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Attentively Efficient Controllers for Event-Triggered Feedback Systems

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***Abstract*— State dependent event-triggered systems sample the system state when the difference between the current state and the last sampled state exceeds a state-dependent threshold. These systems exhibit the *efficient attentiveness property* when the the length of the inter-sampling interval increases monotonically as the sampled state approaches the equilibrium. The efficient attentiveness property may partly explain why event-triggered systems sometimes exhibit inter-sampling intervals that are much longer than those found in comparably performing periodically sampled control systems. This paper establishes sufficient conditions under which an event-triggered system is attentively efficient. These conditions depend on the relative rates of growth in the class $\mathcal{K}$ functions used in dissipative characterizations of the input-to-state stability (ISS) property. Since these functions determine the type of controller used by the system, these results suggest that a suitable choice of controller has a greatly increase the inter-sampling intervals seen in event-triggered control systems. In other words, the design of attentively efficient event-triggers with sufficiently long sampling intervals may really be an issue of nonlinear controller design.**

I. Introduction

Event-triggered control systems are of great interest in the development of *networked control systems* [1]. State dependent event-triggered systems [2] are sampled-data systems that sample the system state when the difference between the current state and the last sampled state exceeds a state-dependent threshold. These systems exhibit the so-called *efficient attentiveness property* [3] where inter-sampling interval goes to infinity as the system state approaches the system's equilibrium. The efficient attentiveness property is of great interest because systems with this property tend to exhibit very long inter-sampling intervals when operated close to the equilibrium point. This property, therefore, may partly explain why event-triggered system sometimes exhibit inter-sampling intervals [4] that are much longer than the periods in comparably performing periodically sampled control systems. Event-triggered systems possessing the efficient attentiveness property, therefore, may be of great practical value in reducing the complexity of the communication infrastructure supporting networked control systems.

The scaling behavior of event-triggered inter-sampling intervals has attracted a great deal of attention. In [2] it was shown that these times could be bounded away from zero in a manner that prevented the occurrence of arbitrarily fast sampling frequencies (also known as Zeno sampling [5]). In [4] a lower bound on the inter-sampling interval was presented which was a function of the past sampled state; thereby suggesting that with the appropriate choice of event-triggering threshold and controller, one might obtain a system exhibiting the efficient attentiveness property [5]. Very precise bounds on the inter-sampling interval were developed in [3] for homogeneous systems without disturbances. These bounds could be scaled with respect to system state in a manner that exhibits the efficient attentiveness property. These prior results suggest that it may be possible to design event-triggered systems that have the efficient attentiveness property. Recent steps in this direction were taken in [6].

The design methods used in [6] represent a first step toward addressing the efficient attentiveness problem in event-triggered systems. That paper seeks controllers that maximize the inter-sampling interval subject to an event-triggering condition, where the inter-sampling interval is estimated using methods from [7]. The method, however, can be computationally intensive.

The approach adopted in this paper seeks an approach that simultaneously designs both the event-triggering rule and the controller so that the efficient attentiveness property is achieved. Unlike the methods in [6], we are less interested in maximizing the inter-sampling intervals, but are more concerned with finding the conditions on the event-trigger under which we can guarantee in a computationally efficient manner that the system possesses the efficient attentiveness property. In particular, this property makes the following contributions.

- We develop event-triggering rules that assure the input-to-state stability (ISS) of a nonlinear system;
- We establish sufficient conditions on the ISS dissipative inequalities that ensure the event-triggered system possesses the efficient attentiveness property;
- We use universal constructions for ISS controllers [8] to develop attentively efficient event-triggered controllers.

It also appears this approach can be used to assure integral input-to-state stability (*i*ISS) [9] of event-triggered systems.

II. Mathematical Preliminaries

Throughout this paper the linear space of real n-vectors will be denoted as $\mathbb{R}^n$ and the set of non-negative reals will be denoted as $\mathbb{R}^+$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ will be denoted as $|x|$. Consider the real-valued function $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$. $x(t)$ denotes the value x takes at time $t \in \mathbb{R}^+$. The $\mathcal{L}_\infty$ norm of this function is defined as $|x|_{\mathcal{L}_\infty} = \text{ess}\sup_{t>0} |x(t)|$, where $|x(t)|$ is the Euclidean norm of the vector $x(t) \in \mathbb{R}^n$. The function x will be said

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to be essentially bounded if $|x|_{\mathcal{L}_\infty} = M < \infty$ and the linear space of all essentially bounded real valued functions will be denoted as $\mathcal{L}_\infty$. A given real valued function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is positive definite if $V(x) > 0$ for all $x \neq 0$. The function will be said to be radially unbounded if $V(x) \to \infty$ as $|x| \to \infty$. The function V will be said to be *smooth* or $\mathcal{C}^\infty$ if all of its derivatives exist and are continuous.

A function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. If α is unbounded then the function is of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is class K for each fixed $t \geq 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$. Given a function $\alpha : \mathbb{R} \to \mathbb{R}$, we call $\alpha(r)$ an infinitesimal as $r \to 0$ if $\lim_{r\to 0} \alpha(r) = 0$; Given two infinitesimals α, β, we say $\beta(r)$ is an infinitesimal of higher order than $\alpha(r)$ if $\lim_{r\to 0} \frac{\beta(r)}{\alpha(r)} = 0$.

