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Analysis of a Retro-PN Guidance Law

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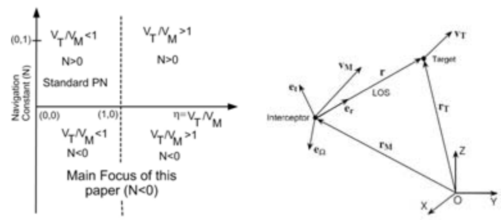
***Abstract*— This paper presents a complete analysis of a three dimensional retro-PN guidance law, suited for intercepting targets that are of higher speeds than the interceptors. It proposes a somewhat counter-intuitive guidance law that uses a negative navigation constant (as against the usual positive one) to show that this modification makes it possible to achieve collision conditions that were inaccessible to the standard law. An analysis for three dimensional engagements is presented in a modified polar coordinate system and complete capturability results are obtained. Simulation results are given to support the theoretical findings.**



(a) Focus of present work (b) Engagement Geometry

Fig. 1. Focus of present work and Basic engagement geometry

I. INTRODUCTION

Intercepting targets having speeds higher than the interceptor speed is a challenging task. There are very few papers in the literature that address this problem. Kuroda and Imado [1] showed that the near-head-on scenario is the best way for an interceptor to achieve small miss distances against a higher speed target. Golan and Shima [2] positioned the interceptor ahead of the higher speed target on its flight trajectory. The standard PN law, with positive navigation constant N, is a widely used guidance law, and has been analyzed extensively in the literature for 2-D engagements with lower speed targets [5], [6], [7]. A 3-D version was discussed by Tyan [3] with the aid of a modified polar co-ordinate system, which was later extended to higher speed targets by Tyan and Shen [4].

Prasanna and Ghose [8], [9], [10] showed that for intercepting a higher speed target, there exists two collision conditions (as against only one for a lower speed target), and one of them can be achieved with positive N (standard PN) while the other can be achieved by using a negative N (retro-PN). However, all these studies were confined to a 2-D space. Since the 3D engagements are more practical, this paper analyzes the capturability of retro-PN guidance law for 3-D engagement scenarios, using a modified polar co-ordinate system given in [3], [4], [11]. Fig. 1(a) gives an idea about the perspective in which this paper makes a contribution.

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II. PRELIMINARIES AND PROBLEM FORMULATION

A. Preliminaries

Consider the engagement geometry shown in Fig. 1(b). In the line-of-sight (LOS) fixed reference frame [4], [3], the relative position vector $\mathbf{r}$ (LOS vector), relative velocity vector $\dot{\mathbf{r}}$ and relative acceleration vector $\ddot{\mathbf{r}}$ between the target and the interceptor are defined as below.

$$\mathbf{r} \triangleq \rho \mathbf{e_r} \triangleq \mathbf{r_T} - \mathbf{r_M} \tag{1}$$

$$\dot{\mathbf{r}} = \dot{\rho}\mathbf{e_r} + \rho\dot{\mathbf{e_r}} = \dot{\rho}\mathbf{e_r} + \rho(\Omega \times \mathbf{e_r}) = \mathbf{v_T} - \mathbf{v_M} \tag{2}$$

$$\ddot{\mathbf{r}} = \ddot{\rho}\mathbf{e_r} + 2\dot{\rho}\dot{\mathbf{e_r}} + \rho\ddot{\mathbf{e_r}} = \mathbf{a_T} - \mathbf{a_M} \tag{3}$$

$\mathbf{e_r}$ is the unit vector along the LOS and $\dot{\mathbf{e_r}} = \Omega \times \mathbf{e_r}$ and the angular velocity of the LOS, Ω is orthogonal to the LOS.

$$\Omega = \rho^{-1}\mathbf{e_r} \times (\mathbf{v_T} - \mathbf{v_M}) \tag{4}$$

Magnitude of the angular velocity of LOS vector with sign

$$\omega \triangleq \begin{cases} \|\Omega\|, \text{when } \mathbf{e_r} \text{ moves anticlockwise} \\ -\|\Omega\|, \text{when } \mathbf{e_r} \text{ moves clockwise} \end{cases} \tag{5}$$

Define $\mathbf{e_t} \triangleq \dot{\mathbf{e}}_\mathbf{r}/\omega$, $\mathbf{e}_\Omega \triangleq \Omega/\omega$. Clearly $\mathbf{e_r} \times \mathbf{e_t} = \mathbf{e}_\Omega$. $\mathbf{e_r}$, $\mathbf{e_t}$ and $\mathbf{e}_\Omega$ are the orthogonal unit vectors of the modified LOS-fixed polar co-ordinate system.

B. Retro-PN Guidance Law

The latax command for the retro-PN guidance law is:

$$\mathbf{a_M} = -(-\beta V_M \mathbf{e_M} \times \Omega) = \beta V_M \omega[(\mathbf{e_M}'\mathbf{e_t})\mathbf{e_r} - (\mathbf{e_M}'\mathbf{e_r})\mathbf{e_t}] \tag{6}$$

where, $-\beta$ is the navigation constant, with $\beta > 0$, V_M is the speed of the interceptor and $\mathbf{e_M}$ is the unit vector along the interceptor velocity vector. Note that in standard PN guidance law, the navigation constant is always positive.

C. Dynamics of relative velocity and direction cosines

Define $u \triangleq \dot{\rho}$, $v \triangleq \rho\omega$, $w \triangleq \rho^{-1}$. Here u and v denote the relative velocity of the target with respect to the interceptor along the LOS and along the transverse direction $\mathbf{e_t}$ respectively and w is the inverse of the range.

$$u = V_T(\mathbf{e_T}'\mathbf{e_r}) - V_M(\mathbf{e_M}'\mathbf{e_r});\; v = V_T(\mathbf{e_T}'\mathbf{e_t}) - V_M(\mathbf{e_M}'\mathbf{e_t}) \quad (7)$$

where, $\mathbf{e_T}$ is the unit vector along the target velocity vector. For a non-maneuvering target, $\mathbf{a_T} = 0$. Transforming the independent variable from t to τ by $d\tau \triangleq wdt$, we obtain,

$$du/d\tau = [v - \beta V_M(\mathbf{e_M}'\mathbf{e_t})]v,\; u(\tau_0) = u_0 \quad (8)$$
$$dv/d\tau = [-u + \beta V_M(\mathbf{e_M}'\mathbf{e_r})]v,\; v(\tau_0) = v_0 \quad (9)$$
$$dw/d\tau = -uw,\; w(\tau_0) = w_0. \quad (10)$$

$v = 0$ is the equilibrium in the (u, v)-plane. The differential equations related to the direction cosines associated with $\mathbf{e_M}$ with respect to θ are obtained as below, where $d\theta \triangleq vd\tau$.

