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Cooperative distributed model predictive control for linear plants subject to convex economic objectives

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***Abstract*— In this paper we propose a cooperative distributed economic model predictive control strategy for linear systems which consist of finite number of subsystems. The suggested control strategy is generating control feedback which converges to the centralized optimal solution and drives the subsystems to the Pareto optimum provided infinite iterations are allowed at each sampling time. Moreover, the control for each subsystem is computed in itself without coordination layer except for a synchronization requirement between subsystems. We first introduce distributed linear systems with 2 subsystems and economic model predictive control, then show the convergence and stability properties of a suboptimal model predictive control strategy for the system.**

I. Introduction

Model predictive control (MPC) is a feedback design technique which computes control actions by taking into account the current state of a plant and all sorts of constraints between input and output variables that need to be fulfilled. Typically, a cost functional is available or is suitably designed so as to find the most appropriate control action by means of real-time optimization. At the same time the control action should steer the plant's operation to a desired operating condition within reasonable amount of time. Recently, as an application for systems which consist of multiple subsystems and/or large-scale systems, distributed MPC has been investigated. See [9] for a recent survey on the subject. The present note further develops the so called *cooperative* Model Predictive Control; this is a particular variant of MPC in which it is assumed that individual subsystems may cooperate towards a common objective. In [5] a solution for cooperative game for distributed systems as suboptimal MPC was suggested.

The contribution of this article is to extend the techniques of [5] to the case of economic Model Predictive Control [8]. This is a variant of standard MPC which aims at achieving both transient and steady-state costs minimization simultaneously. In particular, the MPC control layer directly uses the true economic cost in devising the optimal control action; this entails that cost need not be minimal at the best steady-state and may affect overall stability.

Recently, average performance and stability issues as well as Lyapunov-based analysis techniques were proposed in [7] and [6] respectively.

In this paper we propose a distributed economic MPC problem for linear systems with economic cost function, and also suggest its solution.

II. Distributed linear model

We assume a system that consists of two discrete-time linear subsystems. All arguments can be extended to the case of M subsystems, we limit ourselves to this case to keep notation simpler. Each subsystem has authority over its own input signals which, in turn, also affects the state of other subsystems. Therefore, subsystems are coupled through their inputs. Subsystem 1 is defined as follows

$$x_1^+ = A_1 x_1 + B_{11} u_1 + B_{12} u_2, \tag{1}$$

in which $x_1 \in \mathbb{R}^{n_1}$, $u_1 \in \mathbb{R}^{m_1}$ $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $B_{11} \in \mathbb{R}^{n_1 \times m_1}$, $B_{12} \in \mathbb{R}^{n_1 \times m_2}$. Similarly we define a model of subsystem 2, so that the overall plantwide system is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ + \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} u_1 + \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} u_2. \end{aligned} \tag{2}$$

For the sake of simplicity we denote it as

$$x^+ = Ax + B_1 u_1 + B_2 u_2 \tag{3}$$

in which $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, $B_1 = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$, $B_2 = \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix}$, or even

$$x^+ = Ax + Bu \tag{4}$$

where $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$ and $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

III. Economic model predictive control

For a system as in (4), we introduce pointwise in time state and input constraints, that is we define a discrete-time constrained dynamic system with $x(k)$ and $u(k)$ the state and input at time k, for which the following set of equations and inequalities should hold:

$$x(k+1) = Ax(k) + Bu(k), \ g(x(k), u(k)) \leq 0 \tag{5}$$

at any instant $k \in \mathbb{I}_{\geq 0}$. The function $g : \mathbb{R}^{n_1+n_2} \times \mathbb{R}^{m_1+m_2} \to \mathbb{R}$ is convex so that its sublevel sets are also convex sets.

For this system we assume a convex stage cost function $\ell(x,u) : \mathbb{R}^{n_1+n_2} \times \mathbb{R}^{m_1+m_2} \to \mathbb{R}$ and we would like to find a feedback control law $u(k) = u_e(x(k))$ so that the system remains feasible and minimizes the cost

$$\sum_k \ell\left(x(k), u(k)\right). \tag{6}$$

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Given convexity of ℓ and linearity of (5), it is meaningful to operate, at least asymptotically, in proximity of the best admissible steady-state. The steady-state optimization problem is defined as

$$\min_{x,u} \ell(x,u) \quad \text{s.t} \quad x-(Ax+Bu)=0,\ g(x,u)\leq 0, \tag{7}$$

and, for the sake of simplicity, we assume that it has a unique globally minimizing solution (x_s, u_s). It is worth pointing out that, unlike in standard model predictive control, economic model predictive control deals with stage costs for which points (x,u) satisfying $\ell(x,u) \leq \ell(x_s,u_s)$ and $g(x,u)\leq 0$ may exist which are not steady-states, [6].