Consider a general system of the form

$$\dot{x} = f(x, w) \tag{1}$$

where f is locally Lipschitz and w is an essentially bounded input disturbance. The function $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ that satisfies this system equation is called the *system's state trajectory*.

This system is *input-to-state stable* (ISS) with respect to w if there exists $\gamma \in \mathcal{K}^\infty$ and $\beta \in \mathcal{KL}$ such that for any initial state $x(0)$ and every $w \in \mathcal{L}_\infty$, the system's resulting state trajectory satisfies the following inequality,

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(|w|_{\mathcal{L}_\infty}) \tag{2}$$

for all $t \in \mathbb{R}^+$.

The function $V(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ is called an *ISS-Lyapunov* function if it is positive definite, radially unbounded and smooth such that there exist class $\mathcal{K}_\infty$ functions α, α_1, α_2, and γ such that

$$\begin{aligned} \alpha_1(|x|) &\leq V(x) \leq \alpha_2(|x|) \\ \dot{V} &= \frac{\partial V}{\partial x} f(x, w) < -\alpha(|x|) + \gamma(|w|) \end{aligned} \tag{3}$$

for all $x \in \mathbb{R}^n$ and all $w \in \mathbb{R}^k$. The existence of an ISS-Lyapunov function, V, is necessary and sufficient for the system in equation (1) to be ISS. We will sometimes refer to the inequality in equation (3) as the ISS *dissipative inequality*.

III. Event-Triggered Systems

Let us consider a nonlinear system. The system state $x(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$ satisfies the following differential equation

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ x(0) &= x_0 \end{aligned} \tag{4}$$

where $f(\cdot) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^n$ is locally Lipschitz. The input signal $w(\cdot) : \mathbb{R}^+ \to W \subset \mathbb{R}^l$ is an essentially bounded signal such that $|w|_{\mathcal{L}_\infty} = \overline{w}$, a constant. The *control signal* $u(\cdot) : \mathbb{R}^+ \to \mathbb{R}^m$ is

$$u = k(\hat{x}) \tag{5}$$

where the controller function $k(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is continuous, and the *sampled state*, $\hat{x}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$, is piecewise constant. Given a continuous function $\eta : \mathbb{R}^n \to \mathbb{R}^p$, we define $\tilde{\eta}(t) = \eta(\hat{x}(t)) - \eta(x(t))$. Note that if we choose $\eta(\cdot)$ to be the identity function, then the local error is exactly the measurement error $e(t) = \hat{x}(t) - x(t)$; if $\eta(\cdot) \equiv k(\cdot)$, the error is the *control error* $u(t) - k(x(t))$.

Let us introduce a sequence of *sampling instants*,

$$\mathbb{T} = \{\tau_0, \tau_1, \cdots, \tau_i, \cdots\}$$

where $\tau_i \in \mathbb{R}^+$ and $\tau_i < \tau_{i+1}$ for all $i = 0, 1, 2, \ldots, \infty$. This means that the sampled state is $\hat{x}(t) = x(\tau_i)$ for all $t \in [\tau_i, \tau_{i+1})$ and all $i = 0, 1, 2, \ldots, \infty$. By the definition of $\tilde{\eta}$, one can see that the magnitude of the error $|\tilde{\eta}(\tau_i)| = 0$. For the system in equation (4), the sequence $\mathbb{T} = \{\tau_i\}_{i=0}^\infty$ is generated by an inductive method. Let $\tau_0 = 0$. The $i+1$st sampling instant τ_{i+1} is the first time instant after τ_i whenever

$$|\eta(x(\tau_i)) - \eta(x(t))| < \theta(|x(t)|) \tag{6}$$

is false, where $\theta(\cdot) : \mathbb{R}^+ \to \mathbb{R}$ is any continuous positive definite function. Mathematically, τ_{i+1} is defined by

$$\tau_{i+1} = \min_t \{t > \tau_i \mid |\eta(x(\tau_i)) - \eta(x(t))| \geq \theta(|x(t)|)\}.$$

The inequality in equation (6) is called an *event-trigger*. It represents a *state-dependent* threshold that forces the system in equation (4) to resample the system state whenever the error gets too large. The combination of equations (4) and (6) is called a state-dependent *event-triggered* system [2].

Consider the event-triggered system in equations (4) and (6) which generates the state trajectory x and measured state trajectory $\hat{x}$. Define $\hat{x}_i \in \mathbb{R}^n$ to be equal to the state measurement at time instant τ_i, i.e. $\hat{x}_i = \hat{x}(\tau_i) = x(\tau_i)$. Let $\mathbb{I} = \{(\hat{x}_i, \tau_i)\}_{i=0}^\infty$ be a sequence. The sequence, $\mathbb{I}$ will be called the system's *feedback information* since it represents the *information* transmitted over the control system's feedback channel.