$$d(\mathbf{e_M}'\mathbf{e_r})/d\theta = (\beta+1)\mathbf{e_M}'\mathbf{e_t},\; \mathbf{e_M}'\mathbf{e_r}(\theta_0) = (\mathbf{e_M}'\mathbf{e_r})_0 \quad (11)$$
$$d(\mathbf{e_M}'\mathbf{e_t})/d\theta = -(\beta+1)\mathbf{e_M}'\mathbf{e_r},\; \mathbf{e_M}'\mathbf{e_t}(\theta_0) = (\mathbf{e_M}'\mathbf{e_t})_0 \quad (12)$$
$$d(\mathbf{e_M}'\mathbf{e_\Omega})/d\theta = 0,\; \mathbf{e_M}'\mathbf{e_\Omega}(\theta_0) = (\mathbf{e_M}'\mathbf{e_\Omega})_0 \quad (13)$$

Solving (11) and (12), direction cosines of $\mathbf{e_M}$ are obtained,

$$\begin{bmatrix} \mathbf{e_M}'\mathbf{e_r}(\theta) \\ \mathbf{e_M}'\mathbf{e_t}(\theta) \end{bmatrix} = R[-(\beta+1)(\theta-\theta_0)] \begin{bmatrix} (\mathbf{e_M}'\mathbf{e_r})_0 \\ (\mathbf{e_M}'\mathbf{e_t})_0 \end{bmatrix} \quad (14)$$

where, $R[.] = \begin{bmatrix} \cos(.) & \sin(.) \\ -\sin(.) & \cos(.) \end{bmatrix}$.

The dynamic equations related to u and v in terms of θ are,

$$\begin{bmatrix} du/d\theta \\ dv/d\theta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \beta V_M \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} R[-(\beta+1)(\theta-\theta_0)] \begin{bmatrix} (\mathbf{e_M}'\mathbf{e_r})_0 \\ (\mathbf{e_M}'\mathbf{e_t})_0 \end{bmatrix} \quad (15)$$

Solving (15), u and v are obtained as,

$$\begin{bmatrix} u(\theta) \\ v(\theta) \end{bmatrix} = R[-(\theta-\theta_0)] \begin{bmatrix} u_0 + V_M(\mathbf{e_M}'\mathbf{e_r})_0 \\ v_0 + V_M(\mathbf{e_M}'\mathbf{e_t})_0 \end{bmatrix} - R[-(\beta+1)(\theta-\theta_0)] \begin{bmatrix} V_M(\mathbf{e_M}'\mathbf{e_r})_0 \\ V_M(\mathbf{e_M}'\mathbf{e_t})_0 \end{bmatrix} \quad (16)$$

Using (7), it follows that,

$$\begin{bmatrix} u(\theta) \\ v(\theta) \end{bmatrix} = R[-(\theta-\theta_0)] \begin{bmatrix} V_T(\mathbf{e_T}'\mathbf{e_r})_0 \\ V_T(\mathbf{e_T}'\mathbf{e_t})_0 \end{bmatrix} - R[-(\beta+1)(\theta-\theta_0)] \begin{bmatrix} V_M(\mathbf{e_M}'\mathbf{e_r})_0 \\ V_M(\mathbf{e_M}'\mathbf{e_t})_0 \end{bmatrix} \quad (17)$$

Therefore, direction cosines for $\mathbf{e_T}$ can be written as,

$$\begin{bmatrix} \mathbf{e_T}'\mathbf{e_r}(\theta) \\ \mathbf{e_T}'\mathbf{e_t}(\theta) \end{bmatrix} = R[-(\theta-\theta_0)] \begin{bmatrix} (\mathbf{e_T}'\mathbf{e_r})_0 \\ (\mathbf{e_T}'\mathbf{e_t})_0 \end{bmatrix} \quad (18)$$

D. Trajectories in the normalized (u, v)-plane

The trajectory of the engagement in the normalized (u, v)-plane is obtained as a moving cycloid with fixed radius and varying centre. Let $\eta \triangleq V_T/V_M$, $\bar{u} \triangleq u/V_T$, $\bar{v} \triangleq v/V_T$. Note that $\mathbf{v_T} - \mathbf{v_M}$ has no component in $\mathbf{e_\Omega}$ direction.

$$V_T(\mathbf{e_T}'\mathbf{e_\Omega}) = V_M(\mathbf{e_M}'\mathbf{e_\Omega}) \Rightarrow (\mathbf{e_T}'\mathbf{e_\Omega}) = \eta^{-1}(\mathbf{e_M}'\mathbf{e_\Omega}) \quad (19)$$

By (13), $V_M(\mathbf{e_M}'\mathbf{e_\Omega}) = V_M(\mathbf{e_M}'\mathbf{e_\Omega})_0$ and since $\mathbf{a_T} = 0$, $V_T(\mathbf{e_T}'\mathbf{e_\Omega}) = V_T(\mathbf{e_T}'\mathbf{e_\Omega})_0$. From (7), the trajectories in the $(\bar{u}, \bar{v})$-plane are given in two alternative forms below.

$$[\bar{u} + \eta^{-1}(\mathbf{e_M}'\mathbf{e_r})]^2 + [\bar{v} + \eta^{-1}(\mathbf{e_M}'\mathbf{e_t})]^2 = 1 - \eta^{-2}(\mathbf{e_M}'\mathbf{e_\Omega})_0^2 \quad (20)$$

$$[\bar{u} - (\mathbf{e_T}'\mathbf{e_r})]^2 + [\bar{v} - (\mathbf{e_T}'\mathbf{e_t})]^2 = \eta^{-2} - (\mathbf{e_T}'\mathbf{e_\Omega})_0^2 \quad (21)$$

For the feasibility of trajectories given by (20) and (21) the conditions are $(\mathbf{e_M}'\mathbf{e_\Omega})_0^2 < \eta^2$; $(\mathbf{e_T}'\mathbf{e_\Omega})_0^2 < \eta^{-2}$ respectively.