Now, we define some important sets to clarify the issue of feasibility for the optimization problems associated to MPC.

Definition 1 (Feasible set): We define feasible set $\mathbb{Z}_N$ as the set of $(x,\mathbf{u})$ pairs, i.e.

$$\begin{aligned}\mathbb{Z}_N = \Big\{ &(x,\mathbf{u}) \in \mathbb{R}^n \times \mathbb{R}^{N(m_1+m_2)} \mid x(0)=x, x(N)=x_s, \\ &x(k+1) = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x(k) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k) \\ &g(x(k),u(k)) \leq 0, \forall k \in \mathbb{I}_{0:N-1} \Big\}\end{aligned} \tag{8}$$

The set of feasible states $\mathcal{X}_N$ is the projection of $\mathbb{Z}_N$ onto $\mathbb{R}^n$.

Definition 2 (Feasible states): $\mathcal{X}_N$ is called the set of feasible states, and is defined as follows

$$\mathcal{X}_N = \{x | \exists \mathbf{u} \text{ such that } (x,\mathbf{u}) \in \mathbb{Z}_N)\}. \tag{9}$$

We also define a set of feasible control sequences in terms of $\mathbb{Z}_N$.

Definition 3 (The set of feasible control sequences):

$$\mathcal{U}_N(x) := \{\mathbf{u} | (x,\mathbf{u}) \in \mathbb{Z}_N\} \tag{10}$$

For the feasible set, the following lemma is well-known, so we omit the proof.

Lemma 1: The feasible set $\mathbb{Z}_N$ is convex.

Since for real industrial plants there is no natural termination time of production we proceed as in standard MPC by minimizing (6) over a finite time horizon and refining the optimization in a receding horizon manner [1] at each time instant.

IV. Formulation of Centralized control and cooperative control

Now we define centralized control and cooperative control, and compare them to clarify the role of each subsystem in the plantwide system. Consider subsystem 1, for which we define an objective function $V_1(x,\mathbf{u})$, or $V_1(x,\mathbf{u}_1,\mathbf{u}_2)$ for simplicity, as a sum of stage cost functions $\ell_1(x_1(k),u_1(k),u_2(k))$ [5].

$$V_1(x_1(0),\mathbf{u}_1,\mathbf{u}_2) = \sum_{k=0}^{N-1} \ell_1(x_1(k),u_1(k),u_2(k)) \tag{11}$$

$$\begin{aligned} &\text{where } x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, u(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}, \\ &\mathbf{u}_1 = \begin{bmatrix} u_1(0) \\ u_1(1) \\ \vdots \\ u_1(N-1) \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} u_2(0) \\ u_2(1) \\ \vdots \\ u_2(N-1) \end{bmatrix}, \\ &\text{and } x(k+1) = Ax(k) + Bu(k). \end{aligned} \tag{12}$$

In similar manner, we define an objective function for subsystem 2, so the plantwide objective function to be optimized is a weighted sum of objective functions of subsystems;

$$\begin{aligned} V(x(0),\mathbf{u}) = &\rho_1 V_1(x_1(0),\mathbf{u}_1(k),\mathbf{u}_2(k)) \\ &+ \rho_2 V_2(x_2(0),\mathbf{u}_1(k),\mathbf{u}_2(k)) \end{aligned} \tag{13}$$

where $\rho_1, \rho_2 > 0$ are relative weights. In cooperative MPC individual objective functions are stated separately as subsystems may have conflicting goals, on the other hand, as the aim is that of cooperation (rather than competition), only the global utility function V is needed in order to define the control policies.

The goals of cooperative economic MPC are 1) to minimize the plantwide objective function, 2) to control the plantwide system towards its desired steady-state, and 3) to achieve this by means of decentralized optimization. This departs from non-cooperative MPC in that each subsystem i is supposed to select his input by optimizing only its own objective function $V_i(\cdot)$ rather than the global utility function, [4]. In analogy to non-cooperative MPC, however, each subsystem can only adjust its own input, say $\mathbf{u}_i$.