Definition 3.1: Given a positive constant T, the event-triggered system is *attentively efficient* at the equilibrium $x = 0$, if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $|\hat{x}_i| \leq \delta$,

$$T_i = \tau_{i+1} - \tau_i > T - \epsilon. \tag{7}$$

Moreover if $T_i \to \infty$ as $\hat{x}_i \to 0$, then the event-triggered system is *strictly attentively efficient*.

To be attentively efficient requires that the *inter-sampling interval* $T_i = \tau_{i+1} - \tau_i$ is bounded below by a desired lower bound T as $\hat{x}_i$ (the sampled system state) approaches the system's equilibrium point at the origin. In other words, as the system settles into its equilibrium, the frequency with which information is transmitted over the feedback channel becomes smaller. When the event-triggered system is strictly attentively efficient, it means that T_i goes to infinity as $\hat{x}_i \to 0$. This notion of efficient attentiveness control was introduced in [6]. A good example of attentively efficient event-triggered systems will be found in the homogeneous event-triggered systems found in [3].

The main problem considered in this paper concerns the design of the event-triggering function θ in equation (6) and the state feedback controller $k(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ such that the event-triggered system in equations (4) and (6) is input-to-state stable and attentively efficient.

IV. Lower Bounds on Inter-sampling Interval

Let us consider the system in equation (4) with the controller in equation (5). We can rewrite it as

$$\dot{x} = f(x, k(\hat{x}), w) \tag{8}$$

and $|w|_{\mathcal{L}_\infty} = \bar{w}$.

Assume that there exist a smooth positive definite function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$, a continuous, locally Lipschitz function $\eta(\cdot) : \mathbb{R}^n \to \mathbb{R}^p$, $\chi : \mathbb{R}^+ \to \mathbb{R}^+$, and functions $\alpha_1, \alpha_2, \alpha, \gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that

$$\begin{aligned} &\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \\ &\frac{\partial V}{\partial x} f(x, k(\hat{x}), w) \le -\alpha(|x|) + \chi(|x|)\gamma_1(|\tilde{\eta}|) + \gamma_2(|w|) \end{aligned} \tag{9}$$

where $\tilde{\eta} = \eta(\hat{x}) - \eta(x)$. Note that when $\chi(|x|)$ is a constant, V is an ISS control Lyapunov function (ISS-CLF) for the system with respect to the error $\tilde{\eta}$ and the disturbance w.

Remark 4.1: If $\alpha(|x|)$ is replace by $\alpha(x)$ that is positive definite and $\chi, \gamma_1, \gamma_2 \in \mathcal{K}$ but not $\mathcal{K}_\infty$, the analysis in this paper is still applicable to ensure integral ISS (iISS) of the resulting event-triggered control system [9].

Proposition 4.2: Consider the system in equation (8) with V satisfying equation (9). If the event-triggering function, $\theta \in \mathcal{K}$, in equation (6) takes the following form

$$\theta(|x|) = \gamma_1^{-1}\left(\frac{\sigma\alpha(|x|) + \gamma_3(\bar{w})}{\chi(|x|)}\right)$$

with a given constant $\sigma \in (0, 1)$, where $\gamma_3 \in \mathcal{K}$, then the event-triggered system is ISS w.r.t. w and there exists a positive constant T^* such that

$$|x(t)| \le \alpha_1^{-1} \circ \alpha_2 \circ \alpha^{-1}\left(\frac{\gamma_2(\bar{w}) + \gamma_3(\bar{w})}{1 - \hat{\sigma}}\right) \tag{10}$$

holds for any $t \ge T^*$, where $\hat{\sigma} \in (\sigma, 1)$.

Proof: Under the assumptions, we know that

$$\dot{V} \le -\alpha(|x|) + \chi(|x|)\gamma_1(|\tilde{\eta}|) + \gamma_2(|w|).$$

By the event-triggering rule in equation (6) and the assumed event-trigger θ, we know that $|\tilde{\eta}| < \gamma_1^{-1}\left(\frac{\sigma\alpha(|x|)+\gamma_3(\bar{w})}{\chi(|x|)}\right)$, which implies

$$\begin{aligned} \dot{V} &< -(1-\sigma)\alpha(|x|) + \gamma_3(\bar{w}) + \gamma_2(|w|) \\ &< -(1-\sigma)\alpha(|x|) + \bar{\gamma}(\bar{w}) \end{aligned}$$

with $\bar{\gamma} = \gamma_2 + \gamma_3$. This inequality means that the event-triggered system is ISS with respect to the external disturbance w. Also it suggests inequality (10). ∎

Since the system is ISS and the disturbance is bounded, we know the state trajectory stays in a compact set, denoted by $\Lambda \in \mathbb{R}^n$. If we use the event-trigger in (6) to trigger the next sampling instant, then $|\tilde{\eta}(\tau_{i+1})| = \theta(|\hat{x}_{i+1}|)$ holds. Then a lower bound on the inter-sampling interval $T_i = \tau_{i+1} - \tau_i$ is given by the following proposition.