E. Feasible final conditions

For successful interception, the conditions at the final instant are given by $\bar{v}_f = 0$ and $\bar{u}_f < 0$ [12]. The set of initial conditions that would lead to successful interception by the retro-PN guided interceptor, that is, the above-mentioned feasible final conditions, are called the capture zone of the retro-PN guidance law. Solving (20) and (21) separately with constraint $\bar{v}_f = 0$, we obtain (22) and (23) respectively.

$$\bar{u}_f = -\eta^{-1}(\mathbf{e_M}'\mathbf{e_r})_f \pm \eta^{-1}\sqrt{(\mathbf{e_M}'\mathbf{e_r})_f^2 + (\eta^2 - 1)} \quad (22)$$

$$\bar{u}_f = (\mathbf{e_T}'\mathbf{e_r})_f \pm \sqrt{(\mathbf{e_T}'\mathbf{e_r})_f^2 - (1 - \eta^{-2})} \quad (23)$$

From (22), if $\eta < 1$, then for $\bar{u}_f < 0$, $(\mathbf{e_M}'\mathbf{e_r})_f > 0$ and $\sqrt{1-\eta^2} \le (\mathbf{e_M}'\mathbf{e_r})_f \le 1$. If $\eta > 1$, then for $\bar{u}_f < 0$, $(\mathbf{e_M}'\mathbf{e_r})_f$ can be either positive or negative or zero, but the interception is possible only for one set of initial conditions for which (22) holds with '$-$' sign. From (23), if $\eta < 1$, then at $\bar{u}_f < 0$, $(\mathbf{e_T}'\mathbf{e_r})_f$ can be either positive or negative or zero, but the interception would be possible only for one set of initial conditions for which (23) holds with '$-$' sign. If $\eta > 1$, then at $\bar{u}_f < 0$, $(\mathbf{e_T}'\mathbf{e_r})_f < 0$ and $-1 \le (\mathbf{e_T}'\mathbf{e_r})_f \le -\sqrt{(1-\eta^{-2})}$.

III. Capturability Analysis

A. Some preliminary results

$$x \triangleq \sqrt{(\mathbf{e_M}'\mathbf{e_r})_0^2 + (\mathbf{e_M}'\mathbf{e_t})_0^2} = \sqrt{1 - (\mathbf{e_M}'\mathbf{e_\Omega})_0^2} \quad (24)$$

$$y \triangleq \sqrt{(\mathbf{e_T}'\mathbf{e_r})_0^2 + (\mathbf{e_T}'\mathbf{e_t})_0^2} = \sqrt{1 - (\mathbf{e_T}'\mathbf{e_\Omega})_0^2} \quad (25)$$

Clearly, $0 \le x, y \le 1$.

Lemma 1: $\eta > 1 \Rightarrow x \le y$, $\eta < 1 \Rightarrow x \ge y$. $\eta = 1 \Rightarrow x = y$.

Proof: From (19) and subsequent discussions, we get,

$$1 - x^2 = \eta^2(1 - y^2) \Rightarrow y^2 = (x\eta^{-1})^2 + (1 - \eta^{-2}) \quad (26)$$
$$\Rightarrow (1 - x^2)/(1 - y^2) = \eta^2 \quad (27)$$

For $x = y = 1$, η can be any number greater than or equal to or less than 1. Hence from (26), (27), the lemma follows. ■

Since $\mathbf{e_M}'\mathbf{e_\Omega}$ and $\mathbf{e_T}'\mathbf{e_\Omega}$ remain constant throughout the mission time, x and y would also remain same throughout the engagement time. Define $\cos\theta_{M_0} \triangleq (\mathbf{e_M}'\mathbf{e_r})_0/x$, $\sin\theta_{M_0} \triangleq (\mathbf{e_M}'\mathbf{e_t})_0/x$; $\cos\theta_{T_0} \triangleq (\mathbf{e_T}'\mathbf{e_r})_0/y$, $\sin\theta_{T_0} \triangleq (\mathbf{e_T}'\mathbf{e_t})_0/y$. Therefore, referring to (17), $\bar{u}$ and $\bar{v}$ can be written as,

$$\bar{u} = y\cos(\theta_{T_0} - \triangle\theta) - (x/\eta)\cos(\theta_{M_0} - (\beta+1)\triangle\theta) \quad (28)$$

$$\bar{v} = y\sin(\theta_{T_0} - \triangle\theta) - (x/\eta)\sin(\theta_{M_0} - (\beta+1)\triangle\theta) \quad (29)$$

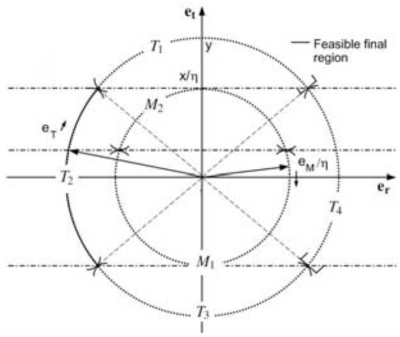

Fig. 2. $(\mathbf{e_r}, \mathbf{e_t})$-plane and feasible region for $\bar{v}_f = 0$

where, $\triangle\theta \triangleq \theta - \theta_0$. Hence, in the $(\mathbf{e_r}, \mathbf{e_t})$-plane the projections of the two vectors $\mathbf{e_T}$ and $\mathbf{e_M}/\eta$ can be thought of as rotating in two circles centered at origin with radius y and x/η respectively. For $\beta > 0$, if $v > 0$, then they rotate in the clockwise direction and if $v < 0$, then they rotate in the anticlockwise direction. The projection of $\mathbf{e_M}/\eta$ rotates $(\beta+1)$ times faster than that of $\mathbf{e_T}$. Hence $\bar{u}$ can be considered as the difference of the projections of $\mathbf{e_T}$ and $\mathbf{e_M}/\eta$ along the LOS, that is in the $\mathbf{e_r}$ direction and $\bar{v}$ as the difference of their components in $\mathbf{e_t}$ direction.

The combinations of v_0 and u_0 (or the combinations of v_0, θ_{T_0} and θ_{M_0}), which lead to the final conditions $\bar{v}_f = 0$ and $\bar{u}_f < 0$, define the capture zone. The combinations of v_0, θ_{T_0} and θ_{M_0}, for which $\bar{v}_0 = 0$ and $\bar{u}_0 < 0$, define the trivial capture zone, since when $\bar{v}_0 = 0$, the system is already at equilibrium and the negative relative velocity along the LOS leads to successful interception. The capture zone, other than the trivial capture zone, would be referred to as the nontrivial capture zone. Define $z \triangleq (x/\eta y)$.

Lemma 2: If $\eta > 1$, then there exists a feasible nontrivial capturability region in terms of θ_{M_0} and θ_{T_0}.

Proof: For successful engagement, the conditions at the final instant are $\bar{v}_f = 0$ and $\bar{u}_f < 0$. By Lemma 1, $\eta > 1 \Rightarrow x < y$ and hence, $z = x/\eta y < 1$. Then with respect to θ_T, the $(\mathbf{e_r}, \mathbf{e_t})$-plane can be segmented into five disjoint regions as shown in Fig.2. $T_1 = \{\theta_T | \sin^{-1} z \le \theta_T < \pi - \sin^{-1} z\}$; $T_2 = \{\theta_T | \pi - \sin^{-1} z < \theta_T < \pi + \sin^{-1} z\}$; $T_3 = \{\theta_T | \pi + \sin^{-1} z < \theta_T < 2\pi - \sin^{-1} z\}$; $T_4 = \{\theta_T | \theta_T \in [0, \sin^{-1} z) \cup [2\pi - \sin^{-1} z, 2\pi)\}$; $T_5 = \{\pi - \sin^{-1} z, \pi + \sin^{-1} z\}$.

