2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

State Estimators for a class of nonlinear Systems

Richard Stadlmayr, Andreas Siuka and Harald Daxberger

***Abstract*— This contribution is dedicated to the state observer design for a certain class of nonlinear dynamic systems. Moreover this approach is intended as an extension to many known controller design methods, where almost all state variables are necessary for the evaluation of the control law, but only a part of the state vector can be measured. The immeasurable parts of state variables have to be estimated for the implementation. In this paper we depart from a given control law, which leads to a (uniformly) asymptotically stable closed loop system. A dynamic extension of the controller by means of an observer provides an estimation for the immeasurable states, but the observer does not compromise the stability of the overall system such that the combination of the nonlinear controller and the state observer is also an asymptotically stable system. During the observer design a linear inhomogeneous set of partial differential equations (pde) have to be solved and we state conditions for the solvability of the pde's, which can be checked in advance in order to get an information, if the pde's are solvable. The observer design procedure is presented for the unstable mechanical benchmark example inertia wheel pendulum and the permanent magnet synchronous drive.**

I. INTRODUCTION

This contribution focuses on the state observer design for a class of nonlinear control systems. In addition to the stabilizing feedback control law the measurement of the required states are important for the implementation. Throughout this paper we assume that a static feedback control law is known for the application and the closed loop dynamics is uniformly asymptotically stable. Let us consider a general nonlinear system of the form

$$\dot{x} = f(t,x,u), \qquad y = h(t,x), \qquad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$, $n \in \mathbb{N}^+$ denotes the state of the dynamic system, $u \in \mathbb{R}^m$, $m \in \mathbb{N}^+$ represents the external control inputs and $y \in \mathbb{R}^m$ is the considered output and the initial condition x_0. A state feedback controller $u(t,x)$ is designed for (1) such that the closed loop dynamics

$$\dot{x} = f_{cl}(t,x) \qquad \lim_{t\to\infty} x(t) = 0 \tag{2}$$

is uniformly asymptotically stable, see [1] or [2].

The authors gratefully acknowledge for the support by the Austrian Center of Competence in Mechatronics (ACCM) and the Linz Center of Mechatronics (LCM).

R. Stadlmayr Linz, Linz Center of Mechatronics GmbH (LCM) and Austrian Center of Competence in Mechatronics (ACCM), Austria. richard.stadlmayr@lcm.at

A. Siuka, Institute of Automatic Control and Control Systems Technology, Johannes Kepler University Linz, Austria. andreas.siuka@jku.at

H. Daxberger, Institute of Automatic Control and Control Systems Technology, Johannes Kepler University Linz, Austrian Center of Competence in Mechatronics (ACCM), Austria. harald.daxberger@jku.at

If a measurement of the entire state is not possible for the evaluation of the control law $u(t,x)$, or the achievable accuracy of the measured variables is not sufficient, then two ways out are thinkable. On the one hand one can suppress selected state variables during the controller design, see [3] or one adds a state estimator. In the special case that (1) is a linear control system the observer design problem is solved, see [4]. In the general case different approaches are currently under research [5]. One finds many systematical design methods based on special normal forms [6] or the internal model approach [7]. Algebraic methods [8], flatness-based methods, also in combination with automatic differentiation [9] can be found in the literature. Furthermore observer design methods based on a Lagrangian approach [10] and many more ideas are already considered in the literature. In this paper we consider the state estimator design as add-on extension for the implementation of control laws. Since the controller design and the stability have been studied in advance it is of strong interest to combine the observer dynamics with the asymptotically stable control system without compromising the stability of the extended system. Roughly speaking some kind of separation should be achieved for the nonlinear case.

The paper is organized as follows. After the introduction we present the idea of the observer design for a certain class of nonlinear systems and we highlight a way to achieve some kind of separation for this problem. Section III deals with the analysis of the arising pdes and the formulation of some solvability conditions for the observer design based on involutive vector fields. Section IV shows the calculations of the state estimator design for two examples together with some simulation results. Finally the paper ends with a short conclusion.

II. STATE OBSERVER DESIGN

The control system (1) with the state vector $x^T = \begin{bmatrix} \eta^T & \mu^T \end{bmatrix} \in \mathbb{R}^n$ can be subdivided in the dynamics of the measurable states $\mu \in \mathbb{R}^{n_\mu}$ and the states $\eta \in \mathbb{R}^{n_\eta}$, which are not accessible via measurement. Clearly the dimension of μ and η coincide with the dimension of x such that $n_\eta + n_\mu = n$ holds and we identify the full state vector x as the so called sensor coordinates. In this contribution we study nonlinear systems of the form

$$\begin{bmatrix} \dot{\eta} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} A_1(t,\mu) \\ A_2(t,\mu) \end{bmatrix} \eta + \begin{bmatrix} f_1(t,\mu,u) \\ f_2(t,\mu,u) \end{bmatrix}, \tag{3}$$

where the matrices $A_1 : \mathbb{R}^{n_\eta} \to \mathbb{R}^{n_\eta}$ and $A_2 : \mathbb{R}^{n_\eta} \to \mathbb{R}^{n_\mu}$ may depend on the measurable states μ as well as the time t. Roughly speaking the estimated variables η of the control system (3) should appear only affine or in combination with the measurable states and the time, otherwise the approach will not work this way. Several physical systems fit to the representation of (3) as we will show in section IV. In general any mechanical system with a constant inertia matrix can be written in this form and the estimation of the generalized velocities is possible under some additional properties. For instance the nonlinear benchmark examples *Ball on Wheel* or the *Inertia Wheel Pendulum* fit to (3). Moreover some important applications like the *hydraulic piston actuator* with uncertain velocity measurement and the *permanent magnet synchronous drive* are included too.