Definition 4: The optimization problem to be solved for cooperative economic MPC of subsystem $i \in \mathbb{I}_{1:2}$ is stated below

$$\begin{aligned} \min_{\mathbf{u}_i} & V(x_1(0),x_2(0),\mathbf{u}_1,\mathbf{u}_2) \text{ subject to} \\ & x(k+1) = Ax(k) + Bu(k),\ k \in \mathbb{I}_{>0} \\ & g(x(k),u(k)) \leq 0,\ k \in \mathbb{I}_{0:N-1} \\ & x(0) = x_0 \\ & x(N) = x_s. \end{aligned} \tag{14}$$

Notice that when $V(\cdot)$ is minimized with respect to $\mathbf{u}_i$, $\mathbf{u}_{3-i}$ plays the role of a known parameter. Compared to cooperative MPC, in centralized MPC scheme we assume that both $\mathbf{u}_1$ and $\mathbf{u}_2$ can be adjusted to find the optimal solution for the objective function.

Definition 5: The optimizing problem we propose to solve for centralized economic MPC is stated

$$\begin{aligned} \min_{\mathbf{u}_1, \mathbf{u}_2} & V(x_1(0), x_2(0), \mathbf{u}_1, \mathbf{u}_2) \text{ subject to} \\ & x(k+1) = Ax(k) + Bu(k), \ k \in \mathbb{I}_{>0} \\ & g(x(k), u(k)) \leq 0, \ k \in \mathbb{I}_{0:N-1} \\ & x(0) = x_0 \\ & x(N) = x_s. \end{aligned} \tag{15}$$

Note that, through centralized optimization, we can find the optimal control for plantwide system since all states and inputs of all subsystems are simultaneously considered.

V. Assumptions and Convergence results

Implementing distributed MPC strategies to converge towards an agreed control action is similar to accomplishing centralized MPC with optimization distributed over many processors [5]. As the time available for computation is finite, instead of converging to the true optimum, we allow each subsystem to inject its own suboptimal control to attain a feasible suboptimal control strategy for the whole system. In this section we introduce the definition of suboptimal MPC, and provide key assumptions to establish its performance and the convergence properties of the resulting closed-loop system.

A. Suboptimal MPC

We define the current state of the systems $x \in \mathbb{R}^{n_1+n_2}$, the trajectory of inputs $\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\} \in \mathbb{R}^{N(m_1+m_2)}$. For a feasible state $x \in \mathcal{X}_N$ we assume a feasible initial input trajectory $\tilde{\mathbf{u}} \in \mathcal{U}_N(x)$. Each subsystem i performs p-times iterations of a feasible path algorithm, and computes a new input sequence $\mathbf{u}_i$, improved with respect to the previous input trajectory. Iterations are synchronized between different subsystems and communication is assumed between the subsystems at each iteration. Of the computed input trajectory $\mathbf{u}$, the first component $u(0)$ is effectively applied to the plant, giving rise to the next state according to the evolution equation $x^+ = Ax + Bu$. For any initial state $x(0)$, we initialize a feasible input trajectory $\tilde{\mathbf{u}}(0) := \mathbf{h}(x(0))$ for some continuous function $\mathbf{h}(\cdot)$. At subsequent times, we denote $\tilde{\mathbf{u}}^+ = \{u(1), u(2), \ldots, u(N-1), u_s\}$ as a warm start [4] for the iterations to be performed by the individual subsystems. Since $\mathbf{u}^+$ is a function of the state x and of $\tilde{\mathbf{u}}^+$, which in turn is function of x and $\mathbf{u}$, the input sequence $\mathbf{u}^+$ can be expressed as a function of $(x, \mathbf{u})$. We denote $h^p(\cdot)$ as the p-times iterates from the warm start through the given iteration algorithm [5], thus $\mathbf{u}^+ = h^p(x, \tilde{\mathbf{u}}^+)$.

B. Assumptions

For subsystems $i \in \mathbb{I}_{1:2}$, following assumptions are used to establish stability.

Assumption 1: For $i \in \mathbb{I}_{1:2}$,

1) The systems (A_i, B_{ij}) are controllable (for j in $\mathbb{I}_{1:2}$).
2) $u_1(k) \in \mathbb{U}_1$ and $u_2(k) \in \mathbb{U}_2$, where $\mathbb{U}_1$ and $\mathbb{U}_2$ are compact and convex sets such that 0 is in the interior of $\mathbb{U}_i \ \forall i \in \mathbb{I}_{1:2}$
3) Every stage cost function $\ell_i(\cdot, \cdot)$ is strongly convex, with bounded sublevel sets.
4) Constraint $g(x, u)$ is convex.

The following strong duality condition is the key assumption for MPC with an economic stage cost function $\ell(x, u)$ [6].