All these regions are defined modulo 2π. For any θ_T, three disjoint regions $M_1 = \{\theta_M | v > 0\}$, $M_2 = \{\theta_M | v < 0\}$, $M_3 = \{\theta_M | v = 0\}$ are defined. Clearly $\theta_{T_0} \in T_1 \Rightarrow v_0 > 0$ and $\theta_{T_0} \in T_3 \Rightarrow v_0 < 0$ for any θ_{M_0}. $\bar{v}_f = 0$ is only possible at $\theta_{T_f} \in T_2 \cup T_4 \cup T_5$. From (28) and (29), we have,

$$[(\bar{u}/y) - \cos(\theta_{T_0} - \triangle\theta)]^2 + [\bar{v}/y - \sin(\theta_{T_0} - \triangle\theta)]^2 = z^2 \quad (30)$$

As $z^2 < 1$, $\bar{u}_f < 0$ if $\pi/2 < \theta_{T_0} - \triangle\theta_f < 3\pi/2$. For $\bar{v}_f = 0$,

$$\bar{u}_f/y = \cos(\theta_{T_0} - \triangle\theta_f) \pm \sqrt{z^2 - \sin^2(\theta_{T_0} - \triangle\theta_f)} \quad (31)$$

From (31), $\left|\sin(\theta_{T_0} - \triangle\theta_f)\right| < z$. Therefore, the conditions of successful interception $\bar{v}_f = 0$ and $\bar{u}_f < 0$ are possible only if $\theta_{T_f} \in T_2 \cup T_5$. By the intermediate value theorem, clearly, given any real β, the normalized relative velocity in the transverse direction $\bar{v}$, being a smooth function of $\triangle\theta$, as given in (29), must cross zero before changing its sign.

In Scenario 1, since $v_0 > 0$, the projection of both $\mathbf{e_T}$ and $\mathbf{e_M}/\eta$ rotate in the $(\mathbf{e_r}, \mathbf{e_t})$-plane in the clockwise direction. $\bar{v}$ has to be zero before the projection of $\mathbf{e_T}$ reaches the T_3 region of θ_T, as in this region $\bar{v} < 0$, that is, $\bar{v}$ would become zero in the T_4 region of θ_T, where, $\bar{u}_f > 0$. Hence, this case would not lead to any feasible interception. In Scenario 5, $v_0 < 0$ and hence the projection of both $\mathbf{e_T}$ and $\mathbf{e_M}/\eta$ rotate in the $(\mathbf{e_r}, \mathbf{e_t})$-plane in the anticlockwise direction. $\bar{v}$ has to be zero before $\mathbf{e_T}$ reaches the T_1 region of θ_T, as in this region $\bar{v} > 0$, that is, $\bar{v}$ would become zero in T_2 or T_4 or T_5 region of θ_T. By selection of proper proportional navigation gain β, this $\bar{v}$ can be made equal to zero in T_2 or T_5 region of θ_T, where, $\bar{u}_f < 0$ leading towards feasible interception.

By similar logic for all scenarios in the disjoint and exhaustive set of initial conditions, mentioned in Table I, if $\eta > 1$, then $\theta_{T_0} \in T_2$ in combination with $\theta_{M_0} \in (M_1 \cup M_2)$ form nontrivial capture zone, while $\theta_{T_0} \in T_2 \cup T_5$ in combination with $\theta_{M_0} \in M_3$ form trivial capture zone. ■

However note that for feasible interception in any scenario, $(\mathbf{e_M}'\mathbf{e_r})_f \le 0$, that is, for $\bar{v}_0 > 0$, $\theta_{M_f} = \pi/2 + \delta$, where, $\delta \in C$ and for $\bar{v}_0 < 0$, $\theta_{M_f} = 3\pi/2 - \delta$, where, $\delta \in D$, where C and D are defined in Lemma 5 and 6 respectively. Following similar arguments, the following results can be proved.

Lemma 3: If $\eta = 1$, then there exists a feasible nontrivial capturability region when $\theta_{T_0} \in T_2$ and $\theta_{M_0} \in [0, 2\pi)$.

Lemma 4: If $\eta < 1$, then there does not exist any nontrivial feasible capturability region in terms of θ_{T_0} and θ_{M_0}.

From Lemma 2, 3 and 4, it is evident that a retro-PN guided interceptor would have a feasible nontrivial capture zone if and only if $\eta \ge 1$, provided $\beta > 0$ in (6). Now the existence of such $\beta > 0$ would be analyzed in the following two lemma for $\bar{v}_0 > 0$ and $\bar{v}_0 < 0$ respectively.

Lemma 5: If $\eta \ge 1$, $\theta_{T_0} \in T_2$ and $\bar{v}_0 > 0$, then

1) *If $\theta_{M_0} \in Q_1 = [0, \pi/2)$, then $\forall\delta \in C, \exists\beta > 0 \ni$*
$3\pi/2 + \theta_{M_0} - \delta = (\beta+1)(\theta_{T_0} - \pi + \sin^{-1}(z\cos\delta))$.
2) *If $\theta_{M_0} \in Q_2 \cup Q_3$, where $Q_2 = (3\pi/2, 2\pi)$ and $Q_3 = [\pi/2, 3\pi/2]$, then $\forall\delta \in C, \exists\beta > 0 \ni$*
$\theta_{M_0} - \pi/2 - \delta = (\beta+1)(\theta_{T_0} - \pi + \sin^{-1}(z\cos\delta))$.

where, $C = [0, \cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0})))$.

Proof: $T_2a = \{\theta_T | \pi - \sin^{-1} z < \theta_T \le \pi\}$, $T_2b = \{\theta_T | \pi < \theta_T < \pi + \sin^{-1} z\}$ and $T_2 = T_2a \cup T_2b$. When $\theta_{T_0} \in T_2a$, then $\bar{v}_0 > 0 \Leftrightarrow \theta_{M_0} \in M_1a \cup M_1b$, where $M_1a = [0, \sin^{-1}(z^{-1}\sin\theta_{T_0}))$, $M_1b = (\pi - \sin^{-1}(z^{-1}\sin\theta_{T_0}), 2\pi)$. When $\theta_{T_0} \in T_2b$, then $\bar{v}_0 > 0 \Leftrightarrow \theta_{M_0} \in M_1b$.

