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Design of penalty functions for optimal control of linear dynamical systems under state and input constraints

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***Abstract*— This paper addresses the problem of solving a *constrained* optimal control for a general single-input single output linear time varying system by means of an *unconstrained* method. The exposed methodology uses a penalty function approach, commonly considered in finite dimensional optimization problem, and extended here it to the considered infinite dimensional (functional optimization) case. The main novelty is that both the bounds on the control variable and on a freely chosen output variable are considered and studied theoretically. It is shown that a relatively simple and constructive choice of penalty functions allows to completely alleviate the usual difficulties of handling such constraints in optimal control. An illustrative example is provided to show the potential of the method.**

I. INTRODUCTION

This paper addresses the problem of solving a *constrained* optimal control for a general single-input single-output (SISO) linear time varying system by means of an *unconstrained* method. The exposed methodology uses a penalty function approach, commonly considered in finite dimensional optimization problem.

In penalty function methods, an augmented performance index is introduced by adding to the original cost of the optimal control problem, so-called *penalty functions* that have some diverging asymptotic behavior when the constraints are approached by any tentative solution. Then, the augmented performance index is optimized, in the absence of constraints, yielding a biased estimate of the solution of the original problem. Gradually, the weight of the penalty functions is reduced to provide a converging sequence, hopefully diminishing the bias.

Computationally, the penalty function methods are appealing, as they yield unconstrained problems for which a vast range of highly effective algorithms are available. In finite dimensional optimization, outstanding algorithms have resulted from the careful analysis of the choice of penalty functions, and the sequence of weights. In particular, the *interior points methods* which are nowadays implemented in successful software packages such as KNITRO [1] have their foundations in these approaches. The interested reader can refer to [2] for an historical perspective starting from the 1960s.

In this article, we are interested in applying similar penalty methods to solve constrained optimal control problems. In numerous domains of engineering, optimal control appears as a very natural formulation of objectives, especially if constraints can be introduced. Unfortunately, these constraints induce some serious difficulties [3], [4], [5]. In particular, constraints bearing on state variables are difficult to characterize, as they generate both constrained and unconstrained arcs along the optimal trajectory. To determine optimality conditions, it is usually necessary to a-priori postulate the sequence and the nature of the arcs constituting the sought-after optimal trajectory. Active or inactive parts of the trajectory split the optimality system in as many algebraic and differential equations. Yet, not much is known on this sequence, and this often results in a high complexity. For this reason, it is often preferred to use a discretization based approach to this problem, and to treat it, e.g. through a collocation method [6], as a finite dimensional problem [7], [8], [9], [10], [11], [12], [13]. Of course, the resulting mathematical problem can be addressed by one of the interior points methods cited above. In this context, interior point algorithms have been applied to optimal control problems by Wright [14], Vicente [15], Leibfritz and Sachs [16], Jockenhövel, Biegler and Wächter [17]. This is not the path that we explore, as we desire to use indirect methods (a.k.a. adjoint methods) to take advantage of their accuracy.

Although there is a well established literature on the mathematical foundations of interior-point methods for finite-dimensional mathematical programming, this is not the case yet for optimal control problems [18], [19]. In [19], the authors investigate the merits of augmenting the cost to be minimized by an integral term penalizing the control when it approaches its upper or lower bound. This very natural idea appears to be a successful way to solve input-constrained optimal control problems but it also proves to require a detailed analysis. The main difficulty is, as noted in [19], that the divergence of the penalty value for constraints-touching (not strictly interior) trajectories is not automatically guaranteed. To illustrate this point, one can simply consider the following case. Let $u \geq 0$ be the constraint that one desires to penalize, using, e.g., a logarithmic penalty which is usually considered as a very good choice in finite dimensional optimization. Over any finite length interval, there exist continuous functions u such that $\min_t(u(t)) = 0$, while $\int \ln(u(t))\,dt < +\infty$. Therefore, there could exist trajectory touching the constraints that could be optimal. This problem of interiority in infinite dimensional optimization has been addressed in [19] for input-constraint optimal control where a formal result is established on such logarithmic functions. Similar interior points methods for

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state constrained optimal control problems have also been used (see [20], [21], [22] in coordination with saturation functions), but the interiority of the trajectory has so far been left as an assumption. In this paper, we consider both input constraints and constraints bearing on state variables. We establish that the trajectory of any penalized problem, provided that certain explicit guidelines are accounted for in the construction of the penalty functions, is strictly interior. The main elements of the proof of interiority in [19] are used. Variations of the trajectory are built allowing us to exhibit sufficient conditions on the penalty functions such that the solution of the penalized problem is interior (i.e. the cost diverges if the essential infimum of the distance between the constrained variable and its constraint is equal to 0). An important technicality differs though. Because of the state constraint, the employed variations cannot be local in time anymore, but are the result of global homotopies on the trajectory.

This theoretical contribution allows us to build a sequence of penalized unconstrained problems, that are numerically easy to solve, and that converge to the solution of the original problem. The interiority result and the obtained algorithm are the main contributions of this article.

This paper is organized as follows: in Section II, the constrained OCP is presented together with two penalized optimal control problem (POCP), a state and input constrained one and an input constrained one. For these two POCPs to be equivalent, two conditions must hold. In Section III, sufficient conditions on the penalty are derived such that the first condition holds. In Section IV, a sufficient condition on the state penalty is given such that the second condition holds as well. In Section V, a constructive choice of the penalty is given such that the two conditions aforementioned hold. There, a completely unconstrained algorithm converging to the solution of the constrained optimal control problem is given. The proposed algorithm is tested on an illustrative example in Section VI. Conclusions and perspectives are given in Section VII.