According to the fundamental idea of the *Luenberger* observer [4] the introduction of the estimator variables $\xi = \eta + K_{\mathcal{O}}(t,\mu)$ and a possibly nonlinear and time variant feedback $K_{\mathcal{O}}$ of the measured output μ is available to solve the problem. In order to get the dynamics of the observer one has to calculate the total time derivative for ξ given by

$$\dot{\xi} = \dot{\eta} + \frac{\partial K_{\mathcal{O}}(t,\mu)}{\partial \mu}\dot{\mu} + \frac{\partial K_{\mathcal{O}}(t,\mu)}{\partial t}. \tag{4}$$

For a compact notation we introduce the symbol $\partial_\mu K_{\mathcal{O}}$ for the Jacobian of the vector-valued function $K_{\mathcal{O}}$ with respect to the state variables μ and $\partial_t K_{\mathcal{O}}$ as the partial derivative with respect to the time t. If one replaces the derivatives of the states $\dot{\mu}$ and $\dot{\eta}$, then one ends up with

$$\begin{aligned}\dot{\xi} = &\left(A_1(t,\mu) + \partial_\mu K_{\mathcal{O}}(t,\mu) A_2(t,\mu)\right)\eta + f_1(t,\mu,u)\\ &\partial_\mu K_{\mathcal{O}}(t,\mu) f_2(t,\mu,u) + \partial_t K_{\mathcal{O}}(t,\mu).\end{aligned} \tag{5}$$

In the sequel the dependencies $A_1(t,\mu) \to A_1$ and so on are suppressed, whenever no confusion is possible. If one adds the vanishing term $\left(A_1 + \partial_\mu K_{\mathcal{O}} A_2\right)(K_{\mathcal{O}} - K_{\mathcal{O}}) = 0$ and modifies (5) in the way

$$\begin{aligned}\dot{\xi} = &\left(A_1 + \partial_\mu K_{\mathcal{O}} A_2\right)(\eta + K_{\mathcal{O}}) + f_1 + \partial_\mu K_{\mathcal{O}} f_2\\ &+\partial_t K_{\mathcal{O}} - \left(A_1 + \partial_\mu K_{\mathcal{O}} A_2\right) K_{\mathcal{O}}\end{aligned} \tag{6}$$

then one gets the dynamic system for the state estimator

$$\begin{aligned}\dot{\xi} = &\left(A_1(t,\mu) + \partial_\mu K_{\mathcal{O}}(t,\mu) A_2(t,\mu)\right)\xi + f_1(t,\mu,u)\\ &- \left(A_1(t,\mu) + \partial_\mu K_{\mathcal{O}}(t,\mu) A_2(t,\mu)\right) K_{\mathcal{O}}(t,\mu)\\ &+\partial_\mu K_{\mathcal{O}}(t,\mu) f_2(t,\mu,u) + \partial_t K_{\mathcal{O}}(t,\mu)\end{aligned} \tag{7}$$

together with the output $\hat{\eta} = \xi - K_{\mathcal{O}}(t,\mu)$ for the estimated state variables. It is easy to see that a measurement of the partial state μ and the knowledge of the input u is sufficient.

The system states $\eta \to \hat{\eta}$ are replaced by the estimated variables in the (static) control law $u(t,\mu,\hat{\eta})$ and the dynamic extension leads to the closed loop system

$$\begin{bmatrix}\dot{\eta}\\ \dot{\mu}\\ \dot{\xi}\end{bmatrix} = \begin{bmatrix} A_1(t,\mu)\eta + f_1(t,\mu,u(t,\mu,\hat{\eta}))\\ A_2(t,\mu)\eta + f_2(t,\mu,u(t,\mu,\hat{\eta}))\\ f_{\mathcal{O}}(t,\xi,\mu,u(t,\mu,\hat{\eta}))\end{bmatrix}. \tag{8}$$

The abbreviation $f_{\mathcal{O}}$ is just a short form for the right hand side of (7). The introduction of the new coordinates

$$\bar{\mu} = \mu \qquad \bar{\eta} = \eta \qquad e_{\mathcal{O}} = \underbrace{\eta - \xi + K_{\mathcal{O}}(t,\mu)}_{\eta - \hat{\eta}}, \tag{9}$$

- including the estimation error $e_{\mathcal{O}}$ - allows an investigation of the convergence. It is worth mentioning that the relation $\partial_\mu K_{\mathcal{O}}(t,\mu) = \partial_{\bar{\mu}} K_{\mathcal{O}}(t,\bar{\mu})$ is met. The inverse of the transformation (9) is given by $\mu = \bar{\mu}$, $\eta = \bar{\eta}$ and $\xi = \bar{\eta} - e_{\mathcal{O}} + K_{\mathcal{O}}(t,\bar{\mu})$ and one ends up with the extended description for the transformed system

$$\begin{bmatrix}\dot{\bar{\eta}}\\ \dot{\bar{\mu}}\\ \dot{e}_{\mathcal{O}}\end{bmatrix} = \begin{bmatrix} A_1(t,\bar{\mu})\bar{\eta} + f_1(t,\bar{\mu},u(t,\bar{\mu},\bar{\eta} - e_{\mathcal{O}}))\\ A_2(t,\bar{\mu})\bar{\eta} + f_2(t,\bar{\mu},u(t,\mu,\bar{\eta} - e_{\mathcal{O}}))\\ \left(A_1(t,\bar{\mu}) + \partial_{\bar{\mu}} K_{\mathcal{O}}(t,\bar{\mu}) A_2(t,\bar{\mu})\right) e_{\mathcal{O}}\end{bmatrix}. \tag{10}$$

In this contribution we are interested in a converging observation error $e_{\mathcal{O}}$, and therefore it is useful to look for functions $K_{\mathcal{O}}(t,\bar{\mu})$ such that the solution of the pdes

$$A_1(t,\bar{\mu}) + \frac{\partial K_{\mathcal{O}}}{\partial \bar{\mu}}(t,\bar{\mu}) A_2(t,\bar{\mu}) = A_{\mathcal{O}}(t) \tag{11}$$

becomes independent of the system variables $\bar{\mu}$. If such a solution exists, then the time variant error system takes the form $\dot{e}_{\mathcal{O}} = A_{\mathcal{O}}(t) e_{\mathcal{O}}$. From now on we call the linear first order pdes (11) design pdes for the rest of the paper.

