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New Consensus Algorithms Based on a Positive Splitting Approach

Valentina Boschian, Maria Pia Fanti, *Senior Member, IEEE*,
Agostino Marcello Mangini, Walter Ukovich, *Member, IEEE*

Abstract — **The paper proposes new consensus protocols for the agreement problem in networks of agents with a discrete time model. A new class of consensus algorithms is introduced on the basis of the positive splitting of the standard iteration matrix. In the framework of non negative matrix theory, some results are proved to guarantee the convergence of the proposed algorithms. In addition, numerous numerical experiments show that the proposed iterative schemes enjoy good rate of convergence even in the cases in which the standard iterative algorithms do not guarantee good performances.**

I. Introduction

THE research related to the topic of networked systems has widely increased during the last years attracting the attention of researchers from different fields such as mathematicians, computer scientists and engineers, [4], [5], [10], [14], [13]. This is due to the recent technological advances in communication and computation following the miniaturization of electronic components which have allowed the realization of large groups of embedded systems, such as sensors and robotic networks.

More in detail, the cooperative control for multi-agent systems can be categorized as either formation control problems with applications to mobile robots, unmanned air vehicles, autonomous underwater vehicles, satellites, aircraft, spacecraft, and automated highway systems, or non formation cooperative control problems such as task assignment, payload transport, role assignment, air traffic control, timing, and search. [14]

In networks of agents, "consensus" means to reach an agreement on the value of a certain quantity of interest that depends on the state of all agents. A consensus algorithm (or protocol) is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network. [11]. By this definition, cooperation can be informally interpreted as "giving consent to providing one's state and following a common protocol that serves the group objective" [11].

Moreover, such networks are intended to be *large-scale*, i.e., the number of connected devices can be large to be able to cover large surface areas. Hence, such networks need scalable algorithms, i.e., algorithms whose computational complexity grows moderately with respect to the number of network nodes, and decentralized algorithms able to solve problems addressing the topological communication network constraints. Some examples of the consensus algorithms are shown and discussed in [11]. The theoretical framework for posing and solving consensus problems for networked dynamic systems are introduced by Olfati-Saber and Murray [12] and Fax and Murray [2]. Moreover, Jadbabaie *et al.* in [8] study the alignment problem involving reaching an agreement and provide convergence results. In such papers as in the analysis performed in [17], the consensus protocols with fixed and switching topologies are proposed mainly using concepts and tools taken from algebraic graph theory. In particular, the network of agents is described by a directed or undirected graph and the associated graph Laplacian matrix L plays an important role in the convergence and alignment analysis [3], [9].

In this paper we consider a sensor network whose nominal state evolution is governed by a discrete time consensus equation. In particular, we start from the discrete time model of consensus networks defined by the equation $x(k+1)=(I-\varepsilon L)x(k)$, where I is the identity matrix and $\varepsilon>0$ is a parameter that is usually called step-size [11]. However, such standard protocols exhibit low speed of reaching a consensus for particular topologies of the digraph. In order to determine new and faster alignment protocols, we propose a class of consensus algorithms that are based on the positive splitting [15] of the matrix $(I-\varepsilon L)$. Moreover, in the framework of non-negative matrix theory some results are proved in order to guarantee the convergence of the proposed algorithms. In addition, for each network topology we determine the positive splitting that allows reaching the group decision value.

Finally, the convergence properties are studied by a set of

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Valentina Boschian is with the Department of Industrial Engineering and Information Technology of the University of Trieste, Trieste, Italy (e-mail valentina.boschian@deei.units.it).

Maria Pia Fanti is with the Department of Electrical and Electronics Engineering of the Polytechnic of Bari, Bari, Italy (phone: +39-080-5963643; fax: +39-080-5963410; e-mail: fanti@deemail.poliba.it).

Agostino M. Mangini is with the Department. of Electrical and Electronics Engineering of the Polytechnic of Bari, Bari, Italy (e-mail: mangini@deemail.poliba.it).