Proposition 4.3: Under the assumptions of Proposition 4.2, there exist positive real constants ρ, λ and δ such that the inter-sampling interval T_i satisfies the following inequality

$$T_i \ge \frac{1}{\rho}\log\left(1 + \frac{\theta(|\hat{x}_{i+1}|)}{\lambda|f(\hat{x}_i, k(\hat{x}_i), 0)| + \delta\bar{w}}\right). \tag{11}$$

Proof: By Proposition 4.2, the event-triggered system is ISS. Since the external disturbance w is bounded, the state trajectory $x(t)$ is inside a compact set for all $t \ge 0$. Let us consider the error system equation for $\dot{e}$ over $[\tau_i, \tau_{i+1})$:

$$\begin{aligned} \dot{e}(t) &= -f(x, k(\hat{x}_i), w) \\ e(\tau_i) &= 0. \end{aligned}$$

Since f is locally Lipschitz with respect to x, $k(\hat{x}_i)$, and w, there exist L_1, $L_2 \in \mathbb{R}^+$ such that

$$\begin{aligned} &\frac{d}{dt}|e(t)| \le |\dot{e}(t)| = |f(\hat{x} - e, k(\hat{x}_i), w)| \\ \le\ &|f(\hat{x}_i, k(\hat{x}_i), 0)| + L_1|e| + L_2\bar{w} \end{aligned} \tag{12}$$

holds for all $t \in [\tau_i, \tau_{i+1})$.

This is a linear differential inequality where $|e(\tau_i)| = 0$. We can therefore integrate it to see that for $t \in [\tau_i, \tau_{i+1})$,

$$|e(t)| \le \frac{|f(\hat{x}_i, k(\hat{x}_i), 0)| + L_2\bar{w}}{L_1}\left(e^{L_1(t-\tau_i)} - 1\right). \tag{13}$$

Since $\eta(\cdot)$ is locally Lipschitz, there exists $L \in \mathbb{R}^+$ so that

$$\begin{aligned} |\tilde{\eta}(t)| &= |\eta(\hat{x}_i) - \eta(x(t))| \\ &\le L|\hat{x}_i - x(t)| = L|e(t)| \end{aligned} \tag{14}$$

holds for all $t \in [\tau_i, \tau_{i+1})$.

Combining equations (13) and (14) yields

$$|\tilde{\eta}(t)| \le \frac{|f(\hat{x}_i, k(\hat{x}_i), 0)| + L_2\bar{w}}{L_1/L}\left(e^{L_1(t-\tau_i)} - 1\right).$$

Note that the next sampling instant occurs when $|\tilde{\eta}(\tau_{i+1})| = \theta(|x(\tau_{i+1})|)$. We can therefore see that

$$\theta(|\hat{x}_{i+1}|) \le (\lambda|f(\hat{x}_i, k(\hat{x}_i), 0)| + \delta\bar{w})\left(e^{\rho T_i} - 1\right)$$

where $\rho = L_1$, $\lambda = \frac{L}{L_1}$, $\delta = \frac{LL_2}{L_1}$, and $T_i = \tau_{i+1} - \tau_i$ is the ith intersampling interval. Solving the above inequality for T_i yields the desired lower bound. ∎

V. Efficient Attentiveness Property

With the bounds derived in Proposition 4.3, we are able to discuss the efficient attentiveness property of the system. To ensure ISS, we selected an event-trigger such that

$$\theta(|\hat{x}_{i+1}|) = \gamma_1^{-1}\left(\frac{\sigma\alpha(|\hat{x}_{i+1}|) + \gamma_3(\bar{w})}{\chi(|\hat{x}_{i+1}|)}\right)$$

where $0 < \sigma < 1$. Recall that γ_1, χ and α are class $\mathcal{K}_\infty$ functions that define $V(x)$ for the original system and γ_3 is any class $\mathcal{K}$ function. Also note that f is locally Lipschitz. Therefore, there must exist a class $\mathcal{K}$ function ϕ such that for any $\hat{x}_i \in \Lambda$,

$$\lambda|f(\hat{x}_i, k(\hat{x}_i), 0)| \le \phi(|\hat{x}_i|) \tag{15}$$

where Λ is the compact set that $x(t)$ stays inside.

We discuss the attentively efficient behavior in two cases: first with disturbances and then without disturbances.

A. Essentially Bounded Disturbances

This case means $|w(t)| \le \bar{w}$ for any $t \ge 0$. According to the bound in equation (11) and the definition of $\theta(|\hat{x}_{i+1}|)$, we have

$$\begin{aligned} T_i &\ge \frac{1}{\rho}\log\left(1+\frac{\theta(|\hat{x}_{i+1}|)}{\lambda|f(\hat{x}_i,k(\hat{x}_i),0)|+\delta\bar{w}}\right) \\ &\ge \frac{1}{\rho}\log\left(1+\frac{\gamma_1^{-1}\left(\frac{\gamma_3(\bar{w})}{\chi(|\hat{x}_{i+1}|)}\right)}{\phi(|\hat{x}_i|)+\delta\bar{w}}\right). \end{aligned} \quad (16)$$

We now need to further discuss χ in three cases:

Case I: When χ is non-increasing, it is easy to see, by inequality (16), $T_i \ge \frac{1}{\rho}\log\left(1+\frac{\gamma_1^{-1}\left(\frac{\gamma_3(\bar{w})}{\chi(0)}\right)}{\phi(|\hat{x}_i|)+\delta\bar{w}}\right)$. Note that we can always choose γ_3 such that this lower bound is larger than a pre-specified constant T as $\hat{x}_i$ approaches the origin. Therefore, the system is attentively efficient. The cost of having this property is the degradation in the level of disturbance attenuation, although ISS is still guaranteed. It is reflected in the disturbance term in the dissipative inequality, which is $\gamma_2(\bar{w})+\gamma_3(\bar{w})$, but not $\gamma_2(\bar{w})$. The system is not strictly attentively efficient because as $\hat{x}_i \to 0$, we see that T_i approaches a finite constant.

Case II: When χ is non-decreasing, we can find an upper bound on x_{i+1}. By Proposition 4.2, we know that there exists a positive constant T^* such that inequality (10) holds for any $t \ge T^*$. Therefore, when $\tau_{i+1} \ge T^*$,

$$|x_{i+1}| \le \alpha_1^{-1}\circ\alpha_2\circ\alpha^{-1}\left(\frac{\gamma_2(\bar{w})+\gamma_3(\bar{w})}{1-\hat{\sigma}}\right).$$

Since χ is non-decreasing, when τ_{i+1} is sufficiently large,

$$T_i \ge \frac{1}{\rho}\log\left(1+\frac{\gamma_1^{-1}\left(\frac{\gamma_3(\bar{w})}{\chi\circ\alpha_1^{-1}\circ\alpha_2\circ\alpha^{-1}\left(\frac{\gamma_2(\bar{w})+\gamma_3(\bar{w})}{1-\hat{\sigma}}\right)}\right)}{\phi(|\hat{x}_i|)+\delta\bar{w}}\right).$$

In this case, if $\chi\circ\alpha_1^{-1}\circ\alpha_2\circ\alpha^{-1}$ grows slower than a linear function, we can still choose $\gamma_3(\bar{w})$ to be large such that the bound in the preceding inequality is close to the desired T and ensure the attentive efficiency.

Case III: When χ is not monotonic, we can still bound T_i. Since $x(t)$ is inside a compact set, there must be a positive constant ξ such that $\chi(|\hat{x}_{i+1}|) \le \xi$. It means $T_i \ge \frac{1}{\rho}\log\left(1+\frac{\gamma_1^{-1}\left(\frac{\gamma_3(\bar{w})}{\xi}\right)}{\phi(|\hat{x}_i|)+\delta\bar{w}}\right)$.

Although this lower bound still approaches a positive constant when x_i is close to zero, this constant may not be arbitrarily specified by choosing γ_3. It is because when we adjust γ_3, the value of ξ also changes. Therefore in this case, we can only say that the inter-sampling intervals are lower bounded by a positive constant, but not the efficient attentiveness property.

Remark 5.1: Note that (strictly) efficient attentiveness property does not implies that T_i goes to the desired T or infinity as i increases. It simply means that the closer the state is to the origin, the less frequent information is transmitted. When the disturbance is present, it is quite possible that $x(t)$ always stay far from the origin due to the disturbance. In this case, frequent data transmission is still necessary, even if the system is attentively efficient.

B. No Disturbances

When $\bar{w}=0$, the event-trigger ensures asymptotic stability of the system. Let

$$\mu(|\hat{x}_{i+1}|) = \gamma_1^{-1}\left(\frac{\sigma\alpha(|\hat{x}_{i+1}|)}{\chi(|\hat{x}_{i+1}|)}\right). \quad (17)$$

Note that μ is not necessarily a class $\mathcal{K}$ function because of χ. The inter-sampling interval satisfies:

$$\begin{aligned} T_i &\ge \frac{1}{\rho}\log\left(1+\frac{\mu(|\hat{x}_{i+1}|)}{\lambda|f(\hat{x}_i,k(\hat{x}_i),0)|}\right) \\ &\ge \frac{1}{\rho}\log\left(1+\frac{\mu(|\hat{x}_{i+1}|)}{\phi(|\hat{x}_i|)}\right). \end{aligned} \quad (18)$$

To ensure strictly attentively efficient behavior, we expect

$$\lim_{|\hat{x}_i|\to 0}\frac{\phi(|\hat{x}_i|)}{\mu(|\hat{x}_{i+1}|)} = 0. \quad (19)$$

Therefore, we need to discuss the relation among μ and ϕ. The results are presented as follows:

Proposition 5.2: Under the assumptions of Proposition 4.2, the following statements are true:

1) If $\mu(s)$ converges to a positive constant a or infinity as s goes to 0, i.e.

$$\lim_{s\to 0}\mu(s) = a \text{ or } \infty, \quad (20)$$

then equation (19) holds.