II. PRESENTATION OF THE PROBLEM

A. Constrained optimal control problem and notations.

In this article, we investigate the following normalized Constrained Optimal Control Problem (COCP)

$$\min_{u \in U^{\mathrm{ad}}} \left[J(x,u,t) = \int_0^1 \ell(x,u,t)dt \right] \tag{1}$$

where $\ell : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ is a Λ-Lipschitz smooth function. The variables $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are respectively the state and the control of the following linear time-varying (LTV) single-input single-output (SISO) dynamics

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) && (2)\\ y(t) &= C(t)x(t) && (3) \end{aligned}$$

with the (fixed) initial condition $x(0) = x_0$. The state space matrices are also assumed to belong in $L^\infty[0,T]$. A solution of (2)-(3) with input u is noted x^u and the corresponding output is noted y^u. Classically, the set of admissible controls U^{ad} is a subset of $L^2[0,1]$ defined by:

$$U^{\mathrm{ad}} = \{u \text{ s.t. } (u(t), y^u(t)) \in [-1,+1]^2\} \tag{4}$$

Now, let us define two useful subsets of U^{ad}

$$\begin{aligned} V^{\mathrm{ad}} &= \{u \text{ s.t. } (u(t), y^u(t)) \in (-1,+1)^2\} && (5)\\ W^{\mathrm{ad}}(\beta) &= \{u \text{ s.t. } (u(t), y^u(t)) \in [-1+\beta, 1-\beta]^2\} && (6) \end{aligned}$$

where $\beta \in [0,+1)$, and $W^{\mathrm{ad}}(0) = U^{\mathrm{ad}}$. Due to the linearity of the dynamics (2)-(3), for all $\beta \in [0,1)$, $W^{\mathrm{ad}}(\beta)$ is a convex set. In what follows, a control is said to be *strictly interior* when it belongs to some $W^{\mathrm{ad}}(\beta)$, with $\beta \in (0,+1)$. We note $\|.\|_\infty$ the usual $L^\infty[0,T]$ norm on $\mathbb{R}$, or on matrix spaces when needed.

Remark: This normalized form of optimal control problem allows to treat, after some changes of variables, numerous cases of interest. It can be made more general by introducing drift terms in the dynamics and biases in the output. This solely complexifies the exposition but leaves the main results unchanged.

B. Presentation of the penalized problem

Following the approach of interior methods in their application to optimal control, we use two penalty functions

$$\begin{aligned} \gamma_y(.) : [-1,1] &\to [0,+\infty) && (7)\\ \gamma_u(.) : [-1,1] &\to [0,+\infty) && (8) \end{aligned}$$

which are assumed to be strictly convex, symmetric, and go to infinity as their argument approaches one of the bounds ± 1. These functions serve to define a (first) Penalized Optimal Control Problem (POCP) with the dynamics (2)-(3)

$$\min_{u \in U^{\mathrm{ad}}} \left[K(u,\epsilon) = \int_0^1 \ell(x,u,t) + \epsilon\left[\gamma_y(y) + \gamma_u(u)\right] dt \right] \tag{9}$$

where $\epsilon > 0$. At this stage, not much has been gained since the POCP (9) is just as difficult to solve as the COCP (1). The main difficulties is the output constraints. This is a well-known fact in optimal control, as discussed in the introduction, stemming from the difficulty to handle the calculus of variations in this case. Interestingly, this point can be alleviated as will be shown. Let us define a second POCP

$$\min_{u \in (-1,1)} \left[K(u,\epsilon) = \int_0^1 \ell(x,u,t) + \epsilon\left[\gamma_y(y) + \gamma_u(u)\right] dt \right] \tag{10}$$

where $\epsilon > 0$. These two POCPs (9) and (10) are not equivalent for two reasons. First, the control in (9) is constrained to belong to $[-1,+1]$, while, on the other hand, the control in (10) belongs to $(-1,+1)$. Second, the output constraint used to define U^{ad} is not present in the formulation of (10). In the following, we wish to show that, provided γ_y and γ_u are suitably chosen, these problems are in fact equivalent. For this, we introduce a preliminary assumption

Assumption 1 (existence, uniqueness): There exists a unique global solution u^* for problem (9).

Under this assumption one obtains the following theorem.

Theorem 1: Let us formulate two conditions (C1) and (C2)

(C1) There exists $\beta \in (0,1)$ such that $u^* \in W^{\text{ad}}(\beta)$.

(C2) For any $\tilde{u} \in (-1,+1)$ such that $\tilde{u} \notin V^{\text{ad}}$, then $K(\tilde{u},\epsilon) = +\infty$ for all $\epsilon > 0$.

Under Assumption 1, if Conditions (C1)-(C2) hold, then there exists a unique solution $u^\sharp$ for problem (10) and one has:

$$u^\sharp = u^* \tag{11}$$

Proof: With β of (C1), from the definitions (4)-(5)-(6), one has

$$W^{\text{ad}}(\beta) \subset V^{\text{ad}} \subset U^{\text{ad}}$$

Thus,

$$\min_{u\in W^{\text{ad}}(\beta)} K(u,\epsilon) \geq \min_{u\in V^{\text{ad}}} K(u,\epsilon) \geq \min_{u\in U^{\text{ad}}} K(u,\epsilon)$$

Using (C1), one has $\min_{u\in W^{\text{ad}}(\beta)} K(u,\epsilon) = \min_{u\in U^{\text{ad}}} K(u,\epsilon) = K(u^*,\epsilon)$. Now, Condition (C2) means that problem (10) is equivalent to $\min_{u\in V^{\text{ad}}} K(u,\epsilon)$. Indeed, if u does not belong to V^{ad}, the cost $K(u,\epsilon)$ is not finite. As a consequence, to prove the theorem, we have to prove the existence and the uniqueness of $u^\sharp$ solution of $\min_{u\in V^{\text{ad}}} K(u,\epsilon)$. Using the fact that u^* is admissible for (10) and that it is the global minimizer of the cost $K(.,\epsilon)$ on $U^{\text{ad}} \supset V^{\text{ad}}$, then u^* is an optimal control for (10). The existence of an optimal control $u^\sharp$ for (10) is thus proven. Now let us consider an optimal control $u_2^\sharp$ for (10) such that $u_2^\sharp \neq u^*$. From Condition (C2), this control belongs to V^{ad}, then it is an admissible control for (9). Moreover, we have proven that the optimal cost for (10) is $K(u^*,\epsilon)$. Thus, $K(u_2^\sharp,\epsilon) = K(u^*,\epsilon)$. Then, $u_2^\sharp$ is an optimal control for (9). Since the optimal control for (9) is unique $u_2^\sharp = u^*$, This contradiction gives the uniqueness of the optimal control for (10) and one has:

$$u^\sharp = u^*$$

■

We now pursue as follows. In Section III, sufficient conditions on the penalty are derived such that Condition (C1) actually holds (Theorem 2). In Section IV, a sufficient condition on the state penalty is given such that Condition (C2) holds as well (Theorem 3). Eventually, in Section V, penalty functions satisfying the conditions from Sections III and IV are exhibited (Theorem 4). From Theorem 1, one has that these penalties guarantee the equivalence of problems (9) and (10).