Walter Ukovich is with the Department of Industrial Engineering and Information Technology of the University of Trieste, Trieste, Italy (e-mail walter.ukovich@deei.units.it).

tests showing that the proposed new consensus protocols exhibit good convergence performances even in the cases in which the standard consensus algorithms converge slowly.

The paper is organized as follows. Section II gives the background of the consensus problems and Section III describes the new class of consensus algorithms based on a positive splitting technique and proves their convergence. Moreover, Section IV compares and explains the benefits of the proposed algorithms by testing the convergence properties in different network topologies. Finally, Section V summarizes the conclusions.

II. Definition and Notation of the Consensus Problem

A. Background

Consider a network of n autonomous agents labeled by an index $i\in V$ with $V=\{1,2,..,n\}$. Let $x_i\in\Re$ denote the state of the agent i that can represent a physical quantity, such as, for instance, altitude, position, temperature, voltage, and so on. The interaction topology of a network of agents is represented using a directed graph $G=(V,E)$ where $V=\{1,2,..,n\}$ is the set of nodes and $E\subseteq V\times V$ is the set of edges. Moreover, matrix $\boldsymbol{A}=[a_{ij}]$ denotes the adjacency matrix and $N_i=\{j\in V: a_{ij}\neq 0\}$ the set of neighbors of agent i. More precisely, agent i communicates with agent j if j is a neighbor of i (hence $a_{ij}\neq 0$). We say that the nodes of a network have reached a consensus if and only if (iff) $x_i=x_j$ for all $i,j\in V$. Whenever the agents of a network are all in agreement, the common value of all nodes is called the agreement state and can be expressed as $x^*=\alpha\mathbf{1}$, where $\mathbf{1}=[1,1,\ldots,1]^T$ and α is a collective decision of the group of the agents.

A well-known consensus algorithm that solves the agreement problem in a network of agents with discrete time model is the following [11]:

$$x_i(k+1)=x_i(k)+\varepsilon\sum_{j\in N_i}a_{ij}(x_j(k)-x_i(k)) \qquad (1)$$

and the algorithm can be written as:

$$x(k+1)=P_\varepsilon x(k) \qquad (2)$$

where matrix $P_\varepsilon=(I-\varepsilon L)$ is the iteration matrix, ε is the step-size parameter, I is the identity matrix and $L=[l_{ij}]$ is the *graph Laplacian* induced by the graph G and defined as:

$$l_{ij}=\begin{cases}\sum_{k=1,k\neq j}^{n}a_{ij} & \text{if } j=i\\ -a_{ij} & \text{if } j\neq i\end{cases} \qquad (3)$$

Denoting by $\Delta=\max_i l_{ii}$ the maximum node out-degree of graph G, P_ε is a nonnegative and stochastic matrix for all $\varepsilon\in(0, 1/\Delta)$. According to the definition of graph Laplacian in (5), all row-sums of L are zero because $\sum_{j=1}^{n}l_{ij}=0$ for $i=1,\ldots,n$.

Now, in order to study the convergence of algorithm (3), we recall the following properties about non-negative matrices.

Definition 1. [15] A $n\times n$ non-negative matrix $B=[b_{ij}]$ is *irreducible* if for every pair i, j of its index set, there exists a positive integer m (that depends on the indices i, j) such that $b_{ij}^m>0$. An irreducible matrix is said to be cyclic (or periodic) with period d, if the period of any one (and so of each one) of its indices satisfies $d>1$, and is said to be acyclic (or aperiodic) if $d=1$.

Note that the graph associated with an irreducible non negative matrix B is a strongly connected graph. Moreover, the graph associated with a cyclic matrix is said *d-periodic* [2] and has the property that the set of all cycle lengths has a common divisor $d>1$.