2) If

$$\lim_{s\to 0}\frac{\phi(s)}{s} = 0, \quad (21)$$

$$\lim_{s\to 0}\frac{\phi(s)}{\mu(s)} = 0, \quad (22)$$

$$\lim_{s\to 0}\mu(s) = 0, \quad (23)$$

then equation (19) holds.

Proof: Consider Statement 1. Since the system is asymptotically stable, $\hat{x}_i \to 0$ means $\hat{x}_{i+1} \to 0$. Therefore, with equations (20) and (18), it implies equation (19).

When $\lim_{s\to 0}\mu(s) = 0$, we show an upper bound on $\frac{\phi(|\hat{x}_i|)}{\mu(|\hat{x}_{i+1}|)}$ that converges to 0 as $|\hat{x}_i| \to 0$. Recall that by the event-trigger,

$$\begin{aligned} \mu(|\hat{x}_{i+1}|) &= |\eta(\hat{x}_i)-\eta(\hat{x}_{i+1})| \\ &\ge L|\hat{x}_i| - L|\hat{x}_{i+1}| \end{aligned}$$

holds, where $L \in \mathbb{R}^+$ is Lipschitz constant of η. It means

$$|\hat{x}_i| \le |\hat{x}_{i+1}| + \frac{1}{L}\mu(|\hat{x}_{i+1}|) \triangleq \psi(|\hat{x}_{i+1}|). \quad (24)$$

Applying this inequality into equation (18) implies

$$\frac{\phi(|\hat{x}_i|)}{\mu(|\hat{x}_{i+1}|)} \le \frac{\phi(\psi(|\hat{x}_{i+1}|))}{\mu(|\hat{x}_{i+1}|)}. \quad (25)$$

To complete the proof, we need to show $\lim_{|\hat{x}_i|\to 0} \frac{\phi(\psi(|\hat{x}_{i+1}|))}{\mu(|\hat{x}_{i+1}|)} = 0$, which is equivalent to showing $\lim_{|\hat{x}_{i+1}|\to 0} \frac{\phi(\psi(|\hat{x}_{i+1}|))}{\mu(|\hat{x}_{i+1}|)} = 0$.

Note that for positive constants s_1, s_2 in a compact set, since $\phi \in \mathcal{K}$, there must exist b_1, b_2, b_3, $b_4 \in \mathbb{R}$ such that $\phi(s_1 + s_2) \leq b_1\phi(b_2 s_1) + b_3\phi(b_4 s_2)$. Therefore, since $\hat{x}_{i+1}$ in a compact set, let $s = |\hat{x}_{i+1}|$ and $\phi(\psi(s)) \leq b_1\phi(b_2 s) + b_3\phi\left(\frac{b_4\mu(s)}{L}\right)$ holds according to the definition of ψ in (24), which means $\frac{\phi(\psi(s))}{\mu(s)} \leq \frac{b_1\phi(b_2 s)}{\mu(s)} + \frac{b_3\phi\left(\frac{b_4\mu(s)}{L}\right)}{\mu(s)}$.

With equations (21) - (23), we know $\lim_{s\to 0^+} \frac{\phi(\psi(s))}{\mu(s)} = 0$. Applying this into equation (25) implies equation (19). ■

Remark 5.3: By the definition of ϕ in (15), equation (21) in fact places a requirement on the system dynamic $f(x, k(x), 0)$. It means that $f(x, k(x), 0)$ must decreases to zero faster than linear functions when x goes to zero. This result is consistent with the work in [3] focusing on homogeneous systems, which is a special case of this work.

Remark 5.4: Equations (22) places the constraints on the event-trigger, where $\mu(|x|)$ is the triggering threshold. It means that if $\mu(s)$ is an infinitesimal, then $\phi(s)$ must be an infinitesimal of higher order than $\mu(s)$. Note that the order of ϕ is greater than that of linear function by equation (21). By the definition of μ, we know that it provides the balance between the orders of γ_1 and $\frac{\alpha(s)}{\chi(s)}$. One way of ensuring equation (22) is to make the order of γ_1 greater than or equal to $\frac{\alpha(s)}{\chi(s)}$. In this case, the order of μ will be less than that of linear functions and therefore less than the order of ϕ by equation (21). Equation (22) therefore establishes the relation between the ISS dissipative inequality in equation (9) and the events ensuring attentively efficient behavior.

VI. Controller Design

This section studies the construction of the feedback law k to guarantee the conditions for attentively efficient behavior. The key is to ensure inequality (9) with some specified α, χ, γ_1. We provide a method to construct k. The idea takes advantage of the universal formula in [8]. One thing worth mentioning is that the proposed feedback law is not the only law for efficient attentiveness property. Our discussion focuses on control-affine systems:

$$\dot{x} = f(x, w) + g(x)u. \tag{26}$$

Given a tuple of (α, χ, γ_1), assume that there is a CLF $V(x)$ such that $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ and

$$\begin{aligned}\inf_u \left\{ \frac{\partial V}{\partial x} f(x, w) + \frac{\partial V}{\partial x} g(x)(u + \tilde{u}) \right\} \\ \leq -3\alpha(|x|) + \chi(|x|)\gamma_1(|\tilde{u}|) + \gamma_2(w)\end{aligned}$$

with $\tilde{u} \in \mathbb{R}$ and some $\alpha_1, \alpha_2, \gamma_2 \in \mathcal{K}_\infty$. Let $a(x, w) = \frac{\partial V}{\partial x} f(x, w)$ and $b(x) = \frac{\partial V}{\partial x} g(x)$. Define

$$c(x) = \max_{\tilde{u}, w} \left\{ a(x, w) + b(x)\tilde{u} - \chi(|x|)\gamma_1(|\tilde{u}|) - \gamma_2(w) \right\}. \tag{27}$$

