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Passive Network Design for Stochastic Vibratory Energy Harvesters

J.T. Scruggs and Q. Li

***Abstract*— This paper considers the use of optimal control theory to design circuits for small-scale, single-transducer vibration energy harvesting applications, in which the external disturbance is a broadband stochastic process. Specifically, we investigate the use of a lossless passive two-port network terminated by a single-directional DC/DC converter, to impose transducer voltage feedback laws on energy harvesting systems. Such an implementation requires external power only to gate one MOSFET in a PWM cycle, and requires no active feedback. The optimization of harvested energy reduces to the optimal design of the input admittance $Y(s)$ of the terminated network, which reduces to a positive-real-constrained, sign-indefinite $\mathcal{H}_2$ optimal control problem. This class of optimization is nonconvex, and a numerically-efficient means of finding its global minimum remains an open problem. Here we introduce conservatism into the problem in such a way as to make the optimization practical, albeit still nonconvex, and we illustrate its solution in the context of a base-excited piezoelectric bimorph cantilever.**

***Index Terms*— Energy harvesting, Stochastic vibration, Passive networks, LMIs, SPR-constrained $\mathcal{H}_2$ optimal control**

I. Introduction

Over the last decade, significant research activity has focused on energy scavenging technology to harvest power from ambient vibration, as a power source for wireless intelligence systems embedded in smart structures [1]. Fig.1a shows a conceptual diagram of the transduction mechanism for such technology, consisting of a passive electromechanical system (with an embedded transducer), transducer terminal with current $i(t)$ and voltage $v(t)$, and an acceleration input $a(t)$. (In the more general case, multiple transducers may be considered [2], but here we restrict our attention to single-transducer systems.) Of the several modes of transduction available for milliwatt-scale applications, piezoelectric approaches have received the most attention. Fig.1b shows the so-called "bimorph" configuration, in which piezoelectric patches are bonded to the upper and lower surfaces of a resonant flexible beam.

Although there are some applications for which satisfactory performance can be achieved by instantaneously delivering the power generated by the harvester directly to an electrical load, in many cases the energy generated by the transducers must be stored for some period of time before it is used. This is the case, for example, where the harvester is used to power a system which operates only intermittently. In such circumstances, power generated by the transducer must be used to continuously charge a storage system, such as a supercapacitor or battery, which is then periodically drained to facilitate infrequent bursts of comparatively high-power activity. The simplest way to accomplish this is to interface the transducer terminals directly with the storage device through a passive diode bridge rectifier [3], as shown in Fig. 2a. However, such techniques have slow recharge times, and also require that the open circuit voltage of $v(t)$ be greater than the bus voltage V_S. Performance can be enhanced using power electronics. Some of the first investigations into the design of such systems were reported in [4]. That work considered the connection of a PWM-controlled DC/DC buck converter to the bridge rectifier, as shown in Fig. 2b. In a follow-up paper [5], the same authors showed that if that converter is operated in discontinuous conduction mode with a constant duty cycle, its input resistance is insensitive to dynamics in its input and output voltages, and is primarily a function of its duty cycle. Subsequent investigations by Lefeuvre et al [6] and Kong et al [7] recognized that if the buck converter is replaced with a buck-boost topology (shown as the converter in Fig. 2d but without the insertion of the lossless two-port), the input resistance theoretically has zero sensitivity to dynamics in the input and output voltages. This enables the effective resistance of the converter to be designed for maximal power absorption, and then for the duty cycle to be found from this resistance afterward. So designed, the converter duty cycle is constant, thus eliminating the need for feedback.

In [2], [8], Scruggs investigated the substitution of a single-directional DC/DC converter for a fully-active H-bridge converter capable of two-way power flow, as illustrated in Fig. 2c. This system can be used to effect explicit feedback control on transducer current. In those studies, it was shown that for $a(t)$ modeled as a broadband stochastic process, the derivation of the optimal effective admittance (i.e., feedback law) $Y(s)$ relating $v(t)$ to $i(t)$ can be found as the solution to a related sign-indefinite $\mathcal{H}_2$ optimal control problem. Those studies illustrate that the optimal $Y(s)$ is not positive-real, and thus cannot be made equivalent to any passive circuit.