Assumption 2 (Strong duality of steady-state problem): For the plantwide stage cost function $\ell(x, u) := \rho_1 \ell_1(x, u) + \rho_2 \ell_2(x, u)$, there exists a multiplier λ_s so that (x_s, u_s) uniquely solves

$$\min_{x,u} \ell(x, u) + [x - (Ax + Bu)]' \lambda_s \text{ s.t. } g(x, u) \leq 0 \tag{16}$$

Furthermore, there exists a K_∞-function β such that the rotated stage cost function

$$L(x, u) := \ell(x, u) + [x - (Ax + Bu)]' \lambda_s - \ell(x_s, u_s) \tag{17}$$

satisfies

$$L(x, u) \geq \beta(|x - x_s|) \tag{18}$$

for all (x, u) satisfying $g(x, u) \leq 0$, and $L(x, u)$ is Lipschitz on $\bigcup_{x \in \mathcal{X}_N} \{x\} \times \mathcal{U}_N(x)$, i.e.,

$$|L(x, u) - L(x_0, u_0)| \leq L_L |(x, u) - (x_0, u_0)| . \tag{19}$$

The existence and uniqueness of the optimal solution for problem (14) is guaranteed by stong convexity of objective functions and strong duality [3]. Therefore, we denote its solution from current state x and assigned the input of the other subsystem $\mathbf{u}_j$ as

$$\mathbf{u}_i^*(x, \mathbf{u}_j) := \arg\min_{\mathbf{u}_i} V(x, \mathbf{u}_i, \mathbf{u}_j) \tag{20}$$

where $i, j \in \mathbb{I}_{1:2}$ and $i \neq j$.

C. Stability and average performance

For standard MPC, defining the cost-to-go variable and using it as a Lyapunov function candidate is a well-known method to analyze of closed-loop stability of equilibria. In our notation, this is:

$$\begin{aligned} W(x) := \min_{\mathbf{u}} & V(x,\mathbf{u}) \\ \text{subject to } & x(k+1) = Ax(k) + Bu(k), k \in \mathbb{I}_{>0} \\ & g(x(k), u(k)) \leq 0, k \in \mathbb{I}_{0:N-1} \\ & x(0) = x \\ & x(N) = x_s \end{aligned} \tag{21}$$

In standard MPC scheme, the assumption $\ell(x_s, u_s) < \ell(x, u)$, which implies $0 = W(x_s) \leq W(x)$, holds for all feasible states. Moreover, along solutions of the closed-loop systems the following inequality holds:

$$W(x(k+1)) - W(x(k)) \leq L(x_s, u_s) - L(x(k), u(k)). \tag{22}$$

This condition implies $V(x(k+1)) \leq V(x(k))$, the monotonicity of the cost-to-go function, evaluated along solutions of closed-loop systems. The monotonicity is important for the proof of asymptotic stability since the cost-to-go function can be used as a Lyapunov function candidate under some mild conditions.

In economic MPC, the inequality (22) does not hold, so Lyapunov-like analysis tools are not generally available. However, in spite of the loss of monotonicity, economic MPC of linear systems subject to strict convex cost functionals and convex constraints, x_s turns out to be asymptotically stable with the same basin of attraction of standard MPC. A proof based on convexity arguments was shown in [8], and a different proof based on Lyapunov argument was derived in [6] recently.

The performance of economic MPC was analyzed in [1]. Its main result is that for a feasible initial state $x \in \mathcal{X}_N$, a closed-loop system has an average performance no worse than that of the best feasible steady state, (x_s, u_s). It might be a major strength of economic MPC that even if the stability is not guaranteed, asymptotic performance is preserved.

It is worth pointing out that the key technical step in the prove of such convergence and performance results is simply the realization that:

$$V(x, \mathbf{u}) \leq V(x, \tilde{\mathbf{u}}), \tag{23}$$

that is, the fact that the (centralized) optimizer yields a solution that is at least as good as the 'warm start' obtained by shifting the feasible solution coming from the previous sample time. In this respect, then, also decentralized cooperative economic MPC will guarantee these same properties even if we do not allow for the optimization iterates to converge to the optimal solution. This is further discussed in the next Section.

VI. Distributed Iterations and Their Convergence Property

We now design a controller, based on iteration, for the cooperative MPC, and show how it works in a plantwide system. The iteration for finding $\mathbf{u}^{p+1}$ in terms of the current input sequence of subsystems $\mathbf{u}_1^p$ and $\mathbf{u}_2^p$ is:

$$\mathbf{u}_1^{p+1} = \alpha_1 \mathbf{u}_1^p + \alpha_2 \mathbf{u}_1^*(x(0), \mathbf{u}_2^p) \tag{24}$$

$$\mathbf{u}_2^{p+1} = \alpha_2 \mathbf{u}_2^p + \alpha_1 \mathbf{u}_2^*(x(0), \mathbf{u}_1^p) \tag{25}$$