To ensure that $c(x)$ is well defined, it is sufficient to demand: (1) $\gamma_2(w)$ grows faster than $a(x, w)$ at infinity for fixed x; (2) γ_1 grows faster than any linear functions at infinity, or γ_1 is linear and $|b(x)| - \chi(|x|) \leq 0$.

As it is in [8], choose $\bar{c}(x)$ such that

$$c(x) + \alpha(|x|) \leq \bar{c}(x) \leq c(x) + 2\alpha(x) \tag{28}$$

holds for any $x \in \mathbb{R}^n$.

The feedback control law $k(x)$ is defined by

$$k(x) = \begin{cases} -\frac{\bar{c}(x) + \sqrt{\bar{c}(x)^2 + |b(x)|^4}}{|b(x)|^2} b^\top(x), & b(x) \neq 0 \\ 0, & b(x) = 0 \end{cases} \tag{29}$$

Note that this feedback law is almost smooth. With this law, we can verify that inequality (9) is satisfied with the pre-specified (α, χ, γ_1) as follows:

$$\begin{aligned}\dot{V} &= a(x, w) + b(x)k(x) + b(x)\tilde{u} \\ &\leq c(x) + b(x)k(x) + \chi(|x|)\gamma_1(|\tilde{u}|) + \gamma_2(w)\end{aligned}$$

where $\tilde{u} = k(x) - k(\hat{x})$. With inequality (28),

$$\dot{V} \leq \bar{c}(x) - \alpha(|x|) + b(x)k(x) + \chi(|x|)\gamma_1(|\tilde{u}|) + \gamma_2(w) \tag{30}$$

It follows from the result in [8] that $\bar{c}(x) + b(x)k(x) \leq 0$. Therefore, inequality (30) implies

$$\dot{V} \leq -\alpha(|x|) + \chi(|x|)\gamma_1(|\tilde{u}|) + \gamma_2(w).$$

Note that in this formulation, $\eta(\cdot) \equiv k(\cdot)$ and the choice of γ_3 is independent of this feedback law. One thing worth mentioning is that when $\bar{w} = 0$ and the pre-specified α, χ satisfies $\frac{\alpha(|x|)}{\chi(|x|)} \to 0$ as $|x| \to 0$, we have to resort to the second statement in Proposition 5.2 to guarantee the attentively efficient behavior. In that case, we still need to check if the feedback law in (29) ensures (21) and (22).

VII. Simulations

This section provides simulation results that illustrate attentively efficient behavior in event-triggered feedback systems. The system under consideration is

$$\begin{aligned}\dot{y}_1 &= -2y_1^3 + y_2^3 + \frac{w}{\sqrt{(2y_1 + y_2)^2 + 1}} \\ \dot{y}_2 &= g(y)(5e^{y_1} - 5 + u) - 2y_2^3 + y_1^3\end{aligned}$$

where $y = (y_1^\top, y_2^\top)^\top$ and $g : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function to be determined. The initial condition satisfies $|y(0)|_\infty \leq 5$.

Consider $V(y) = y_1^2 + y_1 y_2 + y_2^2$:

$$\begin{aligned}\dot{V} &= -3y_1^4 + \frac{(2y_1 + y_2)w}{\sqrt{(2y_1 + y_2)^2 + 1}} \\ &\quad + g(y)(2y_2 + y_1)(5e^{y_1} - 5 + u) - 3y_2^4\end{aligned}$$

We first set $g(y) = y_1$ and $\gamma_1(|\tilde{u}|) = |\tilde{u}|$, $\gamma_2(|w|) = |w|$, $\chi(|y|) = \sqrt{5}|y|^2$, $\alpha(|y|) = |y|^4$. Then we can verify $c(y)$ in equation (27) is well defined and $c(y) = -3y_1^4 - 3y_2^4 + y_1(2y_2 + y_1)(5e^{y_1} - 5)$. With $\bar{c}(y) = c(y) + 1.5\alpha(|y|)$, we have the feedback law $k(\cdot)$ in (29). The disturbance w satisfies $|w|_{\mathcal{L}_\infty} \leq 20$ and $\gamma_3(s) = \frac{s}{20}$. We run the system with the event-trigger in (6), where $\sigma = 0.8$ and $\eta \equiv k$. Figure 1 shows the simulation results. The top plot shows the state trajectories of the system, which oscillate around the origin.

