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# Root Locus for SISO Infinite-dimensional systems

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***Abstract*— The root locus is an important tool for analysing the stability and time constants of linear finite-dimensional systems as a parameter, often the gain, is varied. However, many systems are modelled by partial differential equations or delay equations. These systems evolve on an infinite-dimensional space and their transfer functions are not rational. In this paper we provide a rigorous definition of the root locus and show that it is well-defined for a large class of infinite-dimensional systems. As for finite-dimensional systems, any limit point of a branch of the root locus is a zero. However, the asymptotic behaviour can be quite different from that for finite-dimensional systems. We also show that the familar pole-zero interlacing property for collocated systems generated by a self-adjoint operator extends to infinite-dimensional systems.**

## I. INTRODUCTION

Consider the control system on a Hilbert space $\mathcal{Z}$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned} \tag{1}$$

where for some $b$, $c \in \mathcal{Z}$, $Bu = bu$, $Cx = \langle c, x \rangle$. For real $k$, we are interested in the eigenvalues of $A - kBC$ as $k \to \infty$. A plot of these eigenvalues as $k \to \infty$ is known as a root locus plot. An understanding of the behaviour of these eigenvalues as $k$ varies, or the root locus, is important to understanding the behaviour of the system with feedback

$$u(t) = -ky(t) + v(t),$$

where $v(t)$ is an external signal.

Suppose that the system is finite-dimensional; that is $A \in \mathbb{C}^{n \times n}$. In the 1970's it was shown that if the relative degree of the system is $r$ then there are $r$ eigenvalues going to infinity and the remaining eigenvalues tend to the zeros of the corresponding transfer function $G(s) = C(sI - A)^{-1}B$ [1], [2]. Furthermore, the angle of the asymptotes as $k \to \infty$, in particular whether they are in the left-half-plane, is determined by the relative degree.

Extension of these results, now well-known for finite-dimensions, to infinite-dimensions has been elusive. In [3] the root locus is considered for the case where $A$ is self-adjoint with compact resolvent on a Hilbert space $\mathcal{Z}$, $B$ is a linear bounded operator from $\mathbb{C}^p$ to $\mathcal{Z}$, and $C : D(C) \to \mathbb{C}^p$ where $D(A) \subset D(C)$ is $A$-bounded. A complete analysis of collocated boundary control of parabolic systems on an interval was provided in [4]. The analysis in that paper uses results from differential equations theory and is difficult to extend to more general classes. In [5] high-gain output feedback of infinite-dimensional systems in the case where $A$ generates an analytic semigroup and $B = C^*$ was studied. The zeros of the system are given as the eigenvalues of an operator and a nonlinear stabilizing feedback law is constructed. Zeros of systems where $A$ is self-adjoint and $B = C^*$ are shown to be real and be bounded by $\alpha$ if $A + A^* \leq 2\alpha I$ on $D(A)$ in [6]. If moreover, the system transfer function can be written in spectral form, and an additional technical condition is satisfied, the poles and the zeros interlace on the real axis.

In this paper we provide a definition of the root locus for single-input-single-output infinite-dimensional systems and show that it is well-defined if no invariant zeros are in the spectrum of $A$. Each eigenvalue of $A$ defines a branch of the root locus and these curves are smooth and non-intersecting if the set of zeros of $(A, B, C)$ and the eigenvalues of $A$ do not intersect. Moreover, if any branch converges to a point, that point is a zero of the system. Conversely, each zero is the terminus of a branch of the root locus. We close by generalizing the earlier result of [6] on the interlacing property to a wider class of systems. Furthermore, we show that every branch of the root locus of these collocated systems converges to a zero. We conclude by illustrating the results with a few simple examples.

## II. ROOT LOCUS

In this section we establish that the root locus of a large class of infinite-dimensional systems is well-defined.