Lemma 1: According to [1] (and references therein) the origin of the cascade system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} f_1(t,x_1,x_2)\\ f_2(t,x_2)\end{bmatrix} \qquad x_1 \in R^{n_1}, x_2 \in R^{n_2} \tag{12}$$

is globally uniformly asymptotically stable, if f_1 and f_2 are piecewise continuous in t, locally Lipschitz in x_1, x_2 and $\dot{x}_1 = f_1(t,x_1,x_2)$ is input-to-state stable with x_2 as input as well as $\dot{x}_2 = f_2(t,x_2)$ is globally uniformly asymptotically stable.

This result can be used for the stability analysis of the closed loop system (10), where the identifications $x_1^T = \left[\eta^T, \mu^T\right]$ with $n_1 = n_\eta + n_\mu$ and $x_2 = e_{\mathcal{O}}$ with $n_2 = n_\eta$ are met.

Remark 1: In the time invariant case the closed loop system (10) takes the form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} f_1(x_1,x_2)\\ f_2(x_2)\end{bmatrix} \qquad x_1 \in R^{n_1}, x_2 \in R^{n_2}. \tag{13}$$

The origin of the cascade system is locally asymptotically stable if $\dot{x}_1 = f_1(x_1,0)$ and $\dot{x}_2 = f_2(x_2)$ are locally asymptotically stable and f_1, f_2 at least locally Lipschitz, see [11].

It is worth to mention that the examples presented in section IV are in fact time invariant examples. Nevertheless our approach can be applied to the more general time variant case provided that the conditions of the Lemma 1 are met.

A. Structure of the Design PDEs

The observer design for nonlinear systems has to consider two major points. On the one hand the pde's must have a solution for all the required feedback functions $K_{\mathcal{O}} = \left[k_1(t,\bar{\mu}), \ldots, k_{n_\eta}(t,\bar{\mu})\right]^T$ and on the other hand the dynamic matrix $A_{\mathcal{O}}(t)$ should guarantee the convergence of the estimation error. In many time invariant cases the direct choice of a Hurwitz matrix like $A_{\mathcal{O}} = \mathrm{diag}\{\alpha_i\}$, $i = 1,\ldots,n_\eta$ via its negative eigenvalues $\alpha_i < 0$ is possible, but in special cases a slightly more sophisticated approach is necessary as we will see later on.

If one takes a closer look the first order pdes (11) it becomes clear that the Jacobian $\partial_{\bar{\mu}} K_{\mathscr{O}}$ and the components of A_1, A_2 are smooth scalar functions depending on the measurable states μ and (in the general case) the time t. For an efficient notation we introduce $A_k^{i,j}$ as symbol for the (i,j)-component of the matrix $A_k(t,\bar{\mu})$, $k \in \{1,2,\mathscr{O}\}$.

Let us assume one has chosen a matrix $A_{\mathscr{O}}$, then the expanded design pdes are given in the form

$$\begin{bmatrix} \frac{\partial k_1}{\partial \bar{\mu}_1} & \cdots & \frac{\partial k_1}{\partial \bar{\mu}_{n_\mu}} \\ \vdots & \ddots & \vdots \\ \frac{\partial k_{n_\eta}}{\partial \bar{\mu}_1} & \cdots & \frac{\partial k_{n_\eta}}{\partial \bar{\mu}_{n_\eta}} \end{bmatrix} \begin{bmatrix} A_2^{1,1} & \cdots & A_2^{1,n_\eta} \\ \vdots & \ddots & \vdots \\ A_2^{n_\mu,1} & \cdots & A_2^{n_\mu,n_\eta} \end{bmatrix} = \begin{bmatrix} -A_1^{1,1}+A_{\mathscr{O}}^{1,1} & \cdots & -A_1^{1,n_\mu}+A_{\mathscr{O}}^{1,n_\mu} \\ \vdots & \ddots & \vdots \\ -A_1^{n_\eta,1}+A_{\mathscr{O}}^{n_\eta,1} & \cdots & -A_1^{n_\eta,n_\eta}+A_{\mathscr{O}}^{n_\eta,n_\eta} \end{bmatrix}. \tag{14}$$

For the sake of a compact and clear form we introduce the symbols $A_{1\mathscr{O}}^{i,j}$ for the right hand side of $(-A_1+A_{\mathscr{O}})$. It can be easily verified that (14) is equivalent to the set of pdes

$$\left.\begin{array}{c} \sum\limits_{i=1}^{n_\mu} A_2^{i,1}(t,\bar{\mu}) \frac{\partial k_j}{\partial \bar{\mu}_i} = A_{1\mathscr{O}}^{j,1}(t,\bar{\mu}) \\ \vdots \\ \sum\limits_{i=1}^{n_\mu} A_2^{n_\eta,i}(t,\bar{\mu}) \frac{\partial k_j}{\partial \bar{\mu}_i} = A_{1\mathscr{O}}^{j,n_\eta}(t,\bar{\mu}) \end{array}\right\} \begin{array}{l} \text{for all } k_j(t,\bar{\mu}) \\ j=1,\ldots,n_\eta \end{array} \tag{15}$$

for every feedback function $k_j(t,\bar{\mu})$. The pdes for k_1 to k_{n_η} are independent of each other and it is possible to solve the set of pdes one after an other. It is also of interest to study the question, if the n_η-systems of pdes (15) are solvable. This is the content of the next section.

III. Solvability of first order PDEs

Consider the set of n_η linear inhomogeneous first order pdes (15). Each set can be formally written in the form

$$A(t,x)\left(\partial_x V(t,x)\right)^T = b(t,x)\,. \tag{16}$$

The objective of this section is to state formal conditions, if a pde system of the form (16) is solvable. These conditions can be checked in advance. The solvability analysis is mainly based on the involutivity property of vector fields and we will carry out the calculations for the commonly used variables (x,t) of a control system and the unknown solution $V(t,x)$. An extension of the state vector $x_{ext}^T = [t,x]$ allows us to treat x and t as independent variables, but the variable t has to fulfill some special limitations. Obviously the autonomous dynamics $\dot{t}=1$ has to be considered for a physical control system. Due to the extension of the state vector the result of the time invariant version given in [12] needs to be extended for the general time variant case (16).