The convergence analysis of the discrete-time consensus algorithm relies on the following well-known lemma in matrix theory (Perron-Frobenius- [6]):

Lemma 1 Let B be a primitive (an irreducible stochastic acyclic matrix with only one eigenvalue $\lambda=1$) with left and right eigenvectors w and v, respectively, satisfying $Bv=v$, $w^TB=w^T$, and $v^Tw=1$. Then $\lim_{k\to\infty}B^k=vw^T$.

The convergence and group decision properties of iterative consensus algorithms with row stochastic matrices is stated in the following result proved in [11].

Theorem 1: Consider a network of agents $x_i(k+1)=x_i(k)+u_i(k)$ with topology G applying the distributed consensus algorithm (3) with $0<\varepsilon<1/\Delta$. Let G be a strongly connected graph. Then:

i) a consensus is asymptotically reached for all the initial states;

ii) the group decision value is $x^*=\sum_i w_i x_i(0)$ with $\sum_i w_i=1$;

iii) if the graph is balanced (undirected), an average-consensus is asymptotically reached and $x^*=\sum_i x_i(0)/n$.

Moreover, in [11] it is shown that the decision value is $\lim_{k\to\infty} x(k) = v(w^T x(0))$ where v=**1** is the right eigenvector of P_ε and w is the left eigenvector of P_ε associated with the eigenvalue λ=1.

B. Some comments on the convergence of the consensus algorithm

In this section we analyze the convergence properties of the standard convergence algorithm (1) in relation with the choice of coefficient ε and the network topology.

Proposition 1: Let G be a strongly connected graph. Setting $\varepsilon=1/\Delta$ matrix $P_\varepsilon=(I-\varepsilon L)$ is cyclic iff G is a *periodic* graph and each node of G has the same out-degree.

Proof (if part): Let assume that G is a *d-periodic* graph and each node of G has the same out-degree. Hence, the adjacency matrix A is cyclic and matrix L has equal diagonal entries l_{ii}. If we choose $\varepsilon=1/\Delta=1/l_{ii}$ then we obtain $p_{\varepsilon ii}=0$ for $i=1,\ldots,n$ and $P_\varepsilon=\varepsilon A$. Since A is cyclic, then P_ε is cyclic too.

(*Only if part*): Let assume that matrix P_ε is cyclic. Consequently, it holds $p_{\varepsilon ii}=0$ for $i=1,\ldots, n$ and the diagonal entries of matrix L are $l_{ii}=1/\varepsilon=\Delta$ for $i=1,\ldots, n$. Since $l_{ii} = \sum_{k=1,k\neq j}^{n} a_{ij}$, each node of G has the same outdegree. Moreover, by (5) it holds $P_\varepsilon=\varepsilon A$. Hence, if P_ε is cyclic, then the adjacency matrix A is cyclic too and G is a periodic graph. □

By Lemma 1, if matrix P_ε is cyclic, then the convergence of the iterative scheme (2) is not guaranteed. Hence, Proposition 1 justifies the well-known choice of $\varepsilon\in(0, 1/\Delta)$: the convergence of (1) is assured using $\varepsilon=1/\Delta$ provided that the graph G is not periodic and each node of G has the same outdegree. Moreover, it is shown that in the cases in which the convergence is guaranteed, the maximum value $\varepsilon=1/\Delta$ provides the maximum convergence speed of the iteration scheme (3) [17].

III. A New Class of Consensus Algorithms

In this section we introduce new consensus algorithms that solve agreement problems in a network with fixed or switching topology and zero-communication time delay.

We consider the consensus algorithm (1) and we split matrix P_ε into two non negative square matrices $R\neq0$ and $S\neq0$ such that $P_\varepsilon=R+S$. Each splitting induces the following iterative scheme, [16]:

$$x(k+1) = Rx(k+1) + Sx(k) \qquad k\geq0 \tag{4}$$

and

$$x(k+1) = (I-R)^{-1} Sx(k) \qquad k\geq0 \tag{5}$$

Matrix $\Gamma=(I-R)^{-1}S$ denotes the iteration matrix associated with the positive splitting $P_\varepsilon=R+S$. Since $S\neq0$, by the Perron-Frobenious theorem the spectral radius of R (denoted by $\rho(R)$) is less than one [15]. Therefore $(I-R)$ is non-singular.