Let $A$ be the generator of $C_o$-semigroup on a Hilbert space $\mathcal{Z}$ that has only isolated point spectrum, each with finite multiplicity and of finite type, that is the Riesz projection on the generalized eigenspace is finite dimensional. This implies in particular that the essential spectrum of $A$ is empty and the assumption on the spectrum are satisfied if $A$ has a compact resolvent. Let $\{\lambda_n\}$ be the set of eigenvalues of $A$. Let $B$, $C$ be bounded operators with scalar input and output spaces. This means that $Bu = bu$ and $Cx = \langle c, x \rangle$ for some $b, c \in \mathcal{Z}$. For real $k$, we are interested in the eigenvalues of $A - kBC$. Since $B$ and $C$ are bounded finite rank operators, the spectrum of $A - kBC$ also consists only of point spectrum of finite multiplicity [7, Theorem I.4.1]. There is a family of curves $f_n(k)$ associated with each eigenvalue of $A$ with $f_n(0) = \lambda_n$. The values of $f_n(k)$ are the eigenvalues of $A - kBC$. The *root locus* is the set of curves $f_n(k)$. In this section we show that these curves are well-defined for $k \in [0, \infty)$ and prove some properties of the curves.

Throughout this paper we will use the following notation. For a closed densely defined linear operator $A$ on some

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Hilbert space $\mathcal{Z}$ we denote by $D(A)$, $\sigma(A)$, and $\rho(A)$, the domain, the spectrum, and the resolvent set, respectively.

*Definition 2.1:* Let $(T(t))_{t\geq 0}$ be the $C_o$-semigroup generated by $A$. The system $(A,B,C)$ *is approximately observable* if for every $z \in \mathcal{Z}$ the function $CT(t)z$ is not identically zero on $[0,\infty)$.

Let $G(s) = C(sI-A)^{-1}B$ indicate the characteristic function of $(A,B,C)$. If $(A,B,C)$ is approximately observable, this definition is equivalent to the other definitions of the transfer function for all $s \in \rho(A)$ [8, Cor. 2.8].

*Lemma 2.2:* For any point $s \in \rho(A)$, and $k \neq 0$, $s \in \sigma(A-kBC)$ if and only if $G(s) = -\frac{1}{k}$.

*Proof:* The proof of this lemma is along the lines of that outlined for systems with self-adjoint $A$ and $B = C^*$ in [9, ex. 4.28c(i)]. The point $s \in \sigma(A-kBC)$ if and only if $1 \in \sigma(-k(sI-A)^{-1}BC)$. This occurs if and only if $-\frac{1}{k} \in \sigma((sI-A)^{-1}BC)$, or $-\frac{1}{k} \in \sigma(G(s))$ which is equivalent to $G(s) = -\frac{1}{k}$. □

We need the following result on holomorphic functions.

*Proposition 2.3:* [10, Thm. 7.4] Let $g : \Omega \to \mathbb{C}$ with $\Omega \subset \mathbb{C}$ be holomorphic and $k \in \mathbb{R}$. If $G(s_0) = \frac{1}{k}$ for some $s_0 \in \Omega$ and $m$ be the order of zero which the function $G(s) - \frac{1}{k}$ has at $s_0$. Then there exists for every sufficiently small $\varepsilon > 0$ a neighbourhood $U_\varepsilon$ of $s_0$ such that the function $G(s)|_{U_\varepsilon}$ attains every value $w$ with $0 < |w - \frac{1}{k}| < \varepsilon$ exactly $m$ times.

Similarily to the finite-dimensional case we introduce the notion of transmission zeros and invariant zeros.

*Definition 2.4:* The *transmission zeros of* $(A,B,C)$ are the zeros of $G(s) = C(sI-A)^{-1}B$.

*Definition 2.5:* The *invariant zeros of* $(A,B,C)$ are the set of all $\lambda$ such that

$$\begin{bmatrix} \lambda I - A & b \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

has a solution for some scalar $u$ and non-zero $x \in D(A)$. Denote the set of invariant zeros of a system by $\mathrm{inv}(A,B,C)$.

As in finite-dimensional systems, it is straightforward to show that every transmission zero is an invariant zero. Similarly, every invariant zero $z \in \rho(A)$ is a transmision zero [6].