During the analysis we treat the rows (with components $a_{i,j}$) of $A(t,x)$ as vector fields of the extended system state $[t,x]$, where only the partial derivatives with respect to x appear. According to the row dimension of A one finds r vector fields and one has to analyze the system

$$\sum_{i=1}^{n_\mu} a_{1i}\frac{\partial V}{\partial x_i} = b_1,\ \ldots\ ,\sum_{i=1}^{n_\mu} a_{ri}\frac{\partial V}{\partial x_i} = b_r\,, \tag{17}$$

where $a_{i,j} = a_{i,j}(t,x)$ and $b_i = b_i(t,x)$ have to be considered. First we analyze the case where $A(x,t)$ has locally full rank and a multiplication with its inverse leads to the explicit form

$$\left(\partial_x V\right)^T = \left(A(x,t)\right)^{-1} b(x,t) = \tilde{b}(x,t) \quad . \tag{18}$$

The pde (18) is solvable, if and only if the Jacobian $\partial_x \tilde{b}$ of the right hand side satisfies $\partial_x \tilde{b} = \left(\partial_x \tilde{b}\right)^T$. Provided that the Jacobian $\partial_x \tilde{b}$ is symmetric, then $\tilde{b}$ corresponds to a differential of a potential function and the use of the *Poincaré*-Lemma guarantees the existence of a continuous function V, which is a solution for (18). In the general case of a singular matrix $A(t,x)$ the analysis is a bit more sophisticated. Let us assume that the annihilator N_2 solves the equation $N_2 A = 0$ and the multiplication with a full rank matrix $N^T = [N_1, N_2]$ leads to the reduced pdes $NA\left(\partial_x V\right)^T = Nb$ which read as

$$\begin{bmatrix} N_1 A \\ 0 \end{bmatrix}\left(\partial_x V\right)^T = \begin{bmatrix} \tilde{A} \\ 0 \end{bmatrix}\left(\partial_x V\right)^T = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} b\,. \tag{19}$$

Provided that $\mathrm{span}\{N_2\}$ has its maximal dimension, then (19) is only solvable, if $\mathrm{span}\{NA\} = \mathrm{span}\{Nb\}$ holds and we conclude that $N_2 b = 0$ must be fulfilled. Otherwise no solution can be found. The reduced pdes

$$\begin{bmatrix} \tilde{A} \\ 0 \end{bmatrix}\left(\partial_x V\right)^T = \begin{bmatrix} N_1 b \\ 0 \end{bmatrix} = \tilde{b} \tag{20}$$

have the same structure as (16), but the rows of $\tilde{A}$ are linearly independent. Now, the solvability of (20) can be checked in the following way. We consider a solution of the form $\tilde{V}(\tau,t,x) = \tau + V(t,x)$, where $\partial_x \tilde{V} = \partial_x V$ and $\partial_\tau \tilde{V} = 1$ is satisfied. If we deal with a compact vector field notation $v_i(V) = \tilde{b}_i$ for (20) together with $\partial_\tau \tilde{V} = 1$, then the inhomogeneous pdes can be written in the form

$$\begin{bmatrix} \left(v_i - \tilde{b}_i \partial_\tau\right)\left(\tilde{V}\right) \\ \partial_\tau \tilde{V} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad i = 1,\ldots,\tilde{r}\,. \tag{21}$$

We have introduced $v_i = \sum_{j=1}^{\tilde{r}} a_{i,j}\frac{\partial}{\partial x_j}$, $\tilde{r}$ as the number of the independent columns $\tilde{A}$ of (20) and $\left(\partial_\tau \tilde{V}\right)\tilde{b} = \tilde{b}$. In the sequel we use the abbreviation $a_{i,j}\frac{\partial}{\partial x_j} = \partial_j$ and the involutivity of the vector fields $\tilde{v}_i = \left(v_i - \tilde{b}_i \partial_\tau\right)$ can be checked by calculating the Lie brackets

$$\left[v_i - \tilde{b}_i\partial_\tau, v_j - \tilde{b}_j\partial_\tau\right] = \left[v_i, v_j\right] - \left[v_i, \tilde{b}_j\partial_\tau\right] - \left[\tilde{b}_i\partial_\tau, v_j\right]\,, \tag{22}$$

Obviously $\left[\tilde{b}_i\partial_\tau, \tilde{b}_j\partial_\tau\right] = 0$ vanishes identically. According to [14] the involutivity of the vector fields given by (21) provides necessary and sufficient conditions for the existence of a flat distribution Δ_0. The involutivity of (22) guarantees that one finds a transformation $\breve{w}_i = \sum_{j=1}^{\tilde{r}} \breve{a}_{i,j}(x,t)\, v_j$ as well as a so-called flat distribution $\Delta_0 = \mathrm{span}\{\partial_{\breve{x}_1}, \ldots, \partial_{\breve{x}_r}\}$ for the brackets $\Delta = [\tilde{v}_i, \tilde{v}_j]$ such that (21) becomes

$$\begin{bmatrix} \partial_{\breve{x}_1} \breve{V}(\breve{x}_1, \ldots, \breve{x}_{\tilde{r}}, \breve{x}_{\tilde{r}+1}, \ldots, \breve{x}_n, \breve{t}, \tau) \\ \vdots \\ \partial_{\breve{x}_r} \breve{V}(\breve{x}_1, \ldots, \breve{x}_{\tilde{r}}, \breve{x}_{\tilde{r}+1}, \ldots, \breve{x}_n, \breve{t}, \tau) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} . \quad (23)$$
$$\begin{bmatrix} \partial_{\tau} \breve{V}(\breve{x}_1, \ldots, \breve{x}_{\tilde{r}}, \breve{x}_{\tilde{r}+1}, \ldots, \breve{x}_n, \breve{t}, \tau) \end{bmatrix} \qquad \begin{bmatrix} 1 \end{bmatrix}$$