III. FROM FEASIBLE TO INTERIOR TRAJECTORY

In this section, we determine sufficient conditions on the penalty functions $\gamma_u(.)$ and $\gamma_y(.)$ such that the optimal control u^* of the problem (9) belongs to a set $W^{\text{ad}}(\beta)$ with $\beta \in (0,1)$. This guarantees that Condition (C1) holds. To determine these conditions, we use a proof by contradiction. From u^*, a deviation method is exhibited, such that a constructed control u_{mod} is *strictly interior*. Then, sufficient conditions on the penalty are derived, such that the penalized cost K associated to u^* is greater than the cost K of u_{mod}. This contradicts the optimality of u^* if it is not *strictly interior*.

Section III-A exposes the construction of u_{mod} from u_{ref}. In Section III-B, the conditions on the penalties are exhibited and the main result is given in Theorem 2.

A. Deviation method

Consider any reference control $u_{\text{ref}} \in U^{\text{ad}}$ which can touch the constraint. This reference control may be an optimal control for problem (9), i.e. one can have $u_{\text{ref}} = u^*$.

We now formulate the following (accessibility) assumption on the system

Assumption 2 (accessibility): There exists $\beta_0 \in (0,1)$ such that

$$W^{\text{ad}}(\beta_0) \neq \emptyset \tag{12}$$

Now, pick any $u_i(t) \in W^{\text{ad}}(\beta_0)$. Then, define a modified control as follows

$$u_{\text{mod}}(t) = (1-\alpha)u_{\text{ref}}(t) + \alpha u_i(t), \text{ for a given } \alpha \in (0,1) \tag{13}$$

From this definition, $|u_{\text{mod}}(t)| \leq 1 - \alpha\beta_0$ for all $t \in [0,1]$. Due to the convexity of $U^{\text{ad}} = W^{\text{ad}}(0)$, (13) implies that $u_{\text{mod}} \in U^{\text{ad}}$. The control (13) directly impacts on the value of the state, and consequently on the output y. Using the linearity of the dynamics (2)-(3), a direct computation yields that u_{mod} satisfies

$$u_{\text{mod}}(t) \in W^{\text{ad}}(\alpha\beta_0) \tag{14}$$

This shows that the interpolation (13) generates a *strictly interior* trajectory.

B. Condition guaranteeing the strict interiority of the optimal trajectory

The following result gives an upper estimate on the difference $K(u_{\text{mod}},\epsilon) - K(u_{\text{ref}},\epsilon)$.

Proposition 1: Under Assumption 2, for any $\epsilon > 0$ one has

$$K(u_{\text{mod}},\epsilon) - K(u_{\text{ref}},\epsilon) \leq \alpha\left[U(\epsilon) - L_{u_{\text{ref}}}(\epsilon,\alpha) - L_{y^{u_{\text{ref}}}}(\epsilon,\alpha)\right] \tag{15}$$

with

$$\begin{aligned} U(\epsilon) &\triangleq 4\Lambda + 2\epsilon\left[\gamma_y'(1-\beta_0) + \gamma_u'(1-\beta_0)\right] \\ L_{u_{\text{ref}}}(\epsilon,\alpha) &\triangleq \epsilon(1-\alpha)\beta_0\gamma_u'(1-2\alpha+\beta_0\alpha^2)\mu_{u_{\text{ref}}}(\beta_0\alpha) \\ L_{y^{u_{\text{ref}}}}(\epsilon,\alpha) &\triangleq \epsilon(1-\alpha)\beta_0\gamma_y'(1-2\alpha+\beta_0\alpha^2)\mu_{y^{u_{\text{ref}}}}(\beta_0\alpha) \end{aligned}$$

and, for any measurable function z

$$\mu_z(s) \triangleq \text{meas}\left(\{t \quad \text{s.t.} \quad |z(t)| \geq 1-s\}\right) \tag{16}$$

where meas(.) is the Lebesgue measure (see [23]) of its argument.

Proof: See Appendix A. ■

Finally, using (15), the following result holds.

Lemma 1: Under Assumption 2, if for all $\epsilon > 0$, there exists $\alpha \in (0,1)$ such that

$$L_{u_{\text{ref}}}(\epsilon,\alpha) + L_{y^{u_{\text{ref}}}}(\epsilon,\alpha) > U(\epsilon) \tag{17}$$

then

$$K(u_{\text{mod}},\epsilon) < K(u_{\text{ref}},\epsilon), \quad \forall \epsilon > 0 \tag{18}$$

Using Proposition 1 together with Lemma 1, one has the following result.

Theorem 2: Under Assumptions 1 and 2, consider u^* the optimal control for problem (9) and assume that for all $\epsilon > 0$ there exists $\alpha \in (0,1)$ such that inequality (17) holds for $u_{\text{ref}} = u^*$ and $y^{u_{\text{ref}}} = y^{u^*}$, then there exists $\beta > 0$ such that

$$u^* \in W^{\text{ad}}(\beta) \tag{19}$$

and Condition (C1) holds.

Proof: With the notations of the statement, we wish to prove that u^* is strictly interior. Assume it is not, by picking any $u_i \in W^{\text{ad}}(\beta_0)$, where β_0 is defined by Assumption 2, one can construct $u_{\text{mod}}(t) = (1-\alpha)u^*(t) + \alpha u_i(t)$ with $0 < \alpha \leq 1$. The control u_{mod} belongs to $W^{\text{ad}}(\alpha\beta_0)$. Then, using Proposition 1 and Lemma 1, one directly gets

$$K(u_{\text{mod}},\epsilon) < K(u^*,\epsilon)$$

This contradicts the optimality of u^* and concludes the proof. ∎

Theorem 2 provides a sufficient condition on the penalty functions (under the form of inequalities) such that Condition (C1) from Theorem 1 holds. Now, Condition (C2) must hold as well, this is the subject of the following section.