*Proposition 2.6:* The set $\mathrm{inv}(A,B,C)$ is countable and has no finite accumulation point.

*Proof:* As $G(s)$ is a meromorphic function, the set of transmission zeros of $(A,B,C)$ is countable and has no finite accumulation point. Moreover, by assumption the spectrum of $A$ is countable and has no finite accumulation point. Since the set $\mathrm{inv}(A,B,C)$ is a subset of the union of $\sigma(A)$ and the set of transmission zeros of $(A,B,C)$, $\mathrm{inv}(A,B,C)$ is countable and has no finite accumulation point. □

*Theorem 2.7:* If $\mathrm{inv}(A,B,C) \cap \sigma(A)$ is empty then $\sigma(A) \cap \sigma(A-kBC)$ is empty for all $k$.

*Proof:* Suppose that for some $k \neq 0$ there is $s \in \sigma(A - kBC) \cap \sigma(A)$ and let $x_o \neq 0$, $x_k \neq 0$ be such that

$$sx_o = Ax_o, \qquad sx_k = Ax_k + B(kCx_k). \tag{3}$$

Let $\alpha$ be a scalar such that

$$Cx_o + \alpha Cx_k = 0.$$

If $x_o + \alpha x_k = 0$, then from (3) it follows that $Cx_k = 0$. Since $C$ is linear, then $Cx_o = 0$. This implies that $s$ is an invariant zero of $(A,B,C)$ and so $\mathrm{inv}(A,B,C) \cap \sigma(A)$ is not empty. Suppose now that $x_o + \alpha x_k \neq 0$. Then

$$\begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_o + \alpha x_k \\ kCx_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and so $s$ is an invariant zero of $(A,B,C)$. This implies that $\sigma(A) \cap \mathrm{inv}(A,B,C)$ is not empty. □

We can now show that the root locus for the class of infinite-dimensional functions considered here is well-defined.

*Theorem 2.8:* Consider the root locus functions $f_n(k)$ as defined above. Each $f_n$ is a continuous function of $k$, $k \in [0,\infty)$. Furthermore, if $\mathrm{inv}(A,B,C) \cap \sigma(A) = \phi$ then each branch $f_n$ is a simple non-intersecting curve.

*Proof:* For any finite set of eigenvalues, enclose them by a simple closed curve $\Gamma$ separating this part of the spectrum $\sigma_1(A)$ from the remainder. Let $\mathcal{N}$ be the indices of the eigenvalues of $A$ contained in $\Gamma$. Then [11, IV.5, pg. 213] implies there is a $k^M$ such that $f_n(k)$ is a continuous well-defined curve for all $k \in [0, k^M]$, $n \in \mathcal{N}$. Thus, for each $n$, there is $k_n^M$ so that $f_n$ is a continuous function of $k$ for $k \in [0, k_n^M]$.

Proposition 2.3 together with Lemma 2.2 implies that the root locus curves are defined on the interval $[0,\infty)$. The continuity of the root locus curves follows now from [11, IV.5, pg. 213].

Suppose that for some $n$, $k_1$, $k_2$, $f_n(k_1) = f_n(k_2) = s$. Since $f_n(k_1) = s$ implies that $s \in \sigma(A - k_1BC)$ we have $s \in \rho(A)$ (Theorem 2.7). Lemma 2.2 then implies that $G(s) = -\frac{1}{k_1} = -\frac{1}{k_2}$ and so $k_1 = k_2$. Thus, no curve intersects with itself. □

*Corollary 2.9:* For any point $s$, only finitely many $f_n$ intersect. Multiplicity of the spectrum is preserved at such intersection points.

*Proof:* Let $s \in \mathbb{C}$. If $f_n(k) = s$ for some $n$ and $k$, then $s$ is an eigenvalue of $A - kBC$. As the multiplicity of every spectral point is finite, Proposition 2.3 implies that at most finitely many curves $f_n$ intersect at $s$ and the multiplicity of the spectrum is preserved at these intersection points. □

We also have the following continuity result.