Case 1:$(\theta_{T_0} \in T_2a, \theta_{M_0} \in M_1a \subseteq Q_1)$
Since $\delta < \cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0})) \le \pi/2$ and $\theta_{M_0} \ge 0$, $3\pi/2 + \theta_{M_0} - \delta > 3\pi/2 - \cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0})) \ge \pi$. But, since $\pi - \sin^{-1} z < \theta_{T_0} \le \pi$ and $\delta \ge 0$, $\theta_{T_0} - \pi + \sin^{-1}(z\cos\delta) \le \theta_{T_0} - \pi + \sin^{-1}(z) < \pi$. Hence, if $\theta_{T_0} \in T_2a$, $\theta_{M_0} \in M_1a$, then $\forall\delta \in C$ and for some $\beta > 0$,

$$3\pi/2 + \theta_{M_0} - \delta > \theta_{T_0} - \pi + \sin^{-1}(z\cos\delta)$$
$$\Leftrightarrow 3\pi/2 + \theta_{M_0} - \delta = (\beta+1)(\theta_{T_0} - \pi + \sin^{-1}(z\cos\delta))$$

TABLE I

PRESENCE OF NONTRIVIAL CAPTURE ZONE FOR $x < \eta y$

Scenario No.	Combination of θ_{T_0} and θ_{M_0}	Rotation of $\mathbf{e_T}$ and $\mathbf{e_M}/\eta$	$\bar{v}_f = 0$ at $\theta_{T_f} \in$	$\bar{u}_f$	Feasible Interception	Nontrivial Capture Zone
1	$\{T_1 \cup T_4\} and \{M_1\}$	clockwise	$\{T_4\}$	> 0	No	No
2	$\{T_3 \cup T_4\} and \{M_2\}$	anticlockwise	$\{T_4\}$	> 0	No	No
3	$\{T_4\} and \{M_3\}$	none	$\{T_4\}$	> 0	No	No
4	$\{T_2\} and \{M_1\}$	clockwise	$\{T_2 \cup T_5\}$	< 0	Yes	Yes
5	$\{T_2\} and \{M_2\}$	anticlockwise	$\{T_2 \cup T_5\}$	< 0	Yes	Yes
6	$\{T_2\} and \{M_3\}$	none	$\{T_2\}$	< 0	Yes	No
7	$\{T_5\} and \{M_1\}$	clockwise	$\{T_4\}$	> 0	No	No
8	$\{T_5\} and \{M_2\}$	anticlockwise	$\{T_4\}$	> 0	No	No
9	$\{T_5\} and \{M_3\}$	none	$\{T_5\}$	< 0	Yes	No

Case 2:$(\theta_{T_0} \in T_2a \cup T_2b,\ \theta_{M_0} \in M_1b \subseteq Q_2 \cup Q_3)$
Since $\theta_{M_0} \in M_1b$, $\theta_{M_0} > \pi - \sin^{-1}(z^{-1}\sin\theta_{T_0})$. Here the function $\delta + \sin^{-1}(z\cos\delta)$ is increasing in δ over C, since $x \leq \eta y$. Therefore, for any given $\theta_{T_0} \in T_2$, we have, $\delta + \sin^{-1}(z\cos\delta) < \cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0})) + \pi - \theta_{T_0}$. Thus, $\theta_{M_0} - \pi/2 - \delta - \theta_{T_0} + \pi - \sin^{-1}(z\cos\delta) > 0$. So if $\theta_{T_0} \in T_2a \cup T_2b$, $\theta_{M_0} \in M_1b$, then $\forall\delta \in C$ and for some $\beta > 0$,

$$\theta_{M_0} - \pi/2 - \delta > \theta_{T_0} - \pi + \sin^{-1}(z\cos\delta)$$
$$\Leftrightarrow \theta_{M_0} - \pi/2 - \delta = (\beta+1)(\theta_{T_0} - \pi + \sin^{-1}(z\cos\delta))$$

■

By similar logic the next result for $\bar{v}_0 < 0$ can be proved.

Lemma 6: If $\eta \geq 1$, $\theta_{T_0} \in T_2$ and $\bar{v}_0 < 0$, then

1) *If $\theta_{M_0} \in Q_1 \cup Q_3$, then $\forall\delta \in D, \exists\beta > 0 \ni$* $3\pi/2 - \theta_{M_0} - \delta = (\beta+1)(\pi + \sin^{-1}(z\cos\delta) - \theta_{T_0})$.
2) *If $\theta_{M_0} \in Q_2$, then $\forall\delta \in D, \exists\beta > 0 \ni$* $7\pi/2 - \theta_{M_0} - \delta = (\beta+1)(\pi + \sin^{-1}(z\cos\delta) - \theta_{T_0})$.

where, $D = [0, \pi - \cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0})))$ and Q_1, Q_2, Q_3 are as defined in Lemma 5.

B. Some definitions

Given $\theta_{T_0} \in T_2$, for $\bar{v}_0 < 0$ also, two regions of θ_{M_0}, namely $M_2a = (\max(0, \sin^{-1}(z^{-1}\sin\theta_{T_0})), \pi - \sin^{-1}(z^{-1}\sin\theta_{T_0}))$ and specifically for $\theta_{T_0} \in T_2b$, $M_2b = (2\pi + \sin^{-1}(z^{-1}\sin\theta_{T_0}), 2\pi)$ are defined.

Now for $\eta \geq 1$, define the following regions.

$R_1 = \{\bar{v}_0, \theta_{T_0}, \theta_{M_0} \mid \bar{v}_0 > 0, \theta_{T_0} \in T_2, \theta_{M_0} \in M_1b\}$
$R_2 = \{\bar{v}_0, \theta_{T_0}, \theta_{M_0} \mid \bar{v}_0 > 0, \theta_{T_0} \in T_2a, \theta_{M_0} \in M_1a\}$
$R_3 = \{\bar{v}_0, \theta_{T_0}, \theta_{M_0} \mid \bar{v}_0 < 0, \theta_{T_0} \in T_2, \theta_{M_0} \in M_2a\}$
$R_4 = \{\bar{v}_0, \theta_{T_0}, \theta_{M_0} \mid \bar{v}_0 < 0, \theta_{T_0} \in T_2b, \theta_{M_0} \in M_2b\}$
$R_5 = \{\bar{v}_0, \theta_{T_0}, \theta_{M_0} \mid \bar{v}_0 = 0, \theta_{T_0} \in T_2\}$.