Please note that the constant right hand side of $\partial_\tau \tilde{V} = 1$ of (23) does not affect the bracket operation and the components of $\tilde{v}_i$ do not depend on added variable τ. Consequently the brackets $[\tilde{v}_i, \partial_\tau] \equiv 0$ vanish identically for any $i = 1, \ldots, \tilde{r}$ and the transformation $\breve{w}_i$ does not affect the field in ∂_τ-direction. One can easily convince oneself that the transformed problem (23) has a solution of the form

$$\breve{V} = \tau + \breve{F}(\breve{x}_{\tilde{r}+1}, \ldots, \breve{x}_n, \breve{t}) \quad , \quad (24)$$

where $\breve{F}$ is an arbitrary function which contains only the variables $\breve{x}_{\tilde{r}+1}, \ldots, \breve{x}_n$ and $\breve{t}$. It turns out that the added condition $\partial_\tau \breve{V} = 1$ is also satisfied for the coordinates $(\breve{x}_i, \breve{t})$.

In concluding it may be said that if the brackets $\left[v_i - \tilde{b}_i \partial_\tau, v_j - \tilde{b}_j \partial_\tau\right]$ are involutive and the involutive closure $\tilde{\Delta}$ has the dimension $\dim(\tilde{\Delta}) = n - \tilde{r}$, then the set of pdes (21) is solvable. Especially the existence of a flat distribution Δ_0 follows from *Frobenius'* theorem [14] and it is sufficient to check the involutivity of the extended vector fields $v_i - \tilde{b}_i \partial_\tau$. Please note that the solvability analysis can be done by computer algebra. One gets the information if the pdes have a solution, but the analysis is not constructive to solve the design pdes for concrete applications.

IV. Observer Design for selected Applications

This section is devoted to the observer design for two nonlinear time invariant applications, namely a mechanical example and an electric AC drive. In detail the calculations are shown for the inertia wheel pendulum (IWP) and the permanent magnet synchronous drive (PSM). We skip the controller design and we assume that a control law $u(x)$ is given for each application such that the closed loop dynamics

$$\dot{x} = \begin{bmatrix} \dot{\eta} & \dot{\mu} \end{bmatrix}^T = f(\eta, \mu, u(\mu, \eta)) \quad (25)$$

is (locally) asymptotically stable. Due to the technical restrictions the system variables η are not available as measurement and one has to replace η by the estimated values $\hat{\eta}$ in the control law. Some simulation results are included for an intensive study of the modified control law $u(\mu, \hat{\eta})$ and estimation error $e = \eta - \hat{\eta}$. Here the models, the parameters and the control laws are taken from the literature and we focus only on the design procedure for the observer.

A. The Inertia Wheel Pendulum (IWP)

The inertia wheel pendulum, see Fig. 1 is also known as reaction wheel pendulum [13]. It is a 2-dof mechanical example, which has been investigated with many different control design methods and the IWP has become a benchmark for the control of underactuated systems. The IWP is just a version of the well-known inverted pendulum, which uses a gyroscopic actuation.

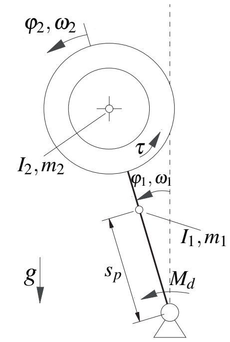

Fig. 1. Inertia Wheel Pendulum (IWP)

We consider a mathematical model

$$M\ddot{q}_e + G(q_e) + D(\dot{q}_e) = M_e \qquad q_e^T = \begin{bmatrix} \varphi_1 & \varphi_2 \end{bmatrix}, \quad (26)$$

with a constant inertia matrix $M \in \mathbb{R}^{2\times 2}$, $G(q_e)$ resulting from the gravity, $D(\dot{q}_e)$ for the dissipative terms and the actuator torque M_e. In detail we consider

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \quad G^T(q_e) = \begin{bmatrix} -m_r \sin(\varphi_1) \\ 0 \end{bmatrix}, \quad (27)$$

together with $D(\dot{q}_e) = \mathrm{diag}\{d_1, d_2\}\dot{q}_e$ for the viscous dissipation and the shortcut $\det(M) = m_{11}m_{22} - m_{12}^2$. The second order model of the IWP is rewritten as a system of first order ode's with the state vector $x^T = \begin{bmatrix} \varphi_1 & \varphi_2 & \omega_1 & \omega_2 \end{bmatrix}$ and the explicit form of (26)

$$\begin{aligned} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{m_{22}d_1}{\det(M)} & \frac{m_{12}d_2}{\det(M)} \\ \frac{m_{12}d_1}{\det(M)} & -\frac{m_{11}d_2}{\det(M)} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} \frac{m_{22}m_r \sin(\varphi_1) - m_{12}u}{\det(M)} \\ \frac{-m_{12}m_r \sin(\varphi_1) + m_{11}u}{\det(M)} \end{bmatrix} \\ \begin{bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \quad (28)$$

fits to (3) with $\eta^T = \begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix}$ and $\mu^T = \begin{bmatrix} \varphi_1 & \varphi_2 \end{bmatrix}$. The angles φ_1 and φ_2 are the measured quantities for the IWP and the angular velocities ω_1 and ω_2 have to be estimated for the control law. For simplicity we choose a Hurwitz matrix $A_{\mathscr{O}} = \mathrm{diag}\{\alpha_1, \alpha_2\}$ as the dynamic matrix of the observer with $\alpha_1, \alpha_2 \in \mathbb{R}^-$. The estimator states are introduced as $\xi_1 = \omega_1 - k_1(\varphi_1, \varphi_2)$ respectively $\xi_2 = \omega_2 - k_2(\varphi_1, \varphi_2)$ and according to (11) the design pdes