*Lemma 2.10:* Let $s_n \to s_o$ as $k_n \to k_o$ where $s_n \in \sigma(A - k_nBC)$. Then $s_o \in \sigma(A - k_oBC)$.

*Proof:* We have, for a sequence $x_n \in \mathcal{Z}$ of normalized eigenvectors of $A - k_nBC$ with eigenvalue $s_n$,

$$\begin{aligned} \|(A - k_oBC)x_n - s_ox_n\| \leq \; & \|(A - k_nBC)x_n - s_nx_n\| \\ & + |(k_n - k_o)| \|BCx_n\| \\ & + |s_n - s_o| \|x_n\|. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \|(A - k_oBC)x_n - s_ox_n\| = 0$$

and so $s_o$ is in the approximate point spectrum of $A - k_oBC$.

Since $\sigma(A)$ consists of only point spectrum, and $k_oBC$ is a bounded perturbation of $A$, then $\sigma(A - k_oBC)$ also contains only point spectrum and $s_o$ is in the point spectrum of $A - k_oBC$. □

## III. Zeros

The following result provides a sufficient condition for $\mathrm{inv}(A, B, C) \cap \sigma(A) = \phi$.

*Proposition 3.1:* If $(A, B, C)$ is approximately observable and controllable and $A$ is Riesz-spectral then $\mathrm{inv}(A, B, C) \cap \sigma(A)$ is empty.

*Proof:* See [9, Ex. 4.28b]. The proof is similar to that for finite-dimensions. □

Although it may be possible to extend this result to a larger class of systems, the assumption of a discrete system is critical.

*Example 3.2:* We consider the transport equation on the interval $[0, \infty)$

$$\frac{\partial w}{\partial t}(\zeta, t) = \frac{\partial w}{\partial \zeta}(\zeta, t) + \chi_{[0,1]}u(t) \quad t \geq 0, \zeta \in [0, \infty) \tag{4}$$

$$w(\zeta, 0) = w_0(\zeta), \quad \zeta \in [0, \infty) \tag{5}$$

$$y(t) = \int_0^1 w(\zeta, t)\, d\zeta, \qquad t \geq 0. \tag{6}$$

The corresponding operator $A : D(A) \subset L^2(0, \infty) \to L^2(0, \infty)$ is given by $Ax := x'$ with $D(A) = H^1(0, \infty)$. The system is approximately controllable and approximately observable, but $\sigma(A) = \sigma(A - kBC) = \{s \in \mathbb{C} \mid \mathrm{Re}\, s \leq 0\}$.

In finite-dimensions, each branch of the root locus $f_n$ converges to either a zero or to infinity as $k \to \infty$. We have obtained some partial results in this direction for infinite-dimensional systems.

*Theorem 3.3:* Let $z \in \rho(A)$ be a transmission zero. Then there exists $s_k \to z$ as $k \to \infty$ such that $s_k \in \sigma(A - kBC)$. Conversely, if $s_k \to s_o$ as $k \to \infty$ where $s_k \in \sigma(A - kBC)$, then $s_o$ is a transmission zero.

*Proof:* By the Open Mapping Theorem, point $z \in \rho(A)$ satisfies $G(z) = 0$ if and only if there is $\{s_n\} \in \mathbb{C} \cap \rho(A)$ and a real-valued sequence such that $s_n \to s$, $k_n \to \infty$ and $G(s_n) = C(s_nI - A)^{-1}B = -\frac{1}{k_n}$. This is (trivially) equivalent to $-\frac{1}{k_n} \in \sigma(C(s_nI - A)^{-1}B)$. By [12, pg. 38 (3)] this is equivalent to $-\frac{1}{k_n} \in \sigma((s_nI - A)^{-1}BC)$ and similarly, since $(s_nI - A)^{-1}B$ and $C$ are bounded operators, $1 \in \sigma(-k_n(s_nI - A)^{-1}BC)$. This is also equivalent to $s_n \in \sigma(A - k_nBC)$ [13, Prop. 4.2,pg. 289]. Since each statement in the preceding argument is an equivalence, the converse follows immediately. That is, if $s_n \to z$ where $s_n \in \sigma(A - k_nBC)$ then $z$ is a transmission zero. □