$$\alpha_1 + \alpha_2 = 1, \ \alpha_1, \alpha_2 > 0 \tag{26}$$

in which $\mathbf{u}_1^*(\cdot)$ and $\mathbf{u}_2^*(\cdot)$ are defined in (20) for given initial condition $x(0)$ and current iterates $\mathbf{u}^p$. If we denote $\mathbf{u}_i^* := \mathbf{u}_i^*(x(0), \mathbf{u}_j^p)$ for simplicity, this iteration is equivalent to

$$\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}^{p+1} = \alpha_1 \begin{bmatrix} \mathbf{u}_1^p \\ \mathbf{u}_2^* \end{bmatrix} + \alpha_2 \begin{bmatrix} \mathbf{u}_1^* \\ \mathbf{u}_2^p \end{bmatrix}. \tag{27}$$

From this iteration, the following lemmas follow immediately.

Lemma 2 (Feasibility): Given a feasible input sequence and state pair $(x, \mathbf{u}^p)$, the next iterate and state pair given in (27) are also feasible. That is,

$$(x, \mathbf{u}^{p+1}) \in \mathbb{Z}_N, \ \forall p \geq 1. \tag{28}$$

Proof: Let $(x, \mathbf{u}^p)$ be feasible. We want to show that if $(x, \mathbf{u}^p) \in \mathbb{Z}_N$, then $(x, \mathbf{u}^{p+1}) \in \mathbb{Z}_N$. Since the feasible set $\mathbb{Z}_N$ is convex from lemma 1, any convex combination of states and input sequence in $\mathbb{Z}_N$ are also belong to the set. From the definition of $\mathbf{u}_1^*$ and $\mathbf{u}_2^*$ in (20), we can easily find $\left(x, \begin{bmatrix} \mathbf{u}_1^p \\ \mathbf{u}_2^* \end{bmatrix}\right)$ and $\left(x, \begin{bmatrix} \mathbf{u}_1^* \\ \mathbf{u}_2^p \end{bmatrix}\right)$ are feasible. Therefore, $(x, \mathbf{u}^{p+1})$ belongs in $\mathbb{Z}_N$ since

$$\begin{aligned} & \alpha_1 \left(x, \begin{bmatrix} \mathbf{u}_1^p \\ \mathbf{u}_2^* \end{bmatrix}\right) + \alpha_2 \left(x, \begin{bmatrix} \mathbf{u}_1^* \\ \mathbf{u}_2^p \end{bmatrix}\right) \\ & = \left((\alpha_1 + \alpha_2)x, \alpha_1 \begin{bmatrix} \mathbf{u}_1^p \\ \mathbf{u}_2^* \end{bmatrix} + \alpha_2 \begin{bmatrix} \mathbf{u}_1^* \\ \mathbf{u}_2^p \end{bmatrix}\right) \\ & = (x, \mathbf{u}^{p+1}) \end{aligned} \tag{29}$$

in which $\alpha_1 > 0$, $\alpha_2 > 0$, and $\alpha_1 + \alpha_2 = 1$. ■

Lemma 3 (Convergence of control sequence): The cost function $V(x(0), \mathbf{u}^p)$ is convergent as $p \to \infty$.

Proof: In [6], it is shown that economic MPC with the objective function $V(\cdot)$ has the same optimizing law of a "rotated MPC" with objective function $\bar{V}(\cdot)$ such that

$$\begin{aligned} & \bar{V}(x, \mathbf{u}_1, \mathbf{u}_2) \\ & := V(x, \mathbf{u}_1, \mathbf{u}_2) - [x - x_s]' \lambda_s - N\ell(x_s, u_s) \\ & = \sum_{k=0}^{N-1} L(x(k), u(k)). \end{aligned} \tag{30}$$

Since $\mathbf{u}_i^*$ is a minimizer of $\bar{V}(\cdot)$, then it holds

$$\begin{aligned}\bar{V}\left(x(0), \begin{bmatrix} \mathbf{u}_1^* \\ \mathbf{u}_2^p \end{bmatrix}\right) \le \bar{V}\left(x(0), \begin{bmatrix} \mathbf{u}_1^p \\ \mathbf{u}_2^p \end{bmatrix}\right), \\ \bar{V}\left(x(0), \begin{bmatrix} \mathbf{u}_1^p \\ \mathbf{u}_2^* \end{bmatrix}\right) \le \bar{V}\left(x(0), \begin{bmatrix} \mathbf{u}_1^p \\ \mathbf{u}_2^p \end{bmatrix}\right)\end{aligned} \tag{31}$$

for all $p \ge 0$. From (31) and convexity of $\bar{V}(\cdot)$, objective function $\bar{V}(\cdot)$ satisfies