In the theory developed in [2], [8], conductive losses in the semiconductor electronics are taken into account in the design process, resulting in a feedback system which maximizes the power transferral from the base excitation to the power bus. In [9], this theory is extended to also account for the gating losses incurred by the MOSFETs in the electronics, which can be of great relevance at small scale or small excitation levels. However, there are two issues which the above theories do not resolve:

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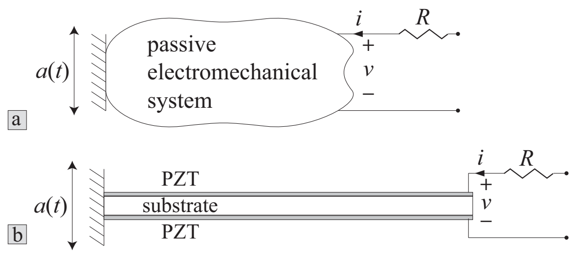

Fig. 1. Generic (a) and PZT bimorph (b) harvesters

a) The gating power necessary to switch the four MOSFETs associated with a fully-active H-bridge may exceed that being extracted by the harvester, resulting in a net power loss for the total system, and
b) Even if the gating power consumption is acceptable, the power consumption necessary merely to compute the optimal voltage feedback law will in general require the use of either analog active circuitry or a DSP. Either way, the power consumption associated with the control intelligence may exceed the power absorbed.

As such, there always exist power and excitation scales, below which active feedback designs will cease to be practical.

The present paper is an investigation of one way to cut down on these "ancillary" power losses, by using a single-directional buck-boost converter as discussed in [6], with a duty cycle $D << 1$ which causes the converter to operate in discontinuous conduction. For frequencies well below the switching frequency, the effective input impedance is approximately resistive, and has the proporationality $R_D \propto D^{-2}$. However, the present paper extends these prior works in two ways. First, we introduce a lossless two-port circuit (comprised of ideal capacitors, inductors, and transformers) between the transducer and the converter, as shown in Fig. 2d. Through proper design of this two-port network, the motivation is to recover as much of the performance of active control as possible, subject to the constraints of what the effectively-passive network can do. Second, we consider the design in a stochastic context, whereas all prior work regarding the operation of these converters was for the case of monochromatic excitation.

II. The stochastic energy harvesting problem

A. Dynamic modeling

Consider the general energy harvester in Fig.1. We assume its electromechanical dynamics can be approximated as linear and finite-dimensional, with state space

$$\dot{\mathbf{x}}_h(t) = \mathbf{A}_h \mathbf{x}_h(t) + \mathbf{B}_h i(t) + \mathbf{G}_h a(t) \quad (1a)$$
$$v(t) = \mathbf{B}_h^T \mathbf{x}_h(t) \quad (1b)$$

We assume that the harvester is a passive system, which implies that there always exists a realization such that $\mathbf{B}_h$ has dual participation in both equations above [10]. More specifically, we assume that the driving point impedance of the harvester, as seen from its electrical terminals, can be realized by an asymptotically-stable equivalent circuit consisting of ideal capacitors, inductors, and resistors. This

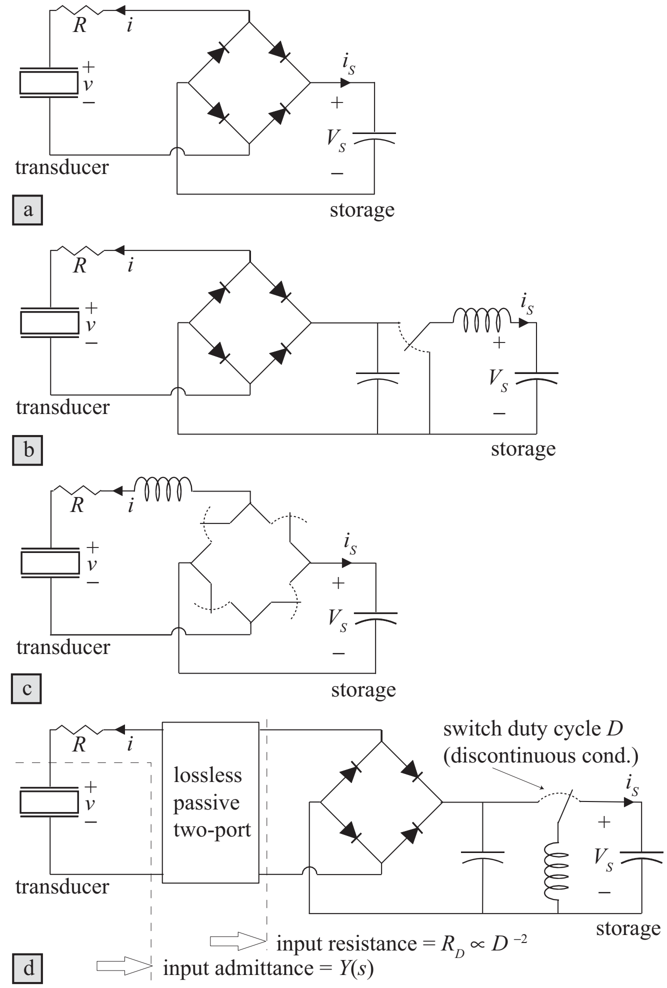

Fig. 2. Various energy harvesting circuit topologies

is equivalent to stating that this impedance has the Weakly-Strict Positive-Real (WSPR) property, which implies that there exists a realization for the above state space for which there exists a matrix $\mathbf{C}$ with full row rank, such that $\mathbf{A}_h + \mathbf{A}_h^T = -\mathbf{C}^T\mathbf{C}$, and for which $(\mathbf{A}_h, \mathbf{C})$ constitutes an observable pair [11]. The power extracted from the harvester, by the electronics, is just $P_e(t) = -i(t)v(t)$.