Clearly, R_5 forms the trivial capture zone. To analyze the nontrivial capture zone define the following set.

$$CR_{RPN} = R_1 \cup R_2 \cup R_3 \cup R_4 \tag{32}$$

Define $f_1, f_2 : C \to \mathbb{R}^+$ and $f_3, f_4 : D \to \mathbb{R}^+$ as
$f_1(\delta) = (\theta_{M_0} - \pi/2 - \delta)/(\theta_{T_0} - \pi + \sin^{-1}(z\cos\delta)) - 1$,
$f_2(\delta) = (3\pi/2 + \theta_{M_0} - \delta)/(\theta_{T_0} - \pi + \sin^{-1}(z\cos\delta)) - 1$,
$f_3(\delta) = (3\pi/2 - \theta_{M_0} - \delta)/(\pi + \sin^{-1}(z\cos\delta) - \theta_{T_0}) - 1$,
$f_4(\delta) = (7\pi/2 - \theta_{M_0} - \delta)/(\pi + \sin^{-1}(z\cos\delta) - \theta_{T_0}) - 1$,
which are convex functions in their respective domains of δ, where, C and D are as mentioned in Lemma 5 and 6 respectively. Therefore, over C, both $f_1(\delta)$ and $f_2(\delta)$ and over D, both $f_3(\delta)$ and $f_4(\delta)$ have unique minima at $\underline{\delta}_1$, $\underline{\delta}_2$, $\underline{\delta}_3$ and $\underline{\delta}_4$ respectively and are positive as ensured by Lemma 5 and 6 respectively. $\underline{\beta}_i = (f_i(\underline{\delta}_i))\,\forall i = 1,2,3,4$.

C. Capturability analysis without LOS turn rate constraint

Theorem 1: Consider a Retro-PN guided ideal interceptor, pursuing a non-maneuvering target with $\eta = V_T/V_M \geq 1$. The necessary and sufficient condition for the existence of nontrivial capture zone with no constraints on LOS turn rate are $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in CR_{RPN}$, where, CR_{RPN} is given by (32) and (1) $if(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_1, \beta \geq \underline{\beta}_1$; (2) $if(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_2$, $\beta \geq \underline{\beta}_2$; (3) $if(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_3$, $\beta \geq \underline{\beta}_3$ and (4) $if(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_4$, $\beta \geq \underline{\beta}_4$.

Proof: From Lemma 2, if $\eta \geq 1$, the nontrivial capture zone exists if and only if $\theta_{T_0} \in T_2$ and $\bar{v}_0 \gtrless 0$. The set $S = \{\bar{v}_0, \theta_{T_0}, \theta_{M_0} \mid \bar{v}_0 \gtrless 0, \theta_{T_0} \in T_2\}$ can be segmented into four mutually disjoint and exhaustive subsets R_1, R_2, R_3, R_4, while by (32), $R_1 \cup R_2 \cup R_3 \cup R_4 = CR_{RPN} = S$. Therefore, $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in CR_{RPN}$ forms one of the necessary and sufficient conditions for existence of nontrivial capture zone.

Consider $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_1$. Note in Fig. 3 that given a $\theta_{T_0} \in T_2$, for feasible interception, $\bar{v}_f$ can be zero for any $\delta \in [0, \cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0}))) = C$. For the proof of necessity of $\beta \geq \underline{\beta}_1$, consider that $\bar{v}_f = 0$ for some $\delta = \delta' \in C$. Therefore, the angle of rotation of the projections of $\mathbf{e_T}$ and $\mathbf{e_M}/\eta$ in the $(\mathbf{e_r}, \mathbf{e_t})$-plane can be related as below.

$$(\theta_{M_0} - \pi/2 - \delta')/(\beta+1) = \theta_{T_0} - \pi + \sin^{-1}(z\cos\delta')$$
$$\Rightarrow \beta' = (\theta_{M_0} - \pi/2 - \delta')/(\theta_{T_0} - \pi + \sin^{-1}(z\cos\delta')) - 1$$

From Lemma 5, $\beta' > 0$ exists and by the definition of $\underline{\delta}_1$ and $\underline{\beta}_1$, we can conclude that $\beta' \geq \underline{\beta}_1$. For the proof of sufficiency of $\beta \geq \underline{\beta}_1$, we need to show that $\exists\delta \in C \ni f_1(\delta) = \beta$. By definition of $\underline{\beta}_1$, for $\beta = \underline{\beta}_1$, it is trivially true. Now consider some $\beta = \beta' \ni \beta' > \underline{\beta}_1$. $f_1(\delta) \to \infty$ as $\delta \to \cos^{-1}(z^{-1}sin(\pi - \theta_{T_0}))$. By the intermediate value theorem, $f_1(\delta)$, being continuous over the domain C, takes all values in the range $[\underline{\beta}_1, \infty)$. Therefore, $\exists\delta = \delta' \in C \ni$ for $\beta = \beta' > \underline{\beta}_1$, $\beta' = f_1(\delta')$. Hence, $\beta \geq \underline{\beta}_1$ is both necessary and sufficient for the existence of the nontrivial capture zone if $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_1$.

By similar analysis of $f_2 : C \to \mathbb{R}^+$, $f_3 : D \to \mathbb{R}^+$ and $f_4 :$

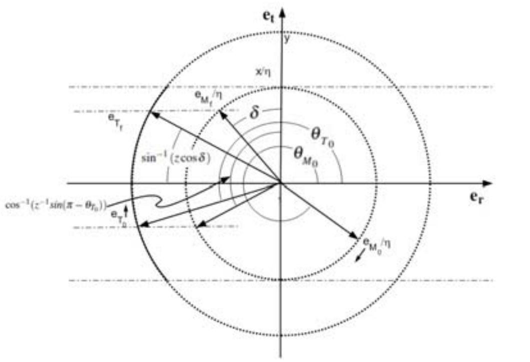

Fig. 3. Rotation of the projections of $\mathbf{e_T}$ and $\mathbf{e_M}/\eta$ in $(\mathbf{e_r}, \mathbf{e_t})$-plane

$D \to \mathbb{R}^+$ in case of $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_2$, $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_3$ and $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_4$ respectively; we obtain that $\beta \geq \underline{\beta}_2$, $\beta \geq \underline{\beta}_3$ and $\beta \geq \underline{\beta}_4$ are necessary and sufficient conditions for existence of nontrivial capture zone in respective cases.

Finally, as it has already been proved that $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_1 \cup R_2 \cup R_3 \cup R_4 = CR_{RPN} = S$ forms a part of the necessary and sufficient conditions of successful interception. ■

D. Capturability analysis with constraint on LOS turn rate

Lemma 7: If $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_1$, *then for* $F_1(\delta) = (f_1(\delta)+2)^2$ *and if* $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_2$, *then for* $F_2(\delta) = (f_2(\delta)+2)^2$, *both defined over C, there exists at least one interval of* δ *in* $C^- = (0, \cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0})))$, *at which* $F_1(\delta)$ *and* $F_2(\delta)$ *are respectively greater than* $g_1(\delta) = 4(1+(\eta^2-1)/(x^2\cos^2(\pi/2+\delta)))$, *defined over* C^-.

Proof: If $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_1 \cup R_2$, then $f_1(\delta)$ and $f_2(\delta)$, as well as $F_1(\delta)$ and $F_2(\delta)$, are defined over the domain $C = [0, \cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0})))$. By construction, $F_1(\delta)$ and $F_2(\delta)$ are continuous convex functions over C and they attain their unique minimum at $\delta = \underline{\delta}_1$ and $\delta = \underline{\delta}_2$ respectively.