IV. FEASIBILITY OF OUPUTS OF THE PENALIZED PROBLEM

In this section, we study how the penalty function $\gamma_y(.)$ can be used to guarantee that any $\tilde{u} \in (-1,+1)$ which does not belong in V^{ad} is such that $K(\tilde{u},\epsilon) = +\infty$ for all $\epsilon > 0$. This is (C2).

Lemma 2: Under Assumption 2, consider problem (10). Assume that the penalty function γ_y is such that the following holds

$$\lim_{\alpha \downarrow 0} \gamma_y(1-\alpha)\mu_y(\alpha) = +\infty \tag{20}$$

where $\mu_y(.)$ is defined in equation (16) and y is any output solution of (2)-(3) with input $u \in (-1,+1)$. Then any control $\tilde{u} \in (-1,+1)$ which does not belong to V^{ad} yields $K(\tilde{u},\epsilon) = +\infty$.

Proof: Since, $\gamma_y \geq 0$, we have

$$\mathcal{I} \triangleq \int_0^1 \gamma_y(y(t))dt \geq \int_{|y(t)|\geq 1-\alpha} \gamma_y(y(t))dt \triangleq \mathcal{J}(\alpha)$$

Since $\gamma_y \geq 0$, $\mathcal{J}(\alpha)$ is an increasing positive function of $\alpha \in [0,1]$, thus $\mathcal{J}(\alpha)$ is minimum in $\alpha = 0$

$$\begin{aligned} \mathcal{J}(0) &= \lim_{\alpha\downarrow 0} \int_{|y(t)|\geq 1-\alpha} \gamma_y(y(t))dt \\ &\geq \lim_{\alpha\downarrow 0} \gamma_y(1-\alpha)\mu_y(\alpha) \end{aligned}$$

with $\mu_y(.)$ the Lebesgue measure defined in equation (16). If (20) holds, then $\mathcal{J}(0) = +\infty$ which implies that $\mathcal{I} = +\infty$. Since $\tilde{u} \in (-1,1)$ and $\tilde{u} \notin V^{\text{ad}}$ the corresponding output $y^{\tilde{u}}$ is such that there exists $\tau \in [0,1]$ such that $|y(\tau)| = 1$. Then, the cost $K(\tilde{u},\epsilon)$ is infinite. This concludes the proof. ∎

Since the measure μ_y appears in equations (17) and (20), it is important to give a lower bound on it. This will be exploited in Section V, in the explicit construction of suitable penalty functions. Noting that y is the output of a LTV system with a bounded input, a lower bound is given by the following Lemma.

Lemma 3: Considering a reference output y, given by (2)-(3) with input $u \in (-1,+1)$, such that there exists $\tau \in [0,1]$ with $|y(\tau)| = 1$. Then, there exists a constant $K < +\infty$ such that the measure $\mu_y(\alpha)$ of the set $\{t \quad \text{s.t.} \quad |y(t)| \geq 1-\alpha\}$ is lower-bounded under the form

$$\mu_y(\alpha) \geq \frac{\alpha}{K} \tag{21}$$

Proof: The proof is given in Appendix B together with the expression of K. ∎

Using Lemmas 2 and 3, one finally has the following result.

Theorem 3: Under Assumption 2, if the state penalty γ_y is such that

$$\lim_{\alpha\downarrow 0} \gamma_y(1-\alpha)\frac{\alpha}{K} = +\infty \tag{22}$$

then Condition (C2) holds.

V. MAIN RESULT AND ALGORITHM

In Section III and IV, conditions have been given, under the form of Theorem 2 and Theorem 3, such that the Conditions (C1)-(C2) required in Theorem 1 hold. These theorems are given under the form of inequalities (17) and (22) depending on a parameter $\alpha \in (0,1)$. In this section, a class of penalty functions γ_y and γ_u are given such that for all $\epsilon > 0$ there exists $\alpha \in (0,1)$ such that inequalities (17) and (22) actually hold.

A. Penalty design

The inequality (17) is now studied. Depending on the nature of the optimal trajectory of (9), the desired strict positivity of $L_{u_{\text{ref}}}(.,.) + L_{y^{u_{\text{ref}}}}(.,.) - U(.)$ may stem from $L_{y^{u_{\text{ref}}}}(.,.)$ or from $L_{u_{\text{ref}}}(.,.)$. When $L_{y^{u^*}}(t)(\epsilon,0) = 0$ (i.e. when the optimal trajectory does not touch the output constraints), our study requires that an assumption on the behavior on the measure $\mu_{u^*}(.)$ is formulated.

Assumption 3 (touching of input constraint): If $\text{supess}_t\, u^*(t) = 1$, then there exists $M > 0$ and $q \geq 0$ such that the asymptotic behavior close to zero of the measure $\mu_{u^*}(.)$ defined in equation (16) satisfies:

$$\mu_{u^*}(\alpha) \geq M\alpha^q \tag{23}$$

Fig. 1 provides an illustration of the possible values $q = 0, 1, 2$ in (23).

We are now ready to state our main result.