$$\begin{aligned}&\bar{V}(x(0), \mathbf{u}^{p+1}) \\ &= \bar{V}\left(x(0), \alpha_1 \begin{bmatrix} \mathbf{u}_1^p \\ \mathbf{u}_2^* \end{bmatrix} + \alpha_2 \begin{bmatrix} \mathbf{u}_1^* \\ \mathbf{u}_2^p \end{bmatrix}\right) \\ &\le \alpha_1 \bar{V}\left(x(0), \begin{bmatrix} \mathbf{u}_1^p \\ \mathbf{u}_2^* \end{bmatrix}\right) + \alpha_2 \bar{V}\left(x(0), \begin{bmatrix} \mathbf{u}_1^* \\ \mathbf{u}_2^p \end{bmatrix}\right) \\ &\le \alpha_1 \bar{V}\left(x(0), \begin{bmatrix} \mathbf{u}_1^p \\ \mathbf{u}_2^p \end{bmatrix}\right) + \alpha_2 \bar{V}\left(x(0), \begin{bmatrix} \mathbf{u}_1^p \\ \mathbf{u}_2^p \end{bmatrix}\right) \\ &= \bar{V}(x(0), \mathbf{u}^p)\end{aligned} \tag{32}$$

The final equality follows from $\alpha_1 + \alpha_2 = 1$. Because $\bar{V}(\cdot)$ is monotonically decreasing and bounded below, it converges [5]. The convergence of $\bar{V}(\cdot)$ ensures convergence of $V(\cdot)$ as $p \to \infty$ the optimal costs associated to the two functionals only differ by a fixed amount. ■

Lemma 4 (Optimality): $V(x(0), \mathbf{u}_1^p, \mathbf{u}_2^p)$ converges to the optimal value $V(x(0), \mathbf{u}_1^0, \mathbf{u}_2^0)$ as $p \to \infty$. Furthermore, the iterates $(\mathbf{u}_1^p, \mathbf{u}_2^p)$ converges to $(\mathbf{u}_1^0, \mathbf{u}_2^0)$, which is the Pareto (centralized) optimal solution.

Proof: The main idea and techniques of the following proof are based on [5]. We may safely deal with $\bar{V}(\cdot)$ instead of $V(\cdot)$ because they agree (up to a constant) on the set of feasible input sequences. From lemma 3, the cost converges, say to $\tilde{V}$. Since $\bar{V}(\cdot)$ is strongly convex, the sublevel sets $lev_{\le a}\bar{V}(x, \mathbf{u}) := \{(x, \mathbf{u}) | \bar{V}(x, \mathbf{u}) \le a\}$ are compact and bounded for all a. Therefore all iterates $\mathbf{u}^p$ belong to the compact sublevel set $lev_{\le V(x,\mathbf{u}^0)}\bar{V}(\cdot)$, so there is at least one accumulation point in it. Choose a subsequence $\mathcal{P} \subset \{1, 2, 3, \ldots\}$ such that $\{\mathbf{u}^p\}_{p\in\mathcal{P}}$ converges to $\bar{\mathbf{u}}$, an arbitrary accumulation point. Clearly $\lim_{p\to\infty} \bar{V}(x(0), \mathbf{u}^p) = \tilde{V}$, and by continuity of $\bar{V}$,

$$\bar{V}(x(0), \bar{\mathbf{u}}) = \lim_{p\in\mathcal{P}, p\to\infty} \bar{V}(x(0), \mathbf{u}^p) = \tilde{V}. \tag{33}$$

Moreover, exploiting conditions (31) we also obtain

$$\begin{aligned}\lim_{p\in\mathcal{P}, p\to\infty} \bar{V}(x(0), \mathbf{u}_1^*(x(0), \mathbf{u}_2^p), \mathbf{u}_2^p) = \tilde{V} \\ \lim_{p\in\mathcal{P}, p\to\infty} \bar{V}(x(0), \mathbf{u}_1^p, \mathbf{u}_2^*(x(0), \mathbf{u}_1^p)) = \tilde{V}.\end{aligned} \tag{34}$$

We suppose for contradiction that $\tilde{V} \ne V(x(0), \mathbf{u}^0)$ and thus $\bar{\mathbf{u}} \ne \mathbf{u}^0$. Since $V(x(0), \cdot)$ is convex, we have

$$\nabla\bar{V}(\bar{u})'(\mathbf{u}^0 - \bar{\mathbf{u}}) \le \Delta\bar{V} := \bar{V}(x(0), \mathbf{u}^0) - \bar{V}(x(0), \bar{\mathbf{u}}) < 0 \tag{35}$$

where $\nabla\bar{V}(x(0), \cdot)$ is the gradient of $\bar{V}(x(0), \mathbf{u})$ with respect to $\mathbf{u}$. This means that at least one of the followings holds.