As $\delta \to \cos^{-1}(z^{-1} sin(\pi - \theta_{T_0}))$, $F_1(\delta), F_2(\delta) \to \infty$. Hence, by the intermediate value theorem, $F_1(\delta) \in [F_1(\underline{\delta}_1), \infty)$ and $F_2(\delta) \in [F_2(\underline{\delta}_2), \infty)$. As $\delta \to 0$, $g_1(\delta) \to \infty$ and as δ increases, $g_1(\delta)$ assumes finite positive values and finally as $\delta \to \cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0}))$, $g_1(\delta) \to g_1(\cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0})))$, which is positive finite. Therefore, there exists at least one $\delta \in C^-$ such that $F_1(\delta) = g_1(\delta)$. Similar argument holds for $F_2(\delta)$ and $g_1(\delta)$. Since $g_1(\delta)$ is convex over its domain C^-, it can intersect $F_1(\delta)$ a finite number of times in C^-. $\mathbb{D}_1 \triangleq \{\delta \in C^- \mid g_1(\delta) = F_1(\delta)\}$, $\mathbb{D}_2 \triangleq \{\delta \in C^- \mid g_1(\delta) = F_2(\delta)\}$ are nonempty finite cardinality sets of positive elements and hence have their maximals. Let $\tilde{\delta}_{11} \triangleq \max\{\mathbb{D}_1\}$, $\tilde{\delta}_{21} \triangleq \max\{\mathbb{D}_2\}$, $\tilde{\delta}_{12} = \tilde{\delta}_{22} \triangleq \sup\{C^-\} = \cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0}))$.

$F_1'(\delta) \triangleq dF_1(\delta)/d\delta$, $F_2'(\delta) \triangleq dF_2(\delta)/d\delta$, $g_1'(\delta) \triangleq dg_1(\delta)/d\delta$. By the smoothness of $F_1'(\delta)$ and $F_2'(\delta)$ over C^-, clearly $\underline{\delta}_1, \underline{\delta}_2 < \pi/2$. Therefore, as $\delta \to \tilde{\delta}_{12} = \tilde{\delta}_{22}$, $F_1'(\delta), F_2'(\delta) > 0$. If $\tilde{\delta}_{12} = \tilde{\delta}_{22} \leq \pi/2$, then $g_1'(\delta) < 0$ as $\delta \to \tilde{\delta}_{12} = \tilde{\delta}_{22}$, in which case trivially $F_1(\delta) > g_1(\delta), \forall \delta \in (\tilde{\delta}_{11}, \tilde{\delta}_{12})$ and $F_2(\delta) > g_1(\delta) \, \forall \delta \in (\tilde{\delta}_{21}, \tilde{\delta}_{22})$. Otherwise, $g_1'(\delta) > 0$; $F_1(\delta), F_2(\delta) \to \infty$; $F_1'(\delta)/g_1'(\delta)$, $F_2'(\delta)/g_1'(\delta) \to \infty$ as $\delta \to \tilde{\delta}_{12} = \tilde{\delta}_{22}$, hence, $F_1(\delta) > g_1(\delta) \, \forall \delta \in (\tilde{\delta}_{11}, \tilde{\delta}_{12})$ and $F_2(\delta) > g_1(\delta) \, \forall \delta \in (\tilde{\delta}_{21}, \tilde{\delta}_{22})$. ■

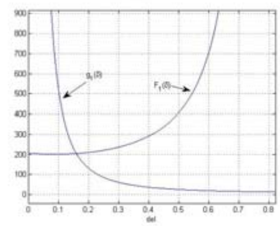

Fig. 4. $F_1(\delta)$ and $g_1(\delta)$

Lemma 8: If $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_3$, *then for* $F_3(\delta) = (f_3(\delta)+2)^2$ *and if* $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_4$, *then for* $F_4(\delta) = (f_4(\delta)+2)^2$, *both defined over D, there exists at least one interval of* δ *in* $D^- = (0, \pi - \cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0})))$, *at which* $F_3(\delta)$ *and* $F_4(\delta)$ *are respectively greater than* $g_2(\delta) = 4(1+(\eta^2-1)/(x^2\cos^2(3\pi/2-\delta)))$, *defined over* D^-.

Proof: By similar analysis of $F_3(\delta)$, $F_4(\delta)$, and $g_2(\delta)$ over D^-, we get, $\exists \tilde{\delta}_{31}, \tilde{\delta}_{41} \in D^- \ni \forall \delta \in (\tilde{\delta}_{31}, \tilde{\delta}_{32})$, $F_3(\delta) > g_2(\delta)$, and $\forall \delta \in (\tilde{\delta}_{41}, \tilde{\delta}_{42})$, $F_4(\delta) > g_2(\delta)$ where $\tilde{\delta}_{32} = \tilde{\delta}_{42} \triangleq \sup\{D^-\} = \pi - \cos^{-1}(z^{-1}\sin(\pi - \theta_{T_0}))$. ■

Define $\tilde{\delta}_i \triangleq \arg\min\{f_i(\delta) \mid \delta \in [\tilde{\delta}_{i1}, \tilde{\delta}_{i2})\} \; \forall i = 1,2,3,4$.

Theorem 2: Consider a Retro-PN guided ideal interceptor, pursuing a non-maneuvering target with $\eta \geq 1$. *The sufficient conditions for the existence of nontrivial capture zone with constraints on finiteness of LOS turn rate are,* $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in CR_{RPN}$, *(given in (32)) and* $\beta \in [f(\tilde{\delta}_i), f(\tilde{\delta}_{i2}))$ *when* $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_i$ *for* $i = 1,2,3,4$.

Proof: By Theorem 1 and the definition of $\underline{\beta}_1$, if $\eta = V_T/V_M \geq 1$ and $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_1$ and $\beta \in [f_1(\tilde{\delta}_1), f_1(\tilde{\delta}_{12}))$, then the retro-PN guided ideal interceptor would intercept the target. From (8)-(10), we obtain,

$$dw/w = -uwdt = (u/(u - \beta V_M(\mathbf{e_M}'\mathbf{e_r})))(dv/v) \qquad (33)$$

Integrating both sides of (33) in the vicinity of θ_f, we get,

$$\Rightarrow \|\Omega\| \rho^{[(-2u_f + \beta V_M(\mathbf{e_M}'\mathbf{e_r})_f)/(-u_f)]} = \text{constant} \qquad (34)$$