As an example, consider the bimorph piezoelectric beam shown in Fig.1. For this example, the model was taken directly from [12]. Using standard Rayleigh-Ritz techniques to arrive at a finite-dimensional beam model, and imposing classical mechanical damping, the state space can be partitioned as

$$\mathbf{x}_h = \begin{bmatrix} q_1 & \dot{q}_1 & \cdots & q_N & \dot{q}_N & p \end{bmatrix}^T \quad (2)$$

where $\{q_k, \dot{q}_k\}$ are generalized mechanical position and velocity coordinates of vibratory mode k, and where p is normalized piezo voltage. With appropriate normalizations, a realization exists for which

$$\mathbf{A}_h = \begin{bmatrix} \mathbf{\Omega} & \mathbf{\Theta} \\ -\mathbf{\Theta}^T & -1/\tau \end{bmatrix} \quad \mathbf{B}_h = \begin{bmatrix} \mathbf{0} \\ \beta \end{bmatrix} \quad \mathbf{G}_h = \begin{bmatrix} \mathbf{N} \\ 0 \end{bmatrix} \quad (3)$$

and where the further partitionings are made in modal form;

TABLE I

EXAMPLE HARVESTER PARAMETERS

ω_1	241rad/s	ω_2	1510rad/s	ω_3	4220rad/s
η_1	$-0.0820\sqrt{\text{kg}}$	η_2	$-0.0454\sqrt{\text{kg}}$	η_3	$-0.0267\sqrt{\text{kg}}$
θ_1	65.8s^{-1}	θ_2	-228s^{-1}	θ_3	375s^{-1}
ζ_1	0.010	ζ_2	0.0435	ζ_3	0.121
τ	2s	β	$1770\sqrt{\Omega/\text{s}}$	R	10Ω
ω_a	249rad/s	ζ_a	1	σ_a	9.81m/s^2

i.e.,:

$$\mathbf{\Omega} = \operatorname*{diag}_{k=1..N} \left\{ \begin{bmatrix} 0 & \omega_k \\ -\omega_k & -2\zeta_k\omega_k \end{bmatrix} \right\} \tag{4}$$

$$\mathbf{\Theta} = \operatorname*{col}_{k=1..N} \left\{ \begin{bmatrix} 0 \\ \theta_k \end{bmatrix} \right\} \quad , \quad \mathbf{N} = \operatorname*{col}_{k=1..N} \left\{ \begin{bmatrix} 0 \\ \eta_k \end{bmatrix} \right\} \tag{5}$$

For the numerical example considered in this paper, the parameters $\{\omega_k, \zeta_k, \theta_k, \eta_k, \beta, \tau\}$ are given in Table I. These correspond to the transducer studied in [12] with the exception of τ, which was assumed to be infinite in that study and is given a finite value here to reflect finite dielectric leakage of the transducer.

We assume the disturbance a is modeled as white noise, sent through a finite-dimensional strictly-proper, minimum-phase filter; i.e.,

$$\dot{\mathbf{x}}_a(t) = \mathbf{A}_a\mathbf{x}_a(t) + \mathbf{B}_a w(t) \tag{6a}$$

$$a(t) = \mathbf{C}_a\mathbf{x}_a(t) \tag{6b}$$

where w is white noise with spectral intensity of 1. For the purpose of example, we further assume this filter to be second-order, with the matrices

$$\mathbf{A}_a = \begin{bmatrix} 0 & \omega_a \\ -\omega_a & -2\zeta_a\omega_a \end{bmatrix} \qquad \mathbf{B}_a = \mathbf{C}_a^T = \begin{bmatrix} 0 \\ b \end{bmatrix} \tag{7}$$

where w_a and ζ_a determine the passband of the noise, and parameter b is chosen such that a has a given standard deviation σ_a, which may be found as $b = \left(4\zeta_a\omega_a\sigma_a^2\right)^{1/4}$.