$$\nabla\bar{V}(x(0), \bar{\mathbf{u}})' \begin{bmatrix} \mathbf{u}_1^0 - \bar{\mathbf{u}}_1 \\ 0 \end{bmatrix} \le \frac{1}{2}\Delta\bar{V} \text{ or} \tag{36}$$

$$\nabla\bar{V}(x(0), \bar{\mathbf{u}})' \begin{bmatrix} 0 \\ \mathbf{u}_2^0 - \bar{\mathbf{u}}_2 \end{bmatrix} \le \frac{1}{2}\Delta\bar{V} \tag{37}$$

Suppose that (36) holds. Applying Taylor's theorem to $\bar{V}$, the following holds:

$$\begin{aligned}&\bar{V}(x(0), \mathbf{u}_1^p + \epsilon(\mathbf{u}_1^0 - \mathbf{u}_1^p), \mathbf{u}_2^p) \\ &= \bar{V}(x(0), \mathbf{u}_1^p, \mathbf{u}_2^p) \\ &+ \nabla\bar{V}(x(0), \mathbf{u}_1^p, \mathbf{u}_2^p) \begin{bmatrix} \epsilon\left(\mathbf{u}_1^0 - \mathbf{u}_1^p\right) \\ 0 \end{bmatrix} \\ &+ \frac{1}{2} \begin{bmatrix} \epsilon\left(\mathbf{u}_1^0 - \mathbf{u}_1^p\right) \\ 0 \end{bmatrix}' \nabla^2\bar{V}(x(0), \mathbf{u}_1^p + \\ &\gamma_p\epsilon(\mathbf{u}_1^0 - \mathbf{u}_1^p), \mathbf{u}_2^p) \begin{bmatrix} \epsilon\left(\mathbf{u}_1^0 - \mathbf{u}_1^p\right) \\ 0 \end{bmatrix}\end{aligned} \tag{38}$$

for some $\gamma_p \in (0, 1)$. By taking limit with respect to p in (38) (and assuming without loss of generality that γ_p converges to γ), we can derive

$$\begin{aligned}&\lim_{p\in\mathcal{P}, p\to+\infty} \bar{V}(x(0), \mathbf{u}_1^p + \epsilon(\mathbf{u}_1^0 - \mathbf{u}_1^p), \mathbf{u}_2^p) \\ &= \tilde{V} + \epsilon\nabla\bar{V}\left(x(0), \bar{\mathbf{u}}\right)' \begin{bmatrix} \mathbf{u}_1^0 - \bar{\mathbf{u}}_1 \\ 0 \end{bmatrix} \\ &+ \frac{1}{2}\epsilon^2 \begin{bmatrix} \mathbf{u}_1^0 - \bar{\mathbf{u}}_1 \\ 0 \end{bmatrix}' \nabla^2\bar{V}\left(x(0), \bar{\mathbf{u}}_1 + \right. \\ &\left.\epsilon\gamma\left(\mathbf{u}_1^0 - \bar{\mathbf{u}}_1\right), \bar{\mathbf{u}}_2\right) \begin{bmatrix} \mathbf{u}_1^0 - \bar{\mathbf{u}}_1 \\ 0 \end{bmatrix} < \tilde{V}\end{aligned} \tag{39}$$

where the last inequality holds for all sufficiently small $\epsilon > 0$. Since $\mathbf{u}_1^*(\mathbf{u}_2^p)$ is optimal for $\bar{V}\left(x(0), (\cdot, \mathbf{u}_2^p)\right)$, we can find

$$\begin{aligned}\tilde{V} &= \lim_{p\in\mathcal{P}, p\to\infty} \bar{V}(x(0), (\mathbf{u}_1^*(\mathbf{u}_2^p), \mathbf{u}_2^p)) \\ &\le \lim_{p\in\mathcal{P}, p\to\infty} \bar{V}(x(0), (\mathbf{u}_1^p + \epsilon(\mathbf{u}_1^0 - \mathbf{u}_1^p), \mathbf{u}_2^p)) \\ &< \tilde{V}\end{aligned} \tag{40}$$

which gives a contradiction. Because we obtain the same result about $\mathbf{u}_2$ with the same logic from (37), we can conclude that $\tilde{V} = \bar{V}(x(0), \mathbf{u}^0)$ and thus $\bar{\mathbf{u}} = \mathbf{u}^0$. Since $\bar{\mathbf{u}}$ is an arbitrary accumulation point of the sequence $\{\mathbf{u}^p\}$, and the sequence is confined in a compact set, the whole sequence converges to $\mathbf{u}^0$. Moreover, from convergence of

