously," *IEEE Transactions Automatic Control*, vol. 53, no. 8, pp. 1826–1838, September 2008.

[6] W. Ren and R. W. Beard, *Distributed Consensus in Multi-vehicle Cooperative Control*, ser. Communications and Control Engineering. London: Springer-Verlag, 2008.

[7] F. Bullo, J. Cortés, and S. Martínez, *Distributed Control of Robotic Networks*, ser. Applied Mathematics Series. Princeton University Press, 2009.

[8] Z. Qu, *Cooperative Control of Dynamical Systems: Applications to Autonomous Vehicles*. Springer-Verlag, 2009.

[9] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[10] J. Cortes, "Finite-time convergent gradient flows with applications to network consensus," *Automatica*, vol. 42, no. 11, pp. 1993–2000, 2006.

[11] Q. Hui, W. M. Haddad, and S. P. Bhat, "Finite-time semistability and consensus for nonlinear dynamical networks," *IEEE Transactions Automatic Control*, vol. 53, no. 8, pp. 1887–1890, September 2008.

[12] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Transactions Automatic Control*, vol. 55, no. 4, pp. 950–955, 2010.

[13] F. Xiao, L. Wang, J. Chen, and Y. Gao, "Finite-time formation control for multi-agent systems," *Automatica*, vol. 45, no. 11, pp. 2605–2611, 2009.

[14] X. Wang and Y. Hong, "Distributed finite-time χ-consensus algorithms for multi-agent systems with variable coupling topology," *Journal of Systems Science and Complexity*, vol. 23, no. 2, pp. 209–218, 2010.

[15] ——, "Finite-time consensus for multi-agent networks with second-order agent dynamics," in *IFAC World Congress*, Soeul, Korea, July 2008, pp. 15 185–15 190.

[16] T. Nishikawa, A. E. Motter, Y.-C. Lai, and F. C. Hoppensteadt, "Heterogeneity in oscillator networks: Are smaller worlds easier to synchronize?" *Physical Review Letters*, vol. 91, no. 1, p. 014101, 2003.

[17] J. Lu, X. Yu, and G. Chen, "Chaos synchronization of general complex dynamic networks," *Physica A*, vol. 334, no. 1–2, pp. 281–302, 2004.

[18] J. Lu and G. Chen, "A time-varying complex dynamical network model and its controlled synchronization criteria," *IEEE Transactions Automatic Control*, vol. 50, no. 6, pp. 841–846, 2005.

[19] J. Zhou, J. an Lu, and J. Lu, "Adaptive synchronization of an uncertain complex dynamical network," *IEEE Transactions Automatic Control*, vol. 51, no. 4, pp. 652–656, April 2006.

[20] C. Wu, *Synchronization in complex networks of nonlinear dynamical systems*. World Scientific, Singapore, 2007.

[21] G. Chen, X. Wang, X. Li, and J. Lu, "Some recent advances in complex networks synchronization," in *Recent Advances in Nonlinear Dynamics and Synchronization*, ser. Studies in Computational Intelligence. Springer, 2009, pp. 3–16.

[22] W. Yu, G. Chen, and M. Cao, "Consensus in directed networks of agents with nonlinear dynamics," *IEEE Transactions Automatic Control*, 2011, in press.

[23] Y. Cao, W. Ren, F. Chen, and G. Zong, "Finite-time consensus of multi-agent networks with inherent nonlinear dynamics," in *Proceedings of the American Control Conference*, San Francesco, CA, June 2011, pp. 4020–4025.

[24] W. Yu, G. Chen, M. Cao, and J. Kurths, "Second-order consensus for multi-agent systems with directed topologies and nonlinear dynamics," *IEEE Transactions on Systems, Man, and Cybernetics – Part B: Cybernetics*, vol. 40, no. 3, pp. 881–891, 2010.

[25] H. Su, G. Chen, X. Wang, and Z. Lin, "Adaptive second-order consensus of networked mobile agents with nonlinear dynamics," *Automatica*, 2010, in press.

[26] A. J. Laub, *Matrix analysis for scientists and engineers*. SIAM, 2005.

[27] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002.