Augmenting the system and disturbance dynamics as $\mathbf{x} = \begin{bmatrix}\mathbf{x}_h^T & \mathbf{x}_a^T\end{bmatrix}^T \in \mathbb{R}^n$, the system dynamics are then

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}i(t) + \mathbf{G}w(t) \tag{8a}$$

$$v(t) = \mathbf{B}^T\mathbf{x}(t) \tag{8b}$$

with $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{G}$ appropriately defined.

B. Energy harvesting as a control objective

Our objective is to design a feedback law $Y : -v \to i$ as the synthetic linear admittance which optimizes the expectation on generated power. As such, our control objective is to maximize

$$\bar{P}_{gen} = \mathcal{E}\left\{-iv - P_d\right\} \tag{9}$$

where P_d is the transmission dissipation in the electrical network. This dissipation function is typically quite complicated, and will depend on the hardware used to realize $Y(s)$, as well as the manner in which this hardware is operated (e.g., its switching frequency, bus voltage, MOSFET gating voltage, and so forth). However for our purposes we will make the simplifying assumption that P_d is in fact modeled as a simple resistive loss associated with the current extracted from the transducer; i.e., $P_d(t) = i^2(t)R$. This assumption is made primarily because it yields the most straight-forward analysis which still accounts for transmission dissipation in some way. Use of more complicated models for P_d can be viewed as augmenting the theory discussed here.

With these assumptions, we have that an equivalent problem is to find the $Y(s)$ which minimizes

$$J = -\bar{P}_{gen} = \frac{1}{2}\mathcal{E}\left\{ \begin{bmatrix} \mathbf{x} \\ i \end{bmatrix}^T \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^T & 2R \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ i \end{bmatrix} \right\} \tag{10}$$

which is a nonstandard (i.e., sign-indefinite) $\mathcal{H}_2$ problem. In general, this problem would not be well-posed (i.e., J would have no minimum) but it turns out that if the harvester is WSPR then J has a unique, finite, and negative minimum. The following theorem is proved in [8].

Theorem 1: Let the system in (1) be WSPR, and let the augmented system $\mathbf{x} \in \mathbb{R}^n$ be as in (8). Then for any causal, stabilizing mapping from v to i, we have that

$$\bar{P}_{gen} = -\tfrac{1}{2}\mathbf{G}^T\mathbf{P}\mathbf{G} - R\,\mathcal{E}\left\{(i - \mathbf{K}\mathbf{x})^2\right\} \tag{11}$$

where $\mathbf{P}$ is the solution to the nonstandard Riccati equation

$$\mathbf{0} = \mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \tfrac{1}{2R}\left(\mathbf{P} + \mathbf{I}\right)\mathbf{B}\mathbf{B}^T\left(\mathbf{P} + \mathbf{I}\right) \tag{12}$$

and

$$\mathbf{K} = -\tfrac{1}{2R}\mathbf{B}^T\left(\mathbf{P} + \mathbf{I}\right) \tag{13}$$

Furthermore, $\mathbf{P} < 0$ and $\mathbf{A} + \mathbf{B}\mathbf{K}$ is Hurwitz.

The above theorem leads directly to the following corollary, the proof of which is standard.

Corollary 1: For the feedback law $\hat{i}(s) = -Y(s)\hat{v}(s)$ with realization

$$Y(s) \sim \left[\begin{array}{c|c} \mathbf{A}_Y & \mathbf{B}_Y \\ \hline \mathbf{C}_Y & D_Y \end{array}\right] \tag{14}$$

of order $n_Y = n$, $\bar{P}_{gen} > -\frac{1}{2}\mathbf{G}^T\mathbf{P}\mathbf{G} - R\gamma$ where

$$\gamma = \mathcal{G}^T\mathcal{S}\mathcal{G} \tag{15}$$

$$0 > \mathcal{C}^T\mathcal{C} + \mathcal{S}\mathcal{A} + \mathcal{A}^T\mathcal{S} \quad , \quad \mathcal{S} = \mathcal{S}^T > 0 \tag{16}$$

$$\mathcal{C} = \begin{bmatrix} (\mathbf{K} + D_Y\mathbf{B}^T + \mathbf{C}_Y)^T \\ (D_Y\mathbf{B}^T + \mathbf{K})^T \end{bmatrix}^T \quad , \quad \mathcal{G} = \begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix} \tag{17}$$

$$\mathcal{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} - \mathbf{B}D_Y\mathbf{B}^T & \mathbf{B}\mathbf{C}_Y \\ -\mathbf{B}_Y\mathbf{B}^T & \mathbf{A}_Y \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{18}$$

III. SPR-CONSTRAINED $\mathcal{H}_2$ DESIGN

Consider again the harvesting system in Fig. 2d. Because the two-port is lossless, optimization of the power it absorbs is equal to that delivered to the converter. We wish to confine the domain of $Y(s) : -v \to i$ to include only those transfer functions realizable with with this particular network.