Thus, in summary, the root locus of any control system in the class considered here is well-defined. Provided that an assumption, such as observability is satisfied so that $\sigma(A) \cap \mathrm{inv}(A, B, C)$ is empty, then each branch is a simple, non-intersecting curve. The limit of any branch is a transmission zero and every transmission zero is the limit of a branch of the root locus. A remaining question is to establish that either every branch $f_n(k) \to z$ where $z$ is a zero or else whether $|f_n(k)| \to \infty$ in some situations. This question is answered below for a special class of systems.

## IV. Collocated Self-Adjoint Systems

Consider the case where $A$ is a self-adjoint, negative semi-definite operator and $B = C^*$. If the underlying state space is finite-dimensional, then it is well-known that the poles and zeros are real, interlace on the negative real axis and furthermore, the system is relative degree one so that there is one asymptote. This asymptote moves along the negative real axis to $-\infty$. A partial generalization for infinite-dimensional systems was obtained in [6]. In that paper the authors show that the poles and zeros of spectral systems that satisfy an additional technical condition interlace on the real axis. The following theorem which uses results provided earlier in this paper, provides a significant generalization of this earlier work.

*Theorem 4.1:* Suppose that $A$ is a self-adjoint, negative semi-definite operator on a infinite-dimensional space and $B = C^*$. If $\mathrm{inv}(A, B, C) \cap \sigma(A)$ is empty then each $f_n(s)$ is real-valued, all the zeros are real, each branch converges to a zero as $k \to \infty$ and the zeros interlace with the poles.

*Proof:* First, since $A - kBB^*$ is self-adjoint and negative semidefinite, all branches of the root locus lie entirely on the negative real axis $\mathrm{Re}\, s \leq 0$. Also, no branch intersects with itself and so each branch starts at a pole and moves monotonically either to the left or right. The assumption $\mathrm{inv}(A, B, C) \cap \sigma(A)$ is empty implies that the root locus does not intersect with $\sigma(A)$ for any value of $k$. Each branch must either converge to a zero or to infinity. Since no branch crosses a eigenvalue, any asymptotic branch must go to $\infty$. However, all branches of the root locus lie entirely on the negative real axis and so this is impossible. Thus, each branch is a bounded monotonic function of $k$ and converges to a zero. The limit is a zero of $(A, B, C)$ and since the root locus is real-valued for all $k$, all the zeros are real. This also implies that a zero lies between each pole. □

Note that if we consider $k \to -\infty$, the above argument yields that there could be one branch of the root locus that converges to $\infty$. Also, if $A$ is defined on a finite-dimensional space, then there are a finite number of eigenvalues and there is one branch of the root locus that converges to $-\infty$ as $k \to \infty$.

## V. Examples

*Example 5.1:* **(Heat flow in a rod)** Consider the problem of controlling the temperature profile in a rod of length 1 with constant thermal conductivity $\kappa$, mass density $\rho$ and specific heat $C_p$. The rod is insulated at the ends $x = 0$, $x = 1$.To simplify, use dimensionless variables so that $\frac{\kappa}{C_p\rho} = 1$. With control applied through some weight $b(x)$, and the temperature is governed by the following problem

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + b(x)u(t), \qquad x \in (0, L), \qquad t \geq 0.$$

$$\frac{\partial z}{\partial x}(0, t) = 0, \qquad \frac{\partial z}{\partial x}(1, t) = 0,$$

where $b(x) \in \mathcal{L}(\prime, \infty)$. The temperature sensor is modelled by

$$y(t) = \int_0^1 b(x)z(x)dx. \tag{7}$$

It is well-known that this can be written as an abstract control system (1) on the Hilbert space $L_2(0,1)$ with

$$Az = \frac{\partial^2 z}{\partial x^2}, \qquad D(A) = \{z \in H^2(0,1),\ z'(0) = z'(1) = 0\}$$