Theorem 4: Under Assumptions 1, 2 and 3, there exists penalty functions $\gamma_y(.)$ and $\gamma_u(.)$ such that problems (9) and

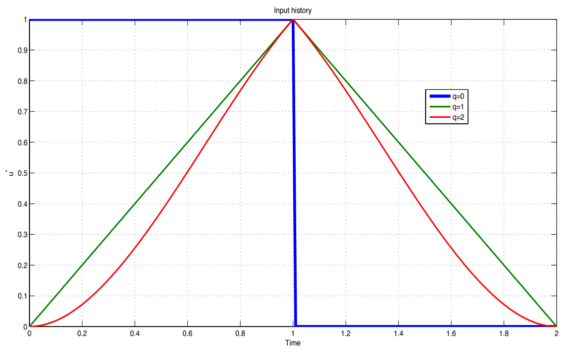

Fig. 1. Local behavior of the input near the constraints for different values of q. A touch point corresponds to $q > 0$, while a constrained arc gives $q = 0$ in (23)

(10) are equivalent: their unique solution u^* and $u^\sharp$ are equal. A particular choice of penalty is:

$$\gamma_y(y) = \left[\frac{1}{2}\left(\frac{2}{\sqrt{1-y^2}} - 1\right)\right]^{n_y} \tag{24}$$

$$\gamma_u(u) = \left[\frac{1}{2}\left(\frac{2}{\sqrt{1-u^2}} - 1\right)\right]^{n_u} \tag{25}$$

with $n_y > 2$ and $n_u > \max\{1, 2(q-1)\}$.

Proof: The existence is proven by showing that (24) and (25) are suitable penalties. First, let us prove that Condition (C2) holds. The penalty (24) is such that equation (22) is satisfied, then Theorem 3 holds. Thus (C2) holds.

Now, let us prove that the optimal solution u^* of (9) belongs to $W^{\mathrm{ad}}(\beta)$, $\beta \in (0,1)$. The proof considers three mutually exclusive cases.

- If there exists $s \in (0,1)$ such that $\mu_{y^{u^*}}(s) = \mu_{u^*}(s) = 0$, then, the optimal trajectory does not touch the constraint, so there exists $\beta \in (0,1)$ such that $u^*(t)$ belongs to $W^{\mathrm{ad}}(\beta)$.
- If $\sup_t |y^{u^*}(t)| = 1$ ($u^* \notin W^{\mathrm{ad}}(\beta)$, $\beta \in (0,1)$), then, using Lemma 3, the state penalty (24) is such that $\lim_{\alpha_\downarrow 0} \gamma'_y(1 - 2\alpha + \beta_0\alpha^2)\mu_{y^{u^*}}(\alpha\beta_0) \geq \lim_{\alpha_\downarrow 0} \gamma'_y(1 - 2\alpha + \beta_0\alpha^2)\frac{\alpha}{K} = +\infty$. Moreover, γ'_y is a continuous function. As a consequence, there always exits $\alpha \in (0,1)$ such that Theorem 2 holds. Then there exists $\beta \in (0,1)$ such that $u^* \in W^{\mathrm{ad}}(\beta)$. This contradiction shows that this case is impossible.
- If there exists $s \in (0,1)$ such that $\mu_{y^{u^*}}(s) = 0$ and $\mathrm{supess}_t |u^*(t)| = 1$ ($u^* \notin W^{\mathrm{ad}}(\beta)$, $\beta \in (0,1)$), then, the control penalty (25) is such that $\lim_{\alpha_\downarrow 0} \gamma'_u(1 - 2\alpha + \beta_0\alpha^2)\mu_{u^*}(\alpha\beta_0) = +\infty$. Moreover, γ'_u and μ_u are continuous functions. As a consequence, there always exists $\alpha \in (0,1)$ such that Theorem 2 holds. Then there exists $\beta \in (0,1)$ such that $u^* \in W^{\mathrm{ad}}(\beta)$. This contradiction shows that this case is impossible.

We have proven that there always exists $\alpha \in (0,1)$ such that Theorems 2 and 3 holds, then Conditions (C1) and (C2) from Theorem 1 hold. This implies that problems (9) and (10) are equivalent. ■

B. Investigation of convergence

Theorem 4 allows us to solve problem (10) instead of problem (9). Our ultimate goal is to solve (1), which as announced earlier in Section II, is approached by a sequence of POCPs (9), or simpler, thanks to the equivalence, a sequence of POCPs (10). One such algorithm is presented below. Now, let us mention a few facts on convergence of the constructed sequence $(u_{\epsilon_n}, \epsilon_n)_{n\in\mathbb{N}}$ where $(\epsilon_n)_{n\in\mathbb{N}}$ is a decreasing sequence converging to zero, and $u^*_{\epsilon_n}$ the solution of (10) for $\epsilon = \epsilon_n$. The proof of convergence of the cost $\lim_{n\to+\infty} P(u^*_{\epsilon_n}, \epsilon_n) = J^*$ follows along the same lines as the proof in [20] and [21]. To prove to convergence of $u^*_{\epsilon_n}$ an assumption on the strong convexity of J can be used. More details can be found in [21].

C. Algorithm

First, to have a completely unconstrained algorithm, the following change in variable is used

$$u \triangleq \phi(\nu) = \tanh(\nu) \tag{26}$$

Where ν is an unconstrained variable such that $\tanh(\nu) \in L^2[0,T]$, and such that the corresponding POCP

$$\min_\nu P(\nu, \epsilon) = \int_0^1 \ell(x, \phi(\nu), t) + \epsilon[\gamma_y(y) + \gamma_u \circ \phi(\nu)]dt \tag{27}$$

is defined with the penalty functions from (24) and (25).

Theorem 5: Under Assumptions 1, 2 and 3, and from Theorem 4, problems (9) and (27) are equivalent in the sense that there exists an optimal solution ν^* of (27) such that

$$u^* = \tanh(\nu^*)$$

where u^* is the optimal solution of (10).

Proof:

$$\begin{aligned}\min_{u\in(-1,+1)} K(u,\epsilon) &= \min_{\phi(\nu)} K(\phi(\nu), \epsilon)\\ &= \min_{\phi(\nu)} P(\nu, \epsilon)\end{aligned}$$

$\phi(.)$ being a bijective mapping, one obtains

$$\min_{u\in(-1,+1)} K(u,\epsilon) = \min_\nu P(\nu, \epsilon)$$

Note $u^\sharp$ the optimal control for (10). Then, the control $\tilde{\nu} = \tanh^{-1}(u^\sharp)$ is an optimal control for (27). Thus, the existence is proven. Now, consider $\nu_2 \neq \tilde{\nu}$ such that $P(\nu_2, \epsilon) = P(\tilde{\nu}, \epsilon)$. Then, the control $u_2 = \tanh(\nu_2) \neq u^\sharp$ is also a global minimizer of (10). This contradicts the uniqueness of $u^\sharp$. Thus, the control $\nu^* = \tilde{\nu}$ is the unique global minimizer of (27) and one has $u^\sharp = \tanh(\nu^*)$. Then, from Theorem 4, one obtains

$$u^* = \tanh(\nu^*)$$

where u^* is the optimal solution of problem (9). ■

The main purpose of the main result of this paper, i.e. Theorem 4 (and Theorem 5 which stems from it), is to allow one to solve a simple OCP (Problem (27)) instead of a constrained OCP (Problem (9)) because they are equivalent.