Lemma 1: An internally asymptotically-stable, rational admittance $Y \in \mathcal{H}_\infty$ is realizable with a lossless two-port terminated by a resistance $R_D > 0$, in series with a resistance $R > 0$, if and only for any minimal state space realization as in (14), $\exists\mathbf{T} = \mathbf{T}^T > 0$ satisfying

$$\begin{bmatrix} \mathbf{A}_Y^T\mathbf{T} + \mathbf{T}\mathbf{A}_Y & & (sym) \\ \mathbf{B}_Y^T\mathbf{T} & -1 & \\ 2R\mathbf{C}_Y & (2RD_Y - 1) & -1 \end{bmatrix} \leqslant 0 \tag{19}$$

Proof: Let the input admittance at the terminals of the two-port network be $Y_2(s) = (Y(s)^{-1} - R)^{-1}$. In 1931, Darlington proved in [13] that $Y_2(s)$ is realizable with a lossless two-port, terminated by a resistance, if and only if it is positive-real (i.e., passive); i.e.,

$$\int_0^T (v(t) + Ri(t))\, i(t) dt \leqslant 0 \ , \ \forall v \in \mathcal{L}_2, \ T > 0 \quad (20)$$

For $R > 0$, this is quivalent to

$$\int_0^T (v(t) + 2Ri(t))^2 \, dt \leqslant \int_0^T v(t)^2 dt \ , \ \forall v \in \mathcal{L}_2, \ T > 0 \quad (21)$$

which, for $Y : -v \to i$ and $Y \in \mathcal{H}_\infty$, is equivalent to $\|1 - 2RY\|_{\mathcal{H}_\infty} \leqslant 1$. Via the Kalman-Yakubovic-Popov lemma [14], this is equivalent to (19) if $Y(s)$ is finite-dimensional and internally asymptotically stable. ∎

We thus arrive at what we will call Design Problem 1:

DP1 :	Problem data:	$\mathbf{A}$, $\mathbf{B}$, $\mathbf{G}$, R, $n_Y = n$
	Minimization:	$\gamma =$ (15)
	Variables:	$\mathbf{A}_Y$, $\mathbf{B}_Y$, $\mathbf{C}_Y$, D_Y, $\mathbf{T}$, $\mathcal{S}$
	Constraints:	(16), (19), $\mathcal{S} > 0$, $\mathbf{T} > 0$

Constraint (19) is more conservative than a positive-real (PR) constraint, and its imposition in DP1 is only superficially different from the imposition of a PR constraint. Moreover, strengthening the inequality to be strict in (19) makes it more conservative than a strictly positive-real (SPR) constraint, and in this paper, we will henceforth assume this.

Standard SPR-constrained $\mathcal{H}_2$ problems have been investigated by numerous researchers over the years. In 1988, Lozano-Leal and Joshi showed that under certain conditions on problem data, unconstrained $\mathcal{H}_2$ controllers turn out to be SPR implicitly [10]. Their result was subsequently used by Kishimoto *et al* to design $\mathcal{H}_2$ controllers which are guaranteed to be SPR [15], [16], as well as by Haddad *et al* in closely related problems involving the design of $\mathcal{H}_2$ controllers with an associated $\mathcal{H}_\infty$ bound [17]. In 1997, Geromel and Gapski [18] found a way of conservatively framing the SPR-constrained $\mathcal{H}_2$ problem as a (convex) LMI problem, and showed that their approach recovered the earlier results in [10] as a special (nonconservative) case. Shimomura and Pullen [19] proposed a related approach to the optimization, which is less conservative than the approach in [18], but which is also nonconvex and must be approached by iterative convex over-bounding BMI solution methods [20], [21]. In an apparently separate research thread, MacMartin and Hall [22] discovered a novel way of imposing a PR constraint on a feedback law, through the imposition of a related closed-loop $\mathcal{H}_\infty$ constraint. Although they did not explicitly pursue it in their paper, their constraint could be used in the context of SPR-constrained $\mathcal{H}_2$ optimal control. All the aforementioned approaches implicitly presume n_Y (usually the same as the order of the plant, with the exception of [22] for which it is twice that), and all successful constrained optimization approaches have also introduced conservatism in the optimization domain by assuming some form of controller/observer separation. Most recently, however, Damaren [23] approached the problem from a direct optimization approach, placing no restrictions on the controller except its order. For the SISO controller case, he then solved the problem for various controller orders, as a nonlinear optimization with linear inequality constraints. The optimal performance he obtained was very close to that of the (suboptimal) method in [19].