A. Convergence properties of the iterative schemes

The following results characterize the convergence properties of the obtained iteration schemes. In particular, we show that under some conditions on the positive splitting, the iterative algorithm (7) converges to the same group decision value $x^* = \sum_i w_i x_i(0)$ of Theorem 1. To this aim we prove the conditions to obtain a primitive iterative matrix Γ, so that the convergence of the consensus algorithm is assured.

Lemma 2: Let $R+S=P_\varepsilon$ be a positive splitting of P_ε. If P_ε is irreducible and stochastic, then matrix $\Gamma=(I-R)^{-1}S$ is stochastic too.

Proof: The matrix $(I-R)^{-1}S$ is the sum of the series $\sum_{k=0}^{\infty} R^k$, that is convergent since matrix R is a positive matrix with spectral radius $\rho(R)<1$ by the Perron-Frobenious theorem [15]. Observing that both S and R are non-negative matrices, it immediately follows that $(I-R)^{-1}$ and Γ are non-negative too.

Since $P_\varepsilon\mathbf{1}=(R+S)\mathbf{1}=\mathbf{1}$, it holds $(I-R)\mathbf{1}=S\mathbf{1}$ and $(I-R)^{-1}S\,\mathbf{1}=\mathbf{1}$. Then v=**1** is the right eigenvector associated with the eigenvalue λ=1 and Γ is a stochastic matrix. □

Theorem 2: Let P_ε be a stochastic irreducible matrix and let $P_\varepsilon=R+S$ be a positive splitting of P_ε. Then, for matrix $\Gamma=(I-R)^{-1}S$, the following statements hold true:

i) $\rho(\Gamma)=1$;

ii) $\lambda_1=1$ is a simple eigenvalue of Γ.

Proof: Statement i) is a direct consequence of Lemma 1.

Statement (ii) follows from the fact that there is a unique right eigenvector v=**1** corresponding to the dominant eigenvalue λ_1=1 of the irreducible matrix $P_\mathcal{E}$. On the other hand, $P_\mathcal{E}$ν=ν implies and is implied by $(I-R)^{-1}S$ ν=ν. Hence matrices $P_\mathcal{E}$ and $(I-R)^{-1}S$ have the same number of independent right eigenvectors associated with the eigenvalue λ_1=1. Therefore the geometric multiplicity of λ_1=1 is the same for both matrices and it equals one. Now, by Lemma 1 the algebraic multiplicity of λ_1=1, as eigenvalue of Γ, equals its geometric multiplicity. Therefore, statement (ii) is proved. □

The following results characterize the left eigenvector of $\Gamma=(I-R)^{-1}S$ associated with the eigenvalue λ=1.

Lemma 3: Let $P_\mathcal{E}$ be a stochastic irreducible matrix and w be the left eigenvector of $P_\mathcal{E}$ associated with the eigenvalue λ=1. Let $R+S=P_\mathcal{E}$ be a positive splitting of $P_\mathcal{E}$. The left eigenvector of matrix $\Gamma=(I-R)^{-1}S$ associated with the eigenvalue λ=1 is $w'^T=w^TS$.

Proof: We assume that vector w is the left eigenvector of $P_\mathcal{E}$ associated with the eigenvalue λ=1, i.e.: $w^T(R+S)=w^T$. We can write:

$$w^TS=w^T(I-R) \tag{6}$$

and

$$w^T(I-R)\ (I-R)^{-1}S=w^T(I-R) \tag{7}$$

and by substituting (6) and (7) we infer

$$w^TS(I-R)^{-1}S=w^TS. \tag{8}$$

Hence w^TS is the left eigenvector of $(I-R)^{-1}S$ associated with the eigenvalue λ=1. □

As it is shown in [7], given a state set and a stochastic matrix there exists a Markov Chain associated with them. Hence, let MC be the Markov Chain associated with the stochastic matrix $(I-R)^{-1}S$. By Theorem 2, Γ has only one eigenvalue equal to 1, consequently MC has only one recurrent class. The following lemmas introduce a sufficient condition assuring Γ primitive, i.e., Γ is irreducible with $|\lambda|<1$ for each eigenvalue $|\lambda|\neq 1$ of Γ.