As $\theta \to \theta_f$, $u \to u_f < 0$ and $v \to v_f = 0$ for feasible interception, $[-2u_f + \beta V_M(\mathbf{e_M}'\mathbf{e_r})_f]$ has to be non-positive for finite LOS turn rate. From (22) and its subsequent discussion, $\beta V_M(\mathbf{e_M}'\mathbf{e_r})_f - 2u_f = \eta^{-1}[(\beta+2)(\mathbf{e_M}'\mathbf{e_r})_f + 2\sqrt{(\mathbf{e_M}'\mathbf{e_r})_f^2 + (\eta^2-1)}]$. Since for a retro-PN guided interceptor, $(\mathbf{e_M}'\mathbf{e_r})_f < 0$ for feasible interception, as indicated in Lemma 2,

$$\begin{aligned} &[-2u_f + \beta V_M(\mathbf{e_M}'\mathbf{e_r})_f] \leq 0 \\ \Leftrightarrow &(\beta+2)^2 \geq 4(1+(\eta^2-1)/(x^2 cos^2(\theta_{M_f}))) \end{aligned} \qquad (35)$$

When $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_1$ or R_2, then by Lemma 7, if $\beta \in [f_1(\tilde{\delta}_1), f_1(\tilde{\delta}_{12}))$ or $\beta \in [f_2(\tilde{\delta}_2), f_2(\tilde{\delta}_{22}))$ respectively, then (35) is satisfied achieving feasible interception with finite LOS turn rate. And when $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_3$ or R_4, then by Lemma 8, if $\beta \in [f_3(\tilde{\delta}_3), f_3(\tilde{\delta}_{32}))$ or $\beta \in [f_4(\tilde{\delta}_4), f_4(\tilde{\delta}_{42}))$ respectively, then (35) is satisfied achieving feasible intercep-

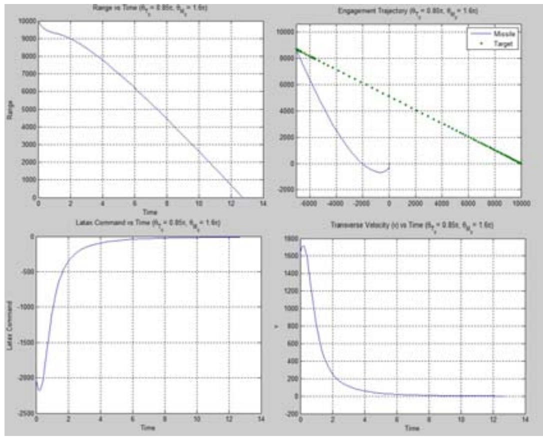

Fig. 5. Engagement Scenario for $\theta_{T_0} = 0.85\pi$, $\theta_{M_0} = 1.60\pi$

tion with finite LOS turn rate. ■

However if $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4$ are singleton sets, the conditions mentioned in Theorem 2 are also necessary for successful interception with finite LOS turn rate in respective cases.

IV. Simulation Results

A. Simulation result for planar engagement for $\eta = 1.5$

The simulation result, shown in Fig. 5, is for the engagement parameters $V_M = 1000\,m/sec$, $V_T = 1500\,m/sec$, $x = 1$, $y = 1$, $\theta_{T_0} = 0.85\pi$, $\theta_{M_0} = 1.60\pi$, Initial Range $(R_0) = 10000\,m$. Clearly, $(\bar{v}_0, \theta_{T_0}, \theta_{M_0}) \in R_1$. Applied navigation constant is $-\beta = -f_1(\tilde{\delta}_1) = -12.3305$. The variation of ρ, $\mathbf{a_M}$ and v with time and the interceptor and target trajectory are shown in Fig. 5. Here $\underline{\delta}_1 = 0.085$, $\tilde{\delta}_{11} = 0.1583 = \tilde{\delta}_1$, $\tilde{\delta}_{12} = 0.8217$ and $\underline{\beta}_1 = 12.205$. Hence applied $\beta > \underline{\beta}_1$ and $\beta = f_1(\tilde{\delta}_1) \in [f_1(\tilde{\delta}_1), f_1(\tilde{\delta}_{12}))$ leading towards successful interception with finite LOS rate.

B. Capture zone of 3D Retro-PN guidance law

Fig. 6 and 7 show the capture zone of the retro-PN guidance law against high speed non-maneuvering target with $\eta = 1.5$ and 1.25 in the $(\bar{u}_0, \bar{v}_0, x)$-space for 3D engagement and in the $(\bar{u}_0, \bar{v}_0)$-plane for planar engagement (where $x = y = 1$) respectively. From Theorem 1, one of the necessary and sufficient conditions for successful interception is $\pi - \sin^{-1} z < \theta_{T_0} < \pi + \sin^{-1} z$. Clearly by (26), as x increases $\sin^{-1} z$ also increases resulting into expansion of the capture zone with respect to feasible zone of θ_{T_0}, which justifies the pattern of the capture zone in the $(\bar{u}_0, \bar{v}_0, x)$-space with respect to x. The capture zone is symmetric $\bar{v}_0$ and is much smaller $\bar{u}_0 > 0$ than for $\bar{u}_0 < 0$. As η increases, the capture zone shrinks since z decreases as η increases, as clearly obtained in (26) and validated in Fig. 6.

V. Conclusion

The conceptualization of retro-PN guidance law has been discussed in this paper. The capturability of this law against high speed non-maneuvering target has been investigated analytically and simulation results have been presented.

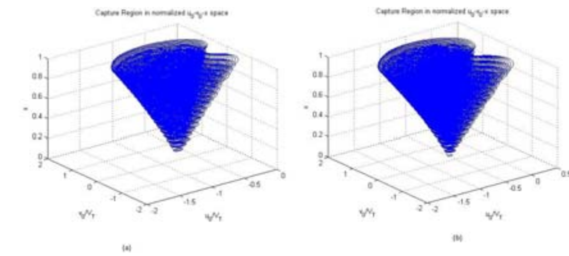

Fig. 6. capture zone of 3D Retro-PN Guidance Law in $(\bar{u}_0, \bar{v}_0, x)$-space for $(a)\,\eta = 1.5$, $(b)\,\eta = 1.25$

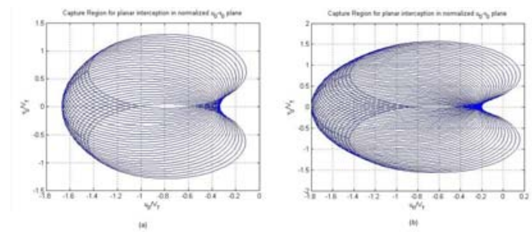

Fig. 7. capture zone of 3D Retro-PN Guidance Law in $(\overline{u_0}, \overline{v_0})$-plane for planar engagement and $(a)\,\eta = 1.5$, $(b)\,\eta = 1.25$

The simulation results were found to be in line with the theoretical analysis. The capture zone of this law has also been depicted in $(\bar{u}_0, \bar{v}_0, x)$-space along with the capture zone for the planar engagement in $(\bar{u}_0, \bar{v}_0)$-plane.

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