Our present analysis is closest to that of [19]; we introduce conservatism in the form of an assumed parametrized structure for $Y(s)$. This is done primarily to make the optimization more tractable. Because the design is conservative, the resultant $Y(s)$ will be sub-optimal over all PR admittances. However, we will require that its peformance exhibit an important bound which justifies the proposed approach. For single-transducer systems such as the one under consideration, it is straight-forward to design a static admittance $Y(s) = M$, and optimize M for maximum $\bar{P}_{gen}$. While nonconvex, the optimization of M can be done through a simple line search on $M \in \mathbb{R}^+$, with performance evaluated via

$$\mathbf{0} = \mathbf{A}_M \mathbf{S} + \mathbf{S}\mathbf{A}_M^T + \mathbf{G}\mathbf{G}^T \quad (22)$$

$$\bar{P}_{gen} = \mathbf{B}^T \mathbf{S} \mathbf{B} \left(M - RM^2 \right) \quad (23)$$

where $\mathbf{A}_M = \mathbf{A} - \mathbf{B}M\mathbf{B}^T$. Any dynamic $Y(s)$ design should be required to perform at least as well as the optimal static case.

We assume $Y(s)$ has the Luenberger observer structure

$$-Y(s) \sim \left[\begin{array}{c|c} \mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{L}\mathbf{B}^T & -\mathbf{L} \\ \hline \mathbf{K} + M\mathbf{B}^T & -M \end{array} \right] \quad (24)$$

where gain matrix $\mathbf{L}$ is a design parameter. For this system, it is straight-forward to verify that γ can be expressed as

$$\gamma = \mathbf{G}^T \mathbf{U}^{-1} \mathbf{G} \quad (25)$$

where $\mathbf{U} > 0$ obeys the Lyapunov inequality

$$\begin{bmatrix} \begin{pmatrix} \mathbf{U}\mathbf{A}_M^T + \mathbf{A}_M \mathbf{U} \\ +\mathbf{L}\mathbf{B}^T\mathbf{U} + \mathbf{U}\mathbf{B}\mathbf{L}^T \end{pmatrix} & (sym) \\ (\mathbf{K} + M\mathbf{B}^T)\mathbf{U} & -1 \end{bmatrix} < 0 \quad (26)$$

Also (19) becomes (with $\mathbf{V} = \mathbf{T}^{-1}$),

$$\begin{bmatrix} \begin{pmatrix} \mathbf{V}\mathbf{A}_K^T + \mathbf{A}_K \mathbf{V} \\ +\mathbf{L}\mathbf{B}^T\mathbf{V} + \mathbf{V}\mathbf{B}\mathbf{L}^T \end{pmatrix} & & (sym) \\ \mathbf{L}^T & -1 & \\ 2R(\mathbf{K} + M\mathbf{B}^T)\mathbf{V} & (2RM - 1) & -1 \end{bmatrix} < 0 \quad (27)$$

where $\mathbf{A}_K = \mathbf{A} + \mathbf{B}\mathbf{K}$, and where we have strengthened this to a strict inequality, as discussed.

Thus we arrive at what we will call Design Problem 2:

DP2 :	Problem data:	$\mathbf{A}$, $\mathbf{B}$, $\mathbf{G}$, R
	Minimization:	$\gamma =$ (25)
	Variables:	$\mathbf{L}$, $\mathbf{U}$, $\mathbf{V}$, M
	Constraints:	(26), (27), $\mathbf{U} > 0$, $\mathbf{V} > 0$

This problem is nonconvex, due to bilinear multiplicative terms involving $\mathbf{U}$, $\mathbf{V}$, $\mathbf{L}$ and M. There is no known way to recover convexity for DP2 without introducing additional

$$\begin{bmatrix} \mathbf{U}\mathbf{A}^T+\mathbf{A}\mathbf{U} & \mathbf{U}\mathbf{K}^T & (\mathbf{U}-\mathbf{U}_k)\mathbf{B} & \mathbf{L}-\mathbf{L}_k-\mathbf{B}(M-M_k) \\ & -1 & 0 & M-M_k \\ & & -w_1 & 0 \\ & (sym) & & -w_1^{-1} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_k\left(\begin{bmatrix}\mathbf{U}\mathbf{B}\\0\end{bmatrix},\begin{bmatrix}\mathbf{L}-\mathbf{B}M\\M\end{bmatrix}\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} < 0 \quad (28)$$