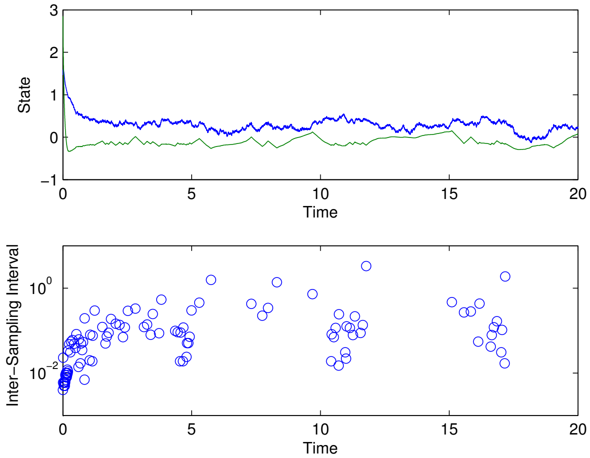

Fig. 1. States and inter-sampling intervals in an event-triggered system with disturbances and the feedback law in (29)

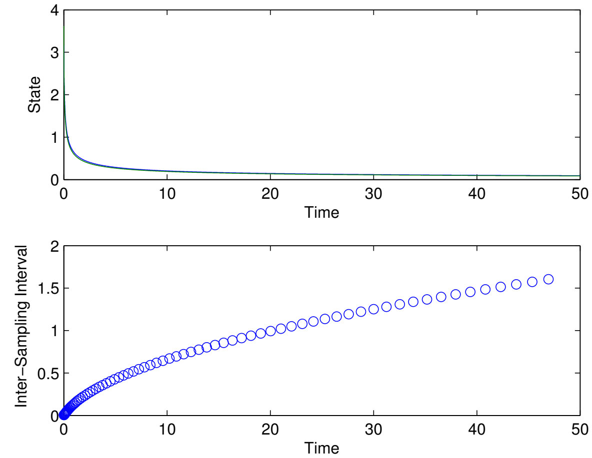

Fig. 2. States and inter-sampling intervals in an event-triggered system without disturbances, $g(y) = y_1$

The bottom plot is the inter-sampling intervals generated by the event-triggering scheme. We can see that although these intervals are not converging to infinity, it remains bounded from below, which is consistent with our theoretic results.

The feedback law proposed in Section VI is not the unique solution to the efficient attentiveness property. A much simpler controller law is

$$u = -5e^{\hat{y}_1} + 5, \quad (31)$$

which implies $|f(y, k(y), 0)| \leq \phi(|y|) = a|y|^3$ with some positive constant a.

Consider $V(y) = y_1^2 + y_1 y_2 + y_2^2$:

$$\dot{V} \quad \leq \quad -1.5|y|^4 + 5\sqrt{5}|y||g(y)||e^{y_1} - e^{\hat{y}_1}| + |w| \quad (32)$$

Then we know that $\alpha(s) = 1.5s^4$, $\chi(s) = 5\sqrt{5}s^2$, $\eta(y) = e^{y_1}$, $\gamma_1(s) = \gamma_2(s) = s$. Still, the disturbance w satisfies $|w|_{\mathcal{L}_\infty} \leq 20$, $\gamma_3(s) = \frac{s}{20}$, and $\sigma = 0.8$. The event-trigger is $|e^{y_1} - e^{\hat{y}_1}| = \frac{1.2|y|^4+1}{5\sqrt{5}|y|^2}$. The simulation results look very similar to the first simulation.

The third simulation considers the case where $w \equiv 0$. Then $\mu(s) = \frac{1.2}{5\sqrt{5}}s^2$. It is easy to verify that the assumptions in Statement 2 of Proposition 5.2 hold. With $\sigma = 0.8$, the event-trigger is $|e^{y_1} - e^{\hat{y}_1}| = \frac{1.2|y|^2}{5\sqrt{5}}$. The simulation result is plotted in Figure 2. Obviously, the states converge to the origin and the inter-sampling intervals go to infinity, which implies the strictly attentively efficient behavior.

VIII. Conclusions

This paper studies the event-triggered feedback systems possessing the efficient attentiveness property. We develop event-triggering rules that assure ISS or iISS of a nonlinear system and establish sufficient conditions that ensure the efficient attentiveness property. A universal construction for ISS (iISS) controllers is to develop attentively efficient event-triggered controllers.

There are still several open problems. For example, when constructing the controller using the method in Section VI for asymptotic stability ($\bar{w} = 0$), if $\frac{\alpha(|x|)}{\chi(|x|)} \to 0$ as $|x| \to 0$, we have to resort to the second statement in Proposition 5.2 for the attentively efficient behavior. That means we have to go back and check if the constructed feedback law ensures equations (21) and (22). One question, therefore, is how to construct a feedback law such that even if $\lim_{|x|\to 0} \frac{\alpha(|x|)}{\chi(|x|)} \to 0$, equations (21) and (22) can be automatically satisfied. A further question is that "is it possible to relax the assumptions in equations (21) and (22)?" Also notice that even with the attentively efficient behavior, the inter-sampling intervals can still be very small when the state is far away from the equilibrium, which may violate scheduling algorithms. These issues will be addressed in the future.

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