$$\underbrace{\begin{bmatrix} \frac{\partial k_1}{\partial \varphi_1} & \frac{\partial k_1}{\partial \varphi_2} \\ \frac{\partial k_2}{\partial \varphi_1} & \frac{\partial k_2}{\partial \varphi_2} \end{bmatrix}}_{(\partial_\mu K_{\mathscr{O}})A_2} = \underbrace{\begin{bmatrix} \frac{m_{22}d_1}{\det(M)} + \alpha_1 & -\frac{m_{12}d_2}{\det(M)} \\ -\frac{m_{12}d_1}{\det(M)} & \frac{m_{11}d_2}{\det(M)} + \alpha_2 \end{bmatrix}}_{A_{\mathscr{O}} - A_1} \quad (29)$$

have a constant right hand side. Due to $A_2 = A_2^{-1} = I$ and the results of section III the Jacobian of the constant right hand vector is always symmetric $\partial_\mu \tilde{b} = \left(\partial_\mu \tilde{b}\right)^T = 0$ and the pdes (29) have a solution. The observer feedback functions

$$\begin{aligned} k_1(\varphi_1,\varphi_2) &= \left(\tfrac{m_{22}d_1}{\det(M)}+\alpha_1\right)\varphi_1 - \tfrac{m_{12}d_2}{\det(M)}\varphi_2 \\ k_2(\varphi_1,\varphi_2) &= -\tfrac{m_{12}d_1}{\det(M)}\varphi_1 + \left(\tfrac{m_{11}d_2}{\det(M)}+\alpha_2\right)\varphi_2 \end{aligned} \tag{30}$$

lead to a time invariant matrix $A_{\mathscr{O}}$ and the error dynamics becomes exponentially stable.

If one drops the dynamics for the angle of the spinning disc $\dot{\varphi}_2=\omega_2$, then the IWP model (28) can be reduced to a 3^{rd}-order system, because the angle φ_2 of the rotating disc does not appear on the right hand side. Now the observer design procedure is also studied with the single measurement of the angle φ_1. Due to the restricted measurement the dynamics of the measurable quantities $\dot{\mu}=A_2\eta+f_2$ is reduced to $\dot{\varphi}_1=\omega_1$ resp. $A_2=\begin{bmatrix}1 & 0\end{bmatrix}$ and $f_2=[0]$. Furthermore the feedback functions k_1 and k_2 can only depend on φ_1 and one ends up with the design pdes

$$\begin{bmatrix} \partial_{\varphi_1}k_1(\varphi_1) & 0 \\ \partial_{\varphi_1}k_2(\varphi_1) & 0 \end{bmatrix} = \begin{bmatrix} A_{\mathscr{O}}^{1,1}+\frac{m_{22}d_1}{\det(M)} & A_{\mathscr{O}}^{1,2}-\frac{m_{12}d_2}{\det(M)} \\ A_{\mathscr{O}}^{2,1}-\frac{m_{12}d_1}{\det(M)} & A_{\mathscr{O}}^{2,2}+\frac{m_{11}d_2}{\det(M)} \end{bmatrix}. \tag{31}$$

Based on the results of section III a similar choice $A_{\mathscr{O}}=\operatorname{diag}\{\alpha_1,\ \alpha_2\}$ leads to pdes, which have no solution. The conflict is apparent for $A_{\mathscr{O}}^{1,2}=0$, because no function $k_1(\varphi_1)$ solves the equation $0=-m_{12}d_2\det(M)^{-1}$ for positive real parameters d_i, $m_{ij}>0$, $i,j\in\{1,2\}$. Despite these restrictions it is possible to choose all the eigenvalues of $A_{\mathscr{O}}$ arbitrary as the following calculation shows. The selection $A_{\mathscr{O}}^{1,2}=m_{12}d_2\det(M)^{-1}$ and $A_{\mathscr{O}}^{2,2}=-m_{11}d_2\det(M)^{-1}$ together with two real constants C_1 and C_2

$$A_{\mathscr{O}}^{1,1}=C_1\det(M)^{-1} \qquad A_{\mathscr{O}}^{2,1}=C_2\det(M)^{-1} \tag{32}$$

guarantee the solvability of (31). The output feedback

$$k_1(\varphi_1)=\frac{C_1+m_{22}d_1}{\det(M)}\varphi_1 \qquad k_2(\varphi_1)=\frac{C_2-m_{12}d_1}{\det(M)}\varphi_1 \tag{33}$$

implies a time invariant system for the estimation error and the calculation of the free constants C_i is strongly linked to the characteristic polynomial $p(A_{\mathscr{O}})=\det(\lambda I-A_{\mathscr{O}})$ of $A_{\mathscr{O}}$. According to the fundamental idea of the pole placement procedure the coefficients of desired characteristic polynomial $p_{des}(A_{\mathscr{O}})=\prod_i(\lambda-\alpha_i)$ have to solve the linear problem

$$\frac{(d_2m_{11}-C_1)}{\det(M)}=\alpha_1+\alpha_2 \qquad \frac{-d_2(m_{12}C_2+C_1m_{11})}{\det(M)}=\alpha_1\alpha_2 \tag{34}$$

in order to get the desired eigenvalues $\{\alpha_1,\alpha_2\}$. Note that the idea is equivalent for a pair of complex eigenvalues.

A simulation result for the controlled IWP and the behavior of the nonlinear observer are included here. Fig. 2 shows the regulated output φ_1 of the stabilized IWP. One can see that the initial error e_1 is eliminated by the tracking controller and an external disturbance F_d is included to study the disturbance behavior too. The parameters of the IWP are taken from a lab setup and we use a combination of feedforward and feedback control as given in [3]. Fig. 3 shows the behavior of the observer for the special choice of the eigenvalues $\alpha_i\in\{-3,-2.25\}$.