$$\begin{bmatrix} \mathbf{V}\mathbf{A}_K^T+\mathbf{A}_K\mathbf{V} & \mathbf{L} & \mathbf{V}\mathbf{K}^T 2R & \mathbf{L}-\mathbf{L}_k & (\mathbf{V}-\mathbf{V}_k)\mathbf{B} \\ & -1 & 2RM-1 & 0 & 0 \\ & & -1 & 2R(M-M_k) & 0 \\ & & & -w_2 & 0 \\ & (sym) & & & -w_2^{-1} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_k\left(\begin{bmatrix}\mathbf{L}\\0\\2RM\end{bmatrix},\begin{bmatrix}\mathbf{V}\mathbf{B}\\0\\0\end{bmatrix}\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} < 0 \quad (29)$$

conservatism, so some form of nonconvex optimization algorithm must be used. Our approach here is analogous to those in [21], [20], as well as [24]. Let $\{\mathbf{L}_k, \mathbf{U}_k, \mathbf{V}_k, M_k\}$ be any set of (suboptimal) design parameters which are feasible in DP2. Let γ_k be the corresponding performance as in (25). Furthermore, for two compatible matrices $\{\mathbf{E}, \mathbf{F}\}$, define the linearization function $\mathcal{B}_k(\mathbf{E}, \mathbf{F})$ as

$$\mathcal{B}_k(\mathbf{E}, \mathbf{F}) \triangleq \mathbf{E}\mathbf{F}_k^T + \mathbf{F}_k\mathbf{E}^T + \mathbf{E}_k\mathbf{F}^T + \mathbf{F}\mathbf{E}_k^T - \mathbf{E}_k\mathbf{F}_k^T - \mathbf{F}_k\mathbf{E}_k^T \quad (30)$$

Then we have the following theorem, which we will not prove here, but which follows along the same lines the results in the above-mentioned references.

Theorem 2: Let $\{w_1, w_2\} \subset \mathbb{R}^+$ be arbitrary weights, and let $\theta > 0$. Then any set of variables $\{\mathbf{L}, M, \mathbf{U}, \mathbf{V}\}$ satisfying $\mathbf{U} > 0$, $\mathbf{V} > 0$, along with (28), (29) and

$$\begin{bmatrix} \theta & \mathbf{G}^T \\ \mathbf{G} & \mathbf{U} \end{bmatrix} > 0 \quad (31)$$

is feasible and satisfies $\gamma < \theta$. Furthermore, there always exists a feasible set $\{\mathbf{L}, M, \mathbf{U}, \mathbf{V}, \theta\}$ such that $\theta < \gamma_k$, except at stationary points in the parameter domain.

The above theorem suggests an iterative approach to the solution of DP2. Prior to the execution of the algorithm, we first find $\mathbf{K}$. At iteration $k = 0$, we start by instantiating initial conditions $\mathbf{L} = \mathbf{L}_0$ and $M = M_0$ in (26) and (27), and finding $\mathbf{U}_0$ and $\mathbf{V}_0$ which minimize γ. (With $\mathbf{L}$ and M fixed, this becomes a convex optimization.) With these variables found, solve the following convex LMI sub-problem for iteration k:

DP2k :

	Problem data:	$\mathbf{A}$, $\mathbf{B}$, $\mathbf{G}$, R
		$\mathbf{L}_k$, M_k, $\mathbf{U}_k$, $\mathbf{V}_k$
	Minimization:	θ
	Variables:	$\mathbf{L}$, M, $\mathbf{U}$, $\mathbf{V}$, θ
	Constraints:	(28), (29), (31) $\mathbf{U} > 0$, $\mathbf{V} > 0$

Then, for the optimal solution variables $\{\mathbf{L}^o, M^o, \mathbf{U}^o, \mathbf{V}^o\}$, set $\mathbf{L}_{k+1} \leftarrow \mathbf{L}^o$, $M_{k+1} \leftarrow M^o$, $\mathbf{U}_{k+1} \leftarrow \mathbf{U}^o$, $\mathbf{V}_{k+1} \leftarrow \mathbf{V}^o$, and advance to iteration $k \leftarrow k+1$.

As such, the overall problem we solve here (DP2) is not guaranteed to converge to a global optimum, as it is a nonconvex algorithm. However on each iteration k, subproblem DP2k is convex, does not fix any of the optimization variables, and is guaranteed to yield an improved solution unless it is initiated at a stationary point. The weights $\{w_1, w_2\}$ can be chosen arbitrarily, but their choice can have a strong influence on the speed of the algorithm, especially in early stages. One useful technique is to redesign these weights upon each iteration, in addition to the other design variables. To do this, the inverses of the weights must be conservatively replaced with linearizations; i.e., $-w_i^{-1} < -2w_{ik}^{-1} + w_{ik}^{-2} w_i$. For the example in the next section, this is what is done.