Lemma 4: Let $P_\mathcal{E}$ be a stochastic irreducible matrix and $R+S=P_\mathcal{E}$ be a positive splitting of $P_\mathcal{E}$. If matrix S has no zero columns, then matrix $\Gamma=(I-R)^{-1}S$ is irreducible.

Proof: Let consider the vector $w'^T=w^TS$ and let MC be the Markov Chain associated with Γ. Since w'^T is the left eigenvector of matrix $\Gamma=(I-R)^{-1}S$ associated with the eigenvalue λ=1, w'^T is proportional to the steady state vector of MC. Now let observe that w>0 is the steady-state probability vector of the recurrent states of the Markov Chain associated with $P_\mathcal{E}$. Hence, the i-th entry of w' is zero iff the i-th column of S has all zero entries. Remarking that only states in recurrent classes can occur with positive steady state probability, Lemma 4 is proved. □

Lemma 5: Let $P_\mathcal{E}$ be a stochastic irreducible matrix. Let $R+S=P_\mathcal{E}$ be a positive splitting of $P_\mathcal{E}$ and let $\lambda\neq 1$ be an eigenvalue of $\Gamma=(I-R)^{-1}S$. If matrix S has no zero columns, then $|\lambda|<1$.

Proof: The proof is given by contradiction. Let us suppose that matrix S has no zero columns and there exists an eigenvalue $\lambda\neq 1$ of $\Gamma=(I-R)^{-1}S$ such that $|\lambda|=1$. Since by Lemma 4 $\Gamma=(I-R)^{-1}S$ is irreducibile, matrix Γ is cyclic [15], and therefore it has all zero entries along the main diagonal. We write:

$$\Gamma=(I-R)^{-1}S=(I+R+R^2+\ldots)S. \tag{9}$$

Since R and S are non-negative matrices, it holds $\Gamma\geq(I+R)S$. Now all the entries along the main diagonal of $(I+R)$ are non-zero. Consequently, an entry of the main diagonal of Γ can equal zero iff the i-th column of S is zero: this contradicts the assumption and the lemma is proved. □

The following theorem guarantees the convergence of the algorithm (5) that is induced by a positive splitting.

Theorem 3: Let $P_\mathcal{E}$ be a stochastic irreducible matrix and w the left eigenvector of $P_\mathcal{E}$ associated with the eigenvalue λ=1. Let $R+S=P_\mathcal{E}$ be a positive splitting of $P_\mathcal{E}$ and S has no zero columns. If there exists μ>0 such that $w^TS=\mu w^T$, i.e., w^T is the left eigenvector of S for an eigenvalue μ>0, then by algorithm (5) a consensus is asymptotically reached for all the initial states and the group decision value is $x^*=vw^Tx(0)$ for all the initial states.

Proof: If $R+S=P_\mathcal{E}$ is a positive splitting of $P_\mathcal{E}$ and S has no zero columns, then by Theorem 2, Lemmas 3, 4 and 5, the matrix $\Gamma=(I-R)^{-1}S$ is primitive with right and left eigenvalues v'=**1** and $w'^T=w^TS$ (such that $\mathbf{1}^Tw'$=1) associated with the eigenvalue λ=1. Consequently, the iterative algorithm (5) converges and gives the decision value $\lim_{k\to\infty}x(k)=v(w^TSx(0))$

Moreover, if there exists μ>0 such that $w^T S = \mu w^T$, then it holds $\lim_{k\to\infty} x(k) = \mu v w^T x(0)$ and by the normalizing condition it holds $\lim_{k\to\infty} x(k) = v w^T x(0)$ This proves the theorem. □