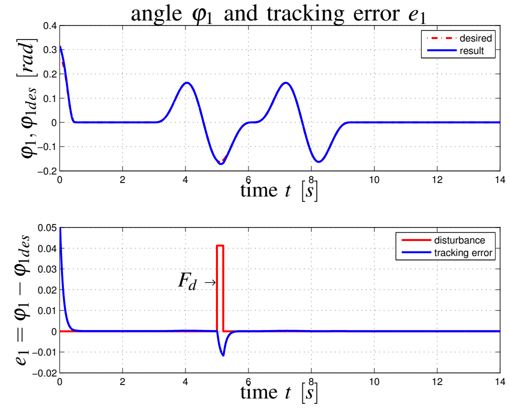

Fig. 2. Results for the regulated output φ_1

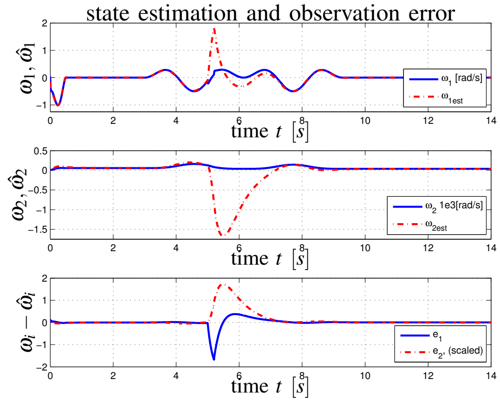

Fig. 3. Estimated states $\hat{\omega}_1$, $\hat{\omega}_2$ and the estimation error $e=\eta-\hat{\eta}$

Obviously the estimation errors $e_i=\omega_i-\hat{\omega}_i$, $i=1,2$ decrease and the closed loop behavior including a stabilizing controller and the estimator is (locally) asymptotically stable.

B. A permanent Magnet Synchronous Drive

A permanent magnetic synchronous motor, or PSM for short, is a 3 phase AC drive with a permanent magnetic material instead of an electrical field winding for the generation of the excitation flux. Synchronous AC drives are widely used in industrial applications especially for moderate power. The mathematical model is more or less a benchmark example in nonlinear MIMO control and numerous papers can be found in the literature. By means of a the well established *Park*-transformation the 3 phase system is transformed into an orthogonal 2-axis system (d,q)

$$\begin{bmatrix} \dot{i}_d \\ \dot{i}_q \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L_d}i_d+n_p\omega\frac{L_qi_q}{L_d}+\frac{1}{L_d}v_d \\ -\frac{R}{L_q}i_q-n_p\omega\left(\frac{L_di_d}{L_q}+\frac{\Psi_0}{L_q}\right)+\frac{1}{L_q}v_q \\ \frac{3n_p}{2J_m}\tau_{el}-\frac{d_m}{J_m}\omega-\frac{1}{J_m}\tau_l \end{bmatrix} \tag{35}$$

with the nonlinear drive torque $\tau_{el}=(\Psi_0+(L_d-L_q)i_d)i_q$. The third order model (35) has to 2 independent voltage inputs $u_{dq}^T=[v_d,v_q]$ and a unknown load τ_l. The constants $(R,\ L_d,\ L_q,\ n_p,\ d_m,\ J_m\in\mathbb{R}^+)$ are positive real numbers and the parameters have a clear physical meaning. The transformed currents i_d and i_q are available as state measurements $\mu^T = \begin{bmatrix} i_d & i_q \end{bmatrix}$, whereas the angular velocity ω and the load torque τ_l are not measured. For this contribution we assume that the load τ_l is almost constant or changes only slowly, which means that $\dot{\tau}_l \approx 0$ is approximately fulfilled. The autonomous dynamics of τ_l is added to the PSM model (35) and the estimated state becomes $\eta^T = [\omega, \tau_l]$.

According to section II the system can be written in the favoured form of (3) with the matrices A_1 and A_2

$$A_1 = \begin{bmatrix} -\frac{d_m}{J_m} & -\frac{1}{J_m} \\ 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} \frac{n_p L_q i_q}{L_d} & 0 \\ -\frac{n_p(L_d i_d + \Psi_0)}{L_q} & 0 \end{bmatrix}. \quad (36)$$

Obviously the matrices depend only on the measured states. It is worth mentioning that the straight forward approach $A_{\mathscr{O}} = \mathrm{diag}\{\alpha_1, \alpha_2\}$ leads to a pde system, which has no solution. In detail one gets the pdes

$$\begin{aligned} \frac{n_p}{L_d} L_q i_q \frac{\partial k_1(i_d,i_q)}{\partial i_d} - \frac{n_p}{L_q}(\Psi_0 + L_d i_d)\frac{\partial k_1(i_d,i_q)}{\partial i_q} &= \frac{d_m + A_{\mathscr{O}}^{1,1}}{J_m} \\ \frac{n_p}{L_d} L_q i_q \frac{\partial k_2(i_d,i_q)}{\partial i_d} - \frac{n_p}{L_q}(\Psi_0 + L_d i_d)\frac{\partial k_2(i_d,i_q)}{\partial i_q} &= \frac{A_{\mathscr{O}}^{2,1}}{J_m} \end{aligned} \quad (37)$$

and the algebraic restrictions $A_{\mathscr{O}}^{1,2} + J_m^{-1} = A_{\mathscr{O}}^{2,2} = 0$ due to the zero column of A_2 resulting from the autonomous dynamics of the load $\dot{\tau}_l = 0$. Apart from the special choice $A_{\mathscr{O}}^{1,2} = -J_m^{-1}$ and $A_{\mathscr{O}}^{2,2} = 0$ one ends up with a contradiction. There are 2 free parameters left for the pole placement approach and the selection $A_{\mathscr{O}}^{1,1} = J_m^{-1} C_1$ and $A_{\mathscr{O}}^{2,1} = J_m^{-1} C_2$ leads to the characteristic polynomial $p(A_{\mathscr{O}}) = \lambda^2 - C_1 J_m^{-1}\lambda + C_2 \left(J_m^{-1}\right)^2$. It is possible to solve the algebraic problem

$$p(A_{\mathscr{O}}) = \det(\lambda I - A_{\mathscr{O}}) = \prod_i (\lambda - \alpha_i) \qquad i = 1,2 \quad (38)$$

for two arbitrary negative eigenvalues $\alpha_i \in \mathbb{R} < 0$ respectively a complex pair of eigenvalues $-\alpha \pm i\beta$.