IV. Numerical example

We now execute the above-described optimization for the model described in Sec. II, and the parameters in Table I. One of the difficulties of the optimization is choosing an initial condition. The obvious choice would be to set M_0 equal to the optimal static admittance, and set $\mathbf{L}_0 = \mathbf{0}$. Theoretically, this would be a feasible initial condition. Furthermore, since each subsequent design $\{\mathbf{L}_k, M_k\}$ is guaranteed to improve upon the performance of the previous design, the algorithm would immediately begin to improve upon the static admittance case. However, this approach has numerical problems, because performance is extremely insensitive to $\mathbf{L}$ when it is initiated at this value, and the algorithm tends to converge (falsely) to the static admittance case. Additionally, for low R it is often the case that the optimal dynamic admittance parameters are far from the static parameters, and can involve a much lower value of M. By starting the algorithm at the static admittance, it may converge to a local, less-favorable minimum. For this example, M_0 was chosen to be $1/10$ its optimal static value, and $\mathbf{L}_0$ was chosen by first designing it to be the Luenberger gain $\mathbf{L}_L$ for an observer with poles evenly placed in the range $(-100, -1000)$, and then obtaining $\mathbf{L}_0 = \alpha \mathbf{L}_L$, where $\alpha \in (0, 1)$ was chosen to be sufficiently small so as to make the resultant controller in (24) adhere to constraint (27). Although not as elegant

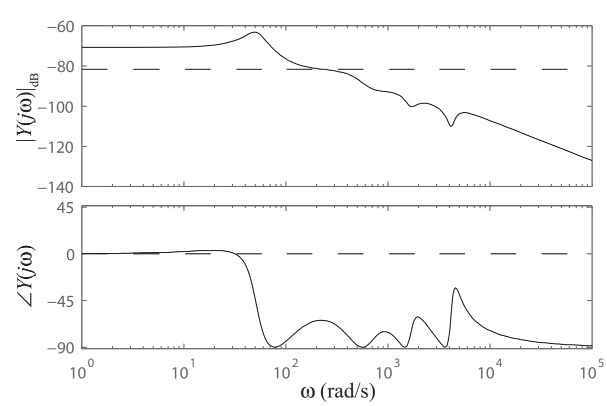

Fig. 3. Bode plots for optimal static (dashed) and dynamic (solid) $Y(s)$

theoretically, this approach appears to be more numerically well-conditioned at the outset of the optimization.

For the algorithm initiated as such, the frequency response for the optimized $Y(s)$ is shown in Fig. 3, together with the optimal static $Y(s) = M$, for reference. The optimized design parameters are $M = 4.112 \times 10^{-11}$ and

$$\mathbf{L} = \begin{bmatrix}0.0607 & 0.2155 & 0.0510 & -0.0025 & -0.0262 \\ ... -0.0012 & -0.3041 & 0.3736 & -6.3610\end{bmatrix}^T \quad (32)$$

The optimal static admittance is $M = 8.232 \times 10^{-5}$.

Power generation for various design cases are listed below:

Active state feedback:	:	1.440 mW
Static admittance:	:	0.8212 mW
Optimal $\{M, \mathbf{L}\}$:	:	1.162 mW

As such, through the introduction of the lossless two-port network, we can recover 24% of the performance of the active case, beyond what can be attained by connecting the harvesting converter directly to the transducer terminals.

V. Comments & Conclusions

The purpose of this paper has been to illustrate that, with appropriate adaptation, concepts from SPR-constrained $\mathcal{H}_2$ optimal control can be used to design the dynamics of passive networks for optimization of harvested energy from vibratory disturbances. The paper has proposed one possible algorithm for synthesizing the optimized feasible admittance of the passive network, using an LMI approach.

There are many avenues for extension of the ideas proposed in this paper. Most immediately, as was noted by Damaren in [23], the nature of the optimal controller for SPR-constraind $\mathcal{H}_2$ control, and related problems, remains an open issue. There is in general no proof that the optimal controller even approximately adheres to a separation principle, which was the assumption which gave rise to the controller structure in (24). Similar optimizations to those performed in this paper, but evaluated over a less-constrained domain, remain an item for future work.

In this paper, we have not discussed the actual *synthesis* of the passive two-port network, from the optimal $Y(s)$. The technique for accomplishing this, which is the classical Darlington synthesis, is a standard result. However, the synthesized network will in general have ideal transformers, which may be challenging to approximate in practice for broadband applications. In general there is no known way to avoid the use of these transformers, without further restriction of the domain for $Y(s)$, and without introduction of additional resistances into the passive network (beyond the single resistance representing the DC/DC converter). The optimization of $Y(s)$, subject to the constraint that it be realizable with a lossless two-port network as in Fig. 2d, but with the additional restriction that no ideal transformers be used in the realization, is a challenging, open problem.

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