and $Bu = b(x)u$, and $C = B^*$ is defined by (7). This system is approximately controllable (and observable) if

$$\int_0^1 b(x)\cos(n\pi x)dx \neq 0 \tag{8}$$

for all integers $n$ [9, Thm. 4.2.1]. The operator $A$ is a self-adjoint, negative semi-definite operator. Thus, the eigenvalues $\lambda$ of $A$ are all real and non-positive. In this case, they are $\lambda = n^2\pi^2$, $n \geq 0$. The invariant zeros depend on $b(x)$, but Theorem 4.1 implies that they are real and negative (since 0 is an eigenvalue of $A$, it cannot be a zero). Furthermore, for any $b(x)$ satifying (8), the zeros interlace with the eigenvalues. The eigenvalues of $A - kBB^*$ converge to the zeros. Thus, the stability of a controlled system is limited by the largest zero, which lies in the interval $[-\pi^2, 0]$.

*Example 5.2:* **(Delay Equation)** Eigenvalues of delay problems are poorly approximated by standard schemes - see for instance, [14]. Furthermore, little is known about zeros or high gain behaviour. Consider a simple delay equation,

$$\begin{aligned} \dot{x}(t) &= ax(t) - x(t-1) + u(t), \\ y(t) &= x(t). \end{aligned}$$

A state-space realization of the form (1) exists on $\mathcal{Z} = \mathbb{C} \times L_2(-1,0)$ with

$$A \begin{bmatrix} r \\ f(\cdot) \end{bmatrix} = \begin{bmatrix} a - f(\cdot(-1) \\ f'(\cdot) \end{bmatrix},$$

$$D(A) = \{(r,f) \in \mathcal{Z};\ f \in H^1(-1,0), f(0) = r\},$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The eigenvalues are given by the roots of

$$\kappa(s) = s - a + e^{-s}. \tag{9}$$

The invariant zeros are the values of $s$ for which there exist a non-trivial solution $(r, f(\cdot) \in D(A)$ to the following:

$$\begin{aligned} sr - ar + f(\cdot - 1) + 1 &= 0, \\ sf(\cdot) - f'(\cdot) &= 0, r = 0. \end{aligned}$$

The only solution to this system of equations is the trivial solution and so there are no invariant zeros. Since

$$\operatorname{rank} \begin{bmatrix} \kappa(s); 1 \end{bmatrix} = 1,$$

the system is approximately controllable [9, Thm. 4.2.10] and since

$$\operatorname{rank} \begin{bmatrix} \kappa(s) \\ 1 \end{bmatrix} = 1,$$

the system is approximately observable [9, Thm. 4.2.6]. Since the systems is approximately controllable and observable, these same conclusions can be found by examining the transfer function

$$G(s) = \frac{1}{\kappa(s)}.$$

The eigenvalues of $A$ form a sequence with $\operatorname{Re}\lambda \to \infty$ as $|\lambda| \to \infty$ and in fact

$$|\lambda| l \leq |a| + e^{\operatorname{Re}\lambda}.$$

[15, Prop. 1.8, Prop. 10]. The eigenvalues of $A - kBB^*$ are the roots of

$$s - a + k + e^{-s} \tag{10}$$

and so they have a similar pattern, for each $k$, as the eigenvalues of $A$.

*Theorem 5.3:* [14, Thm. 6.1] Consider the equation

$$\delta(s) = k_p k_c e^{-s} + 1 + Ts$$

where $T > 0$, $k_p > 0$. All roots of this equation will have negative real parts if

$$\frac{-1}{k_p} < k_c \frac{T}{k_p} \sqrt{z_1 + \frac{1}{T^2}}$$

where $z_1 \in (\frac{\pi}{2}, \pi)$ solves

$$\tan(z) = -Tz.$$

Rewriting the characteristic equation (10) in the above form, we obtain that all the eigenvalues of $A + kBB^*$ are stable for every $k > a$. Since there are no zeros, all branches of the root locus move from the eigenvalues of $A$ to $-\infty$ in the left-half-plane.

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