Each problem (27) penalized by ϵ from a sequence (ϵ_n) can be solved using the calculus of variations. Define the Hamiltonian of the penalized problem (27) as follows

$$H_\epsilon(x,\nu,p,t) \triangleq \ell(x,\phi(\nu),t)+\epsilon\left[\gamma_y(y)+\gamma_u\circ\phi(\nu)\right] +p(t)^T\left[A(t)x(t)+B(t)\phi(\nu)\right] \quad (28)$$

where $p(t)\in\mathbb{R}^n$ is the adjoint state of Pontryagin solution of $\frac{dp}{dt}=-\frac{\partial H_\epsilon}{\partial x}$ and where the penalty functions are chosen according to Theorem 4. The choice of n_u can be made by trial and error which solely depend on the nature of the sought-after (but a-priori unknown) optimal solution u^*. Now, defining a positive decreasing sequence, one can approach the solution of (1).

- Step 1: Initialize the continuous functions $x(t)$ and $p(t)$ such that the initial $|C(t)x(t)|<1$ for all $t\in[0,1]$, and set $\epsilon=\epsilon_0$. Note that $x(t)$ and $p(t)$ need not to satisfy any differential equation at this stage, even if it is better if they do.
- Step 2: Solve for each time $\frac{\partial H_\epsilon}{\partial \nu}=0$, and note ν_ϵ^* the solution.
- Step 3: Solve the $2n$ differential equations $\frac{dx}{dt}=A(t)x(t)+B(t)\phi(\nu_\epsilon^*)$ and $\frac{dp}{dt}=-\frac{\partial H_\epsilon}{\partial x}(x,\nu_\epsilon^*,p,t)$ forming a two point boundary values problem using bvp4c (see [24]), with the following boundary constraints $x(0)=x_0$ and $p(1)=0$.
- Step 4: Decrease ϵ, initialize $x(t)$ and $p(t)$ with the solutions found at Step 3 and restart at Step 2.

Convergence of the state in $L^\infty([0,1])$ and convergence of the control in $L^2([0,1])$ for OCP (1) ([21], [20]) can be established as well.

VI. NUMERICAL EXAMPLE

To illustrate the proposed methodology, we consider the following simple example of constrained OCP

$$\ddot{x}(t)=u(t)-w(t) \quad (29)$$

where $w(t)=3$ if $t\in[3,6]$ and $w(t)=2$ everywhere else. The state constraint is $x(t)\in[x^-(t),x^+(t)]$ and the input constraint is $u(t)\in[u^-(t),u^+(t)]$ where u^- and u^+ are continuous piece-wise affine functions and x^+, x^- are continuous functions. The cost to minimize is $J=\int_0^T\frac{u^2}{2}dt$, $T=10$. The final state is free. Interestingly, the possible initial guess $u(t)\equiv 0$ is not a feasible control because $x^u(t)$ does not satisfy the state constraints. Thus, the trivial solution (for which the adjoint state is equal to zero) is not a feasible solution. The state penalty is chosen according to equation (24) with $n_y=3$, and the control penalty has been chosen according to equation (25) with $n_u=1$. The algorithm has been initialized by setting $x(t)=(x^+(t)+x^-(t))/2$ and $\dot{x}(t)\equiv 0$, which is not a solution of (2). Moreover, the adjoint state is also initialized with $p(t)\equiv 0$. The sequence (ϵ_n) is a sequence of twenty values logarithmically decreasing from 1 to 10^{-10}.

The algorithm of Section V-C generates a sequence of control that are converging to a limit solution. One can see on figures 2, 3 that the solution provided by the proposed

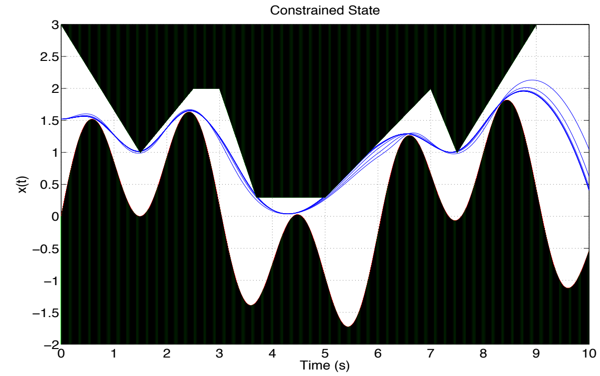

Fig. 2. Histories of the optimal constrained state $x(t)$ while decreasing the penalty parameter ϵ from 1 to 10^{-10}. The white domain is the feasible domain $[x^-(t),x^+(t)]$

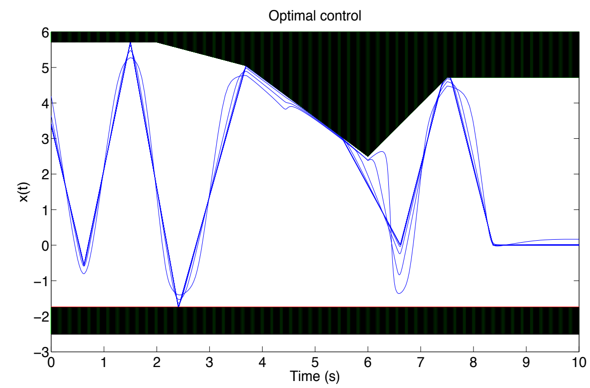

Fig. 3. Histories of the optimal control $u(t)$ while decreasing the penalty parameter ϵ from 1 to 10^{-10}. The white domain is the feasible domain $[u^-(t),u^+(t)]$