B. The proposed consensus algorithms

The algorithm described by equ. (5) is in general difficult to implement. In this subsection we consider the consensus algorithms based on the iterative schemes (4)-(5) that can be described by the following relations:

$$(1-r_{ii})x_i(k+1) = \sum_{j=1}^{i-1} p_{\varepsilon ij} x_j(k+1) + s_{ii} x_i(k) + \sum_{j=i+1}^{n} p_{\varepsilon ij} x_j(k)$$

with $s_{ii} + r_{ii} = p_{\varepsilon ii}$, for i=1,…,n. (10)

In other words, the iterative algorithm (10) establishes an order to update the values of each agent state. More precisely, to update the state at the time k+1, agent i-th uses the already determined values of the states $x_j(k+1)$ for j=1,…,i-1. Hence, the iterative scheme (10) leads to the following set of positive splitting of matrix P_ε:

$Q(e)$={ R≠0, S≠0 | R is a lower non negative triangular matrix, S is an upper non negative triangular matrix and R+S= P_ε =(I-εL) }.

In order to obtain an upper triangular matrix S that satisfies the conditions of Theorem 3, the following set of linear constraints is defined:

$$\Phi(P_\varepsilon, w) = \begin{cases} \sum_{i=1}^{n} w_i s_{ij} - w_j \mu = 0 \text{ for } j = 1,\dots,\ n \\ 0 \le s_{ii} \le p_{\varepsilon ii} \text{ for } i = 1,\dots,n \\ \mathbf{1}^{\mathrm{T}} S > 0 \\ \mu > 0 \end{cases} \quad (11)$$

with s_{ij} =0 for i>j, $s_{ij} = p_{\varepsilon ij}$ for i<j, with i,j=1,…,n.

We remark that (10) with any solution of (11) gives a consensus algorithm associated with the graph G describing the interaction topology of a network of agents. Moreover, if graph G changes, then the consensus can be reached by updating in the iterative scheme (10) the values of s_{ii} and r_{ii} by solving (11).

IV. Algorithm Convergence Properties

In order to evaluate the convergence properties of the proposed algorithms, we consider a network of 20 agents with different topologies. More precisely, the entries of the adjacency matrices of the strongly connected aperiodic graphs describing the network topology are randomly generated equal to 0 or 1 with uniform probability. The asymptotic convergence properties and convergence times are evaluated on 1000 randomly generated adjacency matrices by applying four different consensus algorithms. For each system, the convergence time k* is considered to be the number of broadcasts such that the following condition is satisfied:

$$k^*: \frac{\|x(k^*) - \bar{x}^*\|_2}{\|x(0) - \bar{x}^*\|_2} < 0.01. \quad (12)$$

where $\bar{x}^*$ is the group decision value.

The results of the convergence study are reported in Table I where the first column shows the iterative matrix of the applied consensus algorithm. More precisely, the first iterative matrix is associated to the iterative algorithm (4) with ε=0.5/Δ. The iterative scheme $\Gamma_1 = (1-\varepsilon)I + \varepsilon D^{-1}A$ is proposed in [8] where A is the adjacency matrix of graph G and D is the diagonal matrix whose i-th diagonal element is the valence of vertex i within the graph.

The previous iterative schemes are compared with two algorithms based on the positive splitting of matrix P_ε:

$\Gamma_2 = (I - R_2)^{-1} S_2$ where R_2,S_2∈ Q(0.5/Δ) and R_2 is a strictly lower non negative triangular matrix.

$\Gamma_3 = (I - R_3)^{-1} S_3$ where R_3,S_3∈ Q(0.5/Δ) and S_3 satisfies the set of constraints Φ(P_ε ,w).