A short analysis of the solvability guarantees that the pdes (37) can be solved, because a single vector field is always involutive. A solution of (37) can be calculated with a computer algebra package. The nonlinear feedback function $K_{\mathscr{O}}^T = \begin{bmatrix} k_1(i_d, i_q) & k_2(i_d, i_q) \end{bmatrix}$

$$K_{\mathscr{O}} = \begin{bmatrix} \frac{d_m + C_1}{n_p J_m} \arctan\left(\frac{L_d i_d + \Psi_0}{L_q i_q}\right) + \mathscr{F}(\cdot) + \mathscr{C}_{10} \\ \frac{C_2}{n_p J_m} \arctan\left(\frac{L_d i_d + \Psi_0}{L_q i_q}\right) + \mathscr{F}(\cdot) + \mathscr{C}_{20} \end{bmatrix} \quad (39)$$

lead to a linear autonomous dynamics for the estimation error. The arbitrary function $\mathscr{F}\left(L_d^2 i_d^2 + 2 L_d i_d \Psi_0 + L_q^2 i_q^2\right)$ together with $\mathscr{C}_{10} = \Psi_0^2$ respectively $\mathscr{C}_{20} = \mathscr{C}_{10}$ has a nice physical interpretation. Obviously $\mathscr{F}(\cdot) + \mathscr{C}_{10}$ corresponds to the square of the actual flux of the PSM

$$\Psi_{dq}^T \Psi_{dq} = \Psi_d^2 + \Psi_q^2 = \left(L_d^2 i_d^2 + \Psi_0\right)^2 + \left(L_q i_q\right)^2, \quad (40)$$

which represents a dynamic invariant of the observer system. The dynamic extension for the estimation of the angular rotor speed $\hat{\omega}$ and the load torque $\hat{\tau}_l$ provides some kind of separation and one may interpret the dynamic controller as sensor-less control concept for the speed control of a PSM. The observer system for the angular speed and the load is designed as add-on and one can replace $\omega \to \hat{\omega}$ and $\tau_l \to \hat{\tau}_l$ without compromising the stability of the closed loop system. The investigation for noisy current signals and measurements from a real setup are actually in preparation.

V. Conclusion

The presented observer design approach is intended for a certain class of nonlinear systems. Roughly speaking the procedure is devoted to control systems, where the estimated variables appear only affine or in combination with the measurable variables. Based on these preliminaries one can design a nonlinear observer with an autonomous, possibly time variant error dynamics. The immeasurable states are replaced by the estimated ones $\eta \to \hat{\eta}$ in the control law without compromising the stability. In case of a (uniformly) asymptotically stable closed loop system and a linear time invariant error dynamics together with a couple of Lipschitz conditions one gets some kind of separation principle known from the linear case. Clearly the approach is dedicated to a special class of systems, but one gets an answer in which case this approach leads to the desired result. The arising pde for the observer design can be analyzed in advance, before high efforts are undertaken to solve the pdes. The check of the involutivity of some vector fields is necessary and sufficient and it can be done by computer algebra. Finally the observer design has been shown for two benchmark examples together with simulation results for the closed loop behavior.

VI. ACKNOWLEDGMENTS

The authors gratefully acknowledge for the support of the Austrian Center of Competence in Mechatronics (ACCM) and the Linz Center of Mechatronics (LCM).

References

[1] H. Khalil, *Nonlinear Systems*, New Jersey, US: Prentice Hall, 2002.

[2] S. Sastry, *Nonlinear Systems*, New York, US: Springer Verlag, 2001.

[3] R. Stadlmayr, *On a Combination of Feedforward and Feedback Control for Mechatronic Systems*, Germany, Shaker Verlag, 2009.

[4] D. Luenberger, *Introduction to Dynamic Systems*. New Jersey, United States: John Wiley & Sons Inc., 1979.

[5] A. Astolfi, D. Karagiannis and R. Ortega *Nonlinear and Adaptive Control with Applications*. London, GB: Springer Verlag, 2008.

[6] A. Isidori, *Nonlinear Control Systems, 3rd ed.* London, Great Britain: Springer Verlag, 1995.

[7] A. Isidori, L. Marconi, and A. Serrani, *Robust Autonomous Guidance: An Internal Model Approach*, Great Britain: Springer Verlag, 2003.

[8] M. Fliess, C. Join and H. Sira-Ramirez, "Non-linear estimation is easy", *Int. Journal Modelling Identification and Control*, vol. 4, no. 1, pp. 12–27, 2008.

[9] J. Winkler, S. O. Lindert, K. Röbenack and J. Rudolph "Design of a nonlinear observer using automatic differentiation" *Proc. in Appl. Mathem. and Mechanics*, vol. 4, pp. 147–148, 2004.

[10] N. Aghannan and P. Rouchon, "An Intrinsic Observer for a Class of Lagrangian Systems," *IEEE Trans. on Automatic Control*, vol. 48, no. 6, pp. 936–945, 2003.

[11] A. Isidori, *Nonlinear Control Systems II*, London, Great Britain: Springer Verlag, 1999.

[12] D. Cheng, A. Astolfi, and R. Ortega, "On Feedback Equivalence to Port Controlled Hamiltonian Systems," *Systems & Control Letters*, vol. 54, pp. 911–917, 2005.

[13] M. Spong, P. Corke, and R. Lozano, "Nonlinear Control of the Reaction Wheel Pendulum," *Automatica*, vol. 37, no. 11, pp. 1845–1851, 2001.

[14] H. Nijmeijer and A. van der Schaft, *Nonlinear Dynamical Control Systems*, New York, United States: Springer Verlag, 1990.