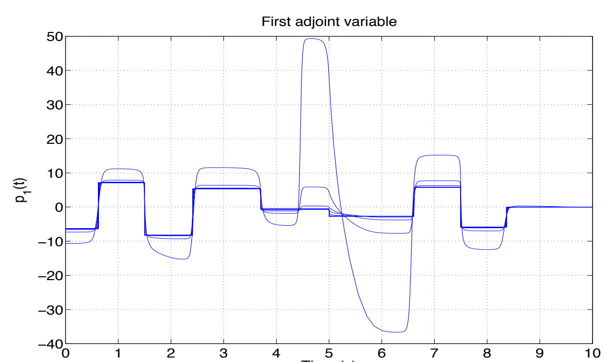

Fig. 4. History of the first adjoint parameter while decreasing the penalty parameter ϵ from 1 to 10^{-10}. Observe the appearance of discontinuities (which are expected as the constraints become active).

unconstrained algorithm computes controls and trajectories that strictly satisfy the constraints. Moreover, one can see in figure 4 that the first adjoint state exhibits some discontinuities when the constrained state reaches a junction between a constrained and an unconstrained arc. The optimal cost given by the optimization procedure is $J^* = 37.846$. It is well-estimated as soon as ϵ get below $10^{-7.5}$ (see Table I).

TABLE I
ITERATIONS

Value of ϵ	Cost
$\epsilon = 1$	$J = 42.5032$
$\epsilon = 10^{-1}$	$J = 39.328$
$\epsilon = 10^{-2}$	$J = 38.257$
$\epsilon = 10^{-3.5}$	$J = 37.902$
$\epsilon = 10^{-4.5}$	$J = 37.861$
$\epsilon = 10^{-5.5}$	$J = 37.851$
$\epsilon = 10^{-6.5}$	$J = 37.848$
$\epsilon = 10^{-7.5}$	$J = 37.846$
$\epsilon = 10^{-8.5}$	$J = 37.846$
$\epsilon = 10^{-10}$	$J = 37.846$

VII. CONCLUSIONS AND FUTURE WORKS

As a result of the proposed study, a practical method to solve constrained optimal control problems for LTV systems has been given. It solely requires the mathematical formulation of a suitably penalized OCP. A constructive choice has been given. Then, this unconstrained problem can be handled using a classic two-point boundary value problem solver. The presented iterative algorithm using an off-the-shelf routine is quite easy to implement and provides satisfactory results. Yet, the constituted algorithm is relatively slow, in particular for small value of the penalty parameter ϵ. Indeed, Step 2 of the algorithm can be hard to solve due to the bad shape of $\frac{\partial H_\epsilon}{\partial \nu}$. Further, no internal information is kept from one value of the penalty parameter to the next. This lack of coordination and the resulting slow-down is not problematic to treat the presented "toy" problem, but reveals costly when considering long horizon or high order optimization. This is our need as we apply the proposed technique to determine optimal energy management strategies (and the dual problem of energetic systems setups) for low consumption buildings over several weeks or months. Therefore, an important objective is to increase the speed of the algorithm.

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APPENDIX

A. Proof of Proposition 1

The difference $K(u_{\text{mod}}, \epsilon) - K(u_{\text{ref}}, \epsilon)$ can be decomposed as follows

$$K(u_{\text{mod}}, \epsilon) - K(u_{\text{ref}}, \epsilon) = K^+ + K^- \tag{30}$$

where $K^+ \geq 0$ (resp. $K^- \leq 0$) represents the possible increase (resp. decrease) on the penalized cost (9) when compared to u_{ref}.

1) An upper bound on the possible increase K^+: To exhibit an upper bound on the possible increase, K^+ is split into three parts itself: the possible increase of the original cost $\int \ell(x,u,t)dt$ and the possible increases due to the state and control penalties, separately.

a) Possible increase of the original cost: There, an upper bound on the possible increase of $\int_0^1 |\ell(x^{u_{\text{mod}}}, u_{\text{mod}}, t)| - |\ell(x^{u_{\text{ref}}}, u_{\text{ref}}, t)|dt$ is exhibited. Let us call K_ℓ this upper bound. Now, let us consider that the cost function $\int \ell(x,u,t)dt$ is Lipschitz with constant Λ, then using equations (13) one has:

$$\begin{aligned} K_\ell &\leq \Lambda\alpha(|y^+ - y^-| + |u^+ - u^-|) \\ K_\ell &\leq \Lambda 4\alpha \end{aligned} \tag{31}$$

b) Possible increase due to the control penalty: Note $K_u \triangleq \int_0^1 \gamma_u(u_{\text{mod}}) - \gamma_u(u_{\text{ref}})dt$. The integrand is positive only when $|u_i(t)| \geq |u_{\text{ref}}|$. Otherwise, the interpolation (13) makes u_{mod} lie further from the saturation than u_{ref}, and the integrand is negative. Using the convexity and symmetry properties of the penalties, one obtains

$$\begin{aligned} K_u &\leq \int_{|u_i| \geq |u_{\text{ref}}|} \gamma_u(u_{\text{mod}}) - \gamma_u(u_{\text{ref}})dt \\ K_u &\leq \int_{|u_i| \geq |u_{\text{ref}}|} \alpha |u_i(t) - u_{\text{ref}}| \gamma_u'(|u_i(t)|)dt \end{aligned}$$

and, then

$$K_u \leq 2\alpha\gamma_u'(1-\beta_0) \tag{32}$$

c) Possible increase due to the state penalty: Note $K_y \triangleq \int_0^1 \gamma_y(y^{u_{\text{mod}}}) - \gamma_y(y^{u_{\text{ref}}})dt$. The calculus follows the exact same lines of (32). One obtains

$$K_y \leq 2\alpha\gamma_y'(1-\beta_0) \tag{33}$$

d) Summary of the upper bounds: Denote $\alpha U(\epsilon)$ the upper bound $K^+ = K_\ell + K_u + K_y$ from equation (30). Then using equation (31), (32) and (33), one has

$$K^+ \leq \alpha U(\epsilon) = \alpha 4\Lambda + 2\epsilon\alpha\left[\gamma_y'(1-\beta_0) + \gamma_u'(1-\beta_0)\right] \tag{34}$$