TABLE I
CONVERGENCE PROPERTIES OF THE CONSENSUS ALGORITHMS

Consensus algorithm	$\bar{k}^*$	s^2	$\bar{x}^*$	E%	$\bar{\lambda}_2$
P_ε ε=0.5/Δ	11	4.98	0.382	0	0.787
Γ_1	4	0.14	0.384	-0.52%	0.265
Γ_2	11	4.88	0.373	2.35%	0.786
Γ_3	8	3.10	0.382	0	0.698

The second and third columns of Table I show respectively the average value $\bar{k}^*$and the variance σ^2 of the convergence time, calculated on the 1000 randomly generated cases. Moreover, the fifth column of Table I reports the average percentage error E% of the consensus value obtained by the corresponding algorithm. In addition, the last column shows the average value $\bar{\lambda}_2$ of the second eigenvalue of the corresponding iterative matrix. Indeed, the average time depends on the largest eigenvalue of the stochastic matrix characterizing the consensus algorithm: the smaller the eigenvalue λ_2 is, the faster the algorithm is [1].

The results show that the proposed algorithms exhibit good performances. In particular, the iterative schemes Γ_2 gives the same average time of P_ε with ε=0.5/Δ, but the decision value exhibits an average error of the 2.35%.

TABLE II
CONVERGENCE PROPERTIES OF THE CONSENSUS ALGORITHMS APPLIED TO PERIODIC DIGRAPHS

Consensus algorithm		d=2	d=3	d=4	d=6	d=12
	k^*	10	8	10	20	72
P_ε ε=0.5/Δ	x^*	0.39	0.39	0.39	0.39	0.39
	λ_2	0.93	0.85	0.73	0.87	0.97
	k^*	9	7	18	38	135
P_ε ε=0.8/Δ	x^*	0.39	0.39	0.39	0.39	0.39
	λ_2	0.89	0.76	0.82	0.92	0.98
	k^*	8	8	11	25	72
Γ_1	x^*	0.39	0.39	0.39	0.39	0.39
	λ_2	0.91	0.80	0.74	0.88	0.97
	k^*	9	8	8	8	17
Γ_2	x^*	0.39	0.40	0.40	0.40	0.39
	λ_2	0.92	0.82	0.68	0.67	0.83
	k^*	8	7	6	6	15
Γ_3	x^*	0.39	0.39	0.39	0.39	0.39
	λ_2	0.88	0.76	0.57	0.54	0.81

Moreover, the iterative schemes Γ_3 improve the convergence obtained by the matrices P_ε with ε=0.5/Δ, since the average number of iterations decreases from $\bar{k}^*$=11 to $\bar{k}^*$=8. Moreover, an efficient numerical behaviour of the iteration process is given by Γ_1, but it converges to a different group decision value (there is an error of -0.52%).

In addition, we analyse a set of cases where the network topologies are described by periodic graphs of 12 nodes. More precisely, we consider 5 cases of periodic graphs with d=2, 3, 4, 6 and 12 and each node has the same outdegree. Table II reports the convergence time and properties: for each value of d the value of k^* decreases using the new algorithm schemes Γ_3. In particular the benefits in applying scheme Γ_3 is evident for the case d=12 in which the standard convergence algorithms are slow. Hence, the performed tests show that the proposed consensus algorithm works very well in all cases, including the cases in which the standard algorithms exhibit low performances.

V. CONCLUSIONS

This paper investigates new and fast alignment protocols that can be applied to the discrete time model of consensus networks. To this aim we propose a class of consensus algorithms that are based on the positive splitting of the standard iteration matrix. The convergence of the proposed discrete-time consensus algorithms is proved in the framework of non-negative matrix theory. Central of our approach is the question of how to use effectively the splitting of the standard iteration matrix in order to improve the rate of the consensus convergence. A set of tests shows the advantages of the proposed protocols: the presented algorithms exhibit good performances even in the cases in which the standard consensus protocols converge slowly.

Future research will focus on the following open important issues: i) determining the conditions under which the problem (13) admits a solution; ii) providing analytical results about the performance analysis of the proposed consensus protocols in relation with the network topology.

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