$\mathbf{u}^p$ to $\mathbf{u}^0$ and continuity of $V(\cdot)$, $V(\cdot)$ also converges to its centralized optimal value. ■

We now establish the stability of the closed-loop system with cooperative MPC scheme as a form of suboptimal MPC. First we define the warm start for each subsystem as

$$\begin{aligned} \tilde{\mathbf{u}}_1^+ &= \{u_1(1), u_1(2), \ldots, u_1(N-1), u_{s1}\} \\ \tilde{\mathbf{u}}_2^+ &= \{u_2(1), u_2(2), \ldots, u_2(N-1), u_{s2}\} \end{aligned} \tag{41}$$

where $u_s = \left[u'_{s_1}, u'_{s_2}\right]'$. The warm start $\tilde{\mathbf{u}}_i^+$ is used as the initial condition for the cooperative MPC problems in subsystems i. By the function h_1^p and h_2^p we represent the outcome of applying the cooperative control iteration (27) p times.

$$\begin{aligned} \mathbf{u}_1^+ &= h_1^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) \\ \mathbf{u}_2^+ &= h_2^p(x_1, x_2, \mathbf{u}_1, \mathbf{u}_2) \end{aligned} \tag{42}$$

The systems evolution is then,

$$\begin{bmatrix} x_1^+ \\ x_2^+ \\ \mathbf{u}_1^+ \\ \mathbf{u}_2^+ \end{bmatrix} = \begin{bmatrix} A_1x_1 + B_{11}u_1 + B_{12}u_2 \\ A_2x_2 + B_{21}u_1 + B_{22}u_2 \\ h_1^p(x_1, x_2, \tilde{\mathbf{u}}_1^+, \tilde{\mathbf{u}}_2^+) \\ h_2^p(x_1, x_2, \tilde{\mathbf{u}}_1^+, \tilde{\mathbf{u}}_2^+) \end{bmatrix} \tag{43}$$

which can equivalently be written as:

$$\begin{bmatrix} x^+ \\ \mathbf{u}^+ \end{bmatrix} = \begin{bmatrix} Ax + B_1u_1 + B_2u_2 \\ h^p(x, \tilde{\mathbf{u}}) \end{bmatrix}. \tag{44}$$

For this closed-loop systems, now the stability of the closed-loop system is derived as follows.

Theorem 1 (Stability): Given Assumption 1, the equilibrium point x_s, which satisfies the steady-state condition (7), is asymptotically stable on the set of feasible states $\mathcal{X}_N$ for the closed-loop system (44).

Proof: From the definition of $\bar{V}(\cdot)$ in (17) and Assumption 2, we can directly derive

$$\begin{aligned} &\bar{V}(x^+, \tilde{\mathbf{u}}^+) - \bar{V}(x, \mathbf{u}) \\ &= \sum_{k=0}^{N-1} L(x^+(k), \tilde{u}^+(k)) - \sum_{k=0}^{N-1} L(x(k), u(k)) \\ &= L(x_s, u_s) - L(x(0), u(0)) = -L(x_0, u_0) \\ &\leq -\beta(|x_0 - x_s|) \end{aligned} \tag{45}$$

The second equality holds since $L(\cdot)$ is uniquely minimized by (x_s, u_s), and by assumption its optimal value is zero. Using this result and applying the controllers in (43) gives

$$\bar{V}(x^+, \mathbf{u}^+) - \bar{V}(x, \mathbf{u}) \leq -L(x_0, u_0) \leq -\beta(|x - x_s|) \tag{46}$$

We also have for all $x \in \mathcal{X}_N$

$$\beta(|x - x_s|) \leq \bar{V}(x, \mathbf{u}) \leq \alpha(|x - x_s|) \tag{47}$$

in which a K_∞ function $\alpha(\cdot) := (|\lambda_s| + L_l L_F)(\cdot) + L_l(1 + L_F)\gamma(\cdot)$, which is established in Appendix of [6]. Hence the closed-loop system is asymptotically stable with $\mathcal{X}_N$ as a region of attraction. ■

VII. Conclusion

We presented a cooperative distributed controller to optimize the same objective function in parallel without a coordinator. The algorithm based on iteration is almost equivalent to a suboptimal centralized controller, which is to be terminated at Pareto optimal at convergence. Even though monotonic decreasing of object function $V(\cdot)$ for each iteration is not guaranteed, the convergence of control sequence $\mathbf{u}^p$ to $\bar{\mathbf{u}}$ and asymptotic stability are satisfied.

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