2) A lower bound on the possible decrease K^-: In this part, a lower bound on $|K^-|$ is exhibited. To do so, two lower bounds are constructed, one on the possible decrease due the control penalty and the other due to the state penalty itself.

a) Possible decrease due to the control penalty: In this paragraph a lower bound $-\alpha L_{u_{\text{ref}}}(\alpha,\epsilon)$ on $K_u \triangleq \epsilon\int_0^1 \gamma_u[u_{\text{mod}}] - \gamma_u[u_{\text{ref}}]dt$ is exhibited. As already discussed, the integrand is negative only when $|u_{\text{mod}}| \leq |u_{\text{ref}}|$. To exhibit the desired lower bound, we restrain the integral on the domain where $|u_{\text{ref}}| \geq 1 - \alpha\beta_0$. The integral over this domain is strictly negative. Using convexity and symmetry properties of the penalty functions and equation (16) one has

$$\begin{aligned} K_u &\leq \epsilon\int_{|u_{\text{ref}}| \geq 1-\alpha\beta_0} \gamma_u(u_{\text{mod}}) - \gamma_u(u_{\text{ref}})dt \\ K_u &\leq -\epsilon\int_{|u_{\text{ref}}| \geq 1-\alpha\beta_0} \alpha|u_i - u_{\text{ref}}|\gamma_u'(|u_{\text{mod}}(t)|)dt \end{aligned}$$

Thus, one gets $K_u \leq -\alpha L_{u_{\text{ref}}}(\alpha,\epsilon)$ with

$$-\alpha L_{u_{\text{ref}}}(\alpha,\epsilon) \triangleq -\alpha\epsilon(1-\alpha)\beta_0\gamma_u'(1-2\alpha+\beta_0\alpha^2)\mu_{u_{\text{ref}}}(\beta_0\alpha) \tag{35}$$

b) Possible decrease due to the state penalty: There, a lower bound $-\alpha L_{y^{u_{\text{ref}}}}(\alpha,\epsilon)$ on K_y is derived. The calculus follows the exact same lines as earlier and one obtains $K_y \leq -\alpha L_{y^{u_{\text{ref}}}}(\alpha,\epsilon)$ with

$$-\alpha L_{y^{u_{\text{ref}}}}(\alpha,\epsilon) \triangleq -\epsilon\alpha(1-\alpha)\beta_0\gamma_y'(1-2\alpha+\beta_0\alpha^2)\mu_{y^{u_{\text{ref}}}}(\beta_0\alpha) \tag{36}$$

c) Summary of the lower bounds: The lower bound on $K^- = K_u + K_y$ from (30) is the sum of the lower bounds $-\alpha L_{u_{\text{ref}}}(\alpha,\epsilon)$ from (35) and $-\alpha L_{y^{u_{\text{ref}}}}(\alpha,\epsilon)$ from (36)

$$0 \geq -\alpha L_{u_{\text{ref}}}(\alpha,\epsilon) - \alpha L_{y^{u_{\text{ref}}}}(\alpha,\epsilon) \geq K^- \tag{37}$$

3) An upper bound on $K(u_{mod},\epsilon) - K(u_{ref},\epsilon)$: Gathering (34) and (37), one finally obtains

$$K(u_{\text{mod}},\epsilon) - K(u_{\text{ref}},\epsilon) \leq \alpha\left[U(\epsilon) - L_{u_{\text{ref}}}(\epsilon,\alpha) - L_{y^{u_{\text{ref}}}}(\epsilon,\alpha)\right] \tag{38}$$

This concludes the proof of Proposition 1.

B. Proof of Lemma 3

Considering the state space representation (2)-(3), we have:

$$\|Ax + Bu\|_\infty \quad \leq \quad G(\|x\|_\infty + 1)$$

with $G \triangleq \sup\{n\|A\|_\infty, \|Bu\|_\infty\}$. Using the integral representation of (2), one has $\|x(t)\|_\infty \leq \|x_0\|_\infty + \int_0^t G(1 + \|x(s)\|_\infty)ds$. Thus, $1 + \|x(t)\|_\infty \leq 1 + \|x_0\|_\infty + \int_0^t G(1 + \|x(s)\|_\infty)ds$. Then using Gronwall's Lemma [26] we have $(1 + \|x(t)\|_\infty) \leq (1 + \|x(0)\|_\infty)e^{Gt}$. Thus, we obtain:

$$R(T) = (1 + \|x(0)\|_\infty)e^{GT} - 1 \geq \|x(t)\|_\infty \quad \forall t \in [0,T] \tag{39}$$

Now, let us consider $t_1 \in [0,1]$ such that $y(t_0) = 1 - \alpha$ and $[0,1] \ni t_1 > t_0$ such that $y(t_1) = 1$. Then, $y(t_1) = y(t_0) + C\int_{t_0}^{t_1} A(s)x(s) + B(s)u(s)ds$. Thus, we have

$$\alpha \leq \|C\|_{L^\infty}\int_{t_0}^{t_1} n\|A\|_{L^\infty}\|x(s)\|_\infty + \|Bu\|_{L^\infty}ds \tag{40}$$

Inserting equation (39) into (40), one has:

$$\alpha \leq \|C\|_{L^\infty}\left[n\|A\|_{L^\infty}R(T) + \|Bu\|_{L^\infty}\right](t_1 - t_0) \tag{41}$$

Defining the following constant K

$$K \triangleq \|C\|_{L^\infty}\left[n\|A\|_{L^\infty}R(T) + \|Bu\|_{L^\infty}\right]$$

one obtains

$$(t_1 - t_0) \geq \frac{\alpha}{K}$$

Since the measure $\mu_y(\alpha)$ cannot be lower than the minimal time needed to reach the constraint starting from $y(t_0) = 1 - \alpha$, we finally obtain

$$\mu_y(\alpha) \geq t_1 - t_0 \geq \frac{\alpha}{K}$$

This concludes the proof.