V. Appendix

Lemma 5.1: Let $f(t,x,\lambda)$ be continuous in (t,x,λ) and locally Lipschitz of order χ in x (uniformly in t and λ) on $[t_0,t_1]\times D\times\{\|\lambda-\lambda_0\|\le c\}$, where $D\subseteq\mathbb{R}^n$ is an open connected set. Let $y(t,\lambda_0)$ be a (unique) solution of $\dot{x}=f(t,x,\lambda_0)$ with $y(t_0,\lambda_0)=y_0\in D$. Suppose that $y(t,\lambda_0)$ is defined and belongs to D for all $t\in[t_0,t_1]$. Then, given $\epsilon>0$, there is $\delta>0$ such that if $\|z_0-y_0\|<\delta$ and $\|\lambda-\lambda_0\|<\delta$, then there is a unique solution $z(t,\lambda)$ of $\dot{x}=f(t,x,\lambda)$ defined on $[t_0,t_1]$, with $z(t_0,\lambda)=z_0$, and $z(t,\lambda)$ satisfies $\|z(t,\lambda)-y(t,\lambda_0)\|<\epsilon$ for all $t\in[t_0,t_1]$.

Proof: The proof is similar to that of Theorem 3.5 in [27] by studying $\|z(t,\lambda)-y(t,\lambda_0)\|$ when $\|\lambda-\lambda_0\|$ is small enough. Due to the continuity of f with respect to λ, for any $\alpha>0$, there exists $\beta>0$ such that

$$\|f(t,x,\lambda)-f(t,x,\lambda_0)\|<\alpha,\ \forall(t,x)\in U,\ \forall\,\|\lambda-\lambda_0\|<\beta,$$

where $U\triangleq\{(t,x)\in[t_0,t_1]\times\mathbb{R}^n|\,\|x-y(t,\lambda_0)\|\le\epsilon\}$. Let $\alpha\le\epsilon$ and $\|y(0)-z(0)\|\le\alpha$. Suppose that $f(t,x,\lambda)$ is Lipschitz of order χ in x on U with a Lipschitz of order χ constant L. By following the analysis in the proof of Theorem 3.5 in [27], it follows that

$$\begin{aligned}
&\|z(t,\lambda)-y(t,\lambda_0)\|\\
\le&\|y(0)-z(0)\|+\int_{t_0}^{t}\|f(s,y(s))-f(s,z(s))\|\,ds\\
&+\int_{t_0}^{t}\|f(s,z(s),\lambda)-f(s,z(s),\lambda_0)\|\,ds\\
\le&\alpha+\alpha(t-t_0)+\int_{t_0}^{t}L\,\|y(s)-z(s)\|^{\chi}\,ds.
\end{aligned}$$

Let $\alpha=\frac{1}{\kappa(1+t-t_0)}$. When κ is sufficiently large (*i.e.,* α is sufficiently small), it follows that $\|z(t,\lambda)-y(t,\lambda_0)\|$ is sufficiently small. Therefore, there exists a positive $\alpha^\star$ such that $\|z(t,\lambda)-y(t,\lambda_0)\|<\epsilon$ when $\alpha\le\alpha^\star$. Then the proof completes by choosing $\delta=\min\{\alpha^\star,\beta\}$. ■

Lemma 5.2: Consider the following vector differential equation $\dot{z}=f(t,z)$, where $z\triangleq[z_1,\dots,z_p]^T\in\mathbb{R}^p$, and $f(t,z)\triangleq[f_1(t,z),\dots,f_p(t,z)]^T$ is defined such that $f_i(t,z)$, $i=1,\dots,p$, is continuous in t and locally Lipschitz of order χ in z_i, $i=1,\dots,p$, for all $t\ge t_0$ and all $z\in J\subseteq\mathbb{R}^p$. Let $[t_0,T)$ (T could be infinity) be the maximal interval of existence of the solution z, and suppose that $z\in J$ for all $t\in[t_0,T)$. Let $\omega\triangleq[\omega_1,\dots,\omega_p]^T\in\mathbb{R}^p$ be a continuous function whose upper right-hand derivative $D^+\omega$ satisfies the differential inequality $D^+\omega\le f(t,\omega)$ with $\omega(t_0)\le z(t_0)$, where $\omega\in J$ for all $t\in[t_0,T)$. Then $\omega(t)\le z(t)$ for all $t\in[t_0,T)$.

Proof: The proof is similar to that of Lemma 3.2 in [23] based on Lemma 5.1. ■