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L_1 Gain Analysis of Linear Positive Systems and Its Application

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***Abstract*— In this paper, we focus on L_1 gain analysis problems of linear time-invariant continuous-time positive systems. A positive system is characterized by the strong property that its output is always nonnegative for any nonnegative input. Because of this peculiar property, it is natural to evaluate the magnitude of positive systems by the L_1 gain (i.e. the L_1 induced norm) in terms of the input and output signals. In contrast with the standard L_1 gain, in this paper, we are interested in L_1 gains with weightings on the input and output signals. It turns out that the L_1 gain with weightings plays an essential role in the stability analysis of interconnected positive systems. More precisely, as a main result of this paper, we show that an interconnected positive system is stable if and only if there exists a set of weighting vectors that renders the L_1 gain of each positive subsystem less than unity. As such, using a terminology in the literature, the weighting vectors work as 'separators,' and thus we establish solid separator-based conditions for the stability of interconnected positive systems. We finally illustrate that these separator-based conditions are effective particularly when we deal with robust stability analysis of positive systems against both L_1 gain bounded and parametric uncertainties.**

Keywords: positive system, L_1 gain, stability, interconnection, separator.

I. Introduction

A linear time-invariant system is said to be positive (or more accurately, *internally* positive) if its state and output are both nonnegative for any nonnegative initial state and nonnegative input [2], [6]. This property can be seen naturally in biology, network communications, economics and probabilistic systems. Even though practical systems in these fields are nonlinear in nature, linear positive system models are still valid in several applications, ex., in age-structured population models in demography [2].

Due to the nonnegative property, it would be natural to evaluate the *magnitude* of positive systems via the L_1 gain (i.e., the L_1 induced norm) in terms of the input and output signals. In general, a properly defined system-gain is useful for quantitative evaluation of the system performance. Indeed, it is shown in [3] that the L_1 gain of positive system plays an important role in robust stability analysis against dynamical and parametric uncertainties. In recent studies on switched positive systems [9], [10], the L_1 gain is also employed as a performance index to be minimized.

In contrast with the standard L_1 gain employed in the literature, we focus on L_1 gains with *weightings* on the input and output signals in this paper. As a preliminary result, we first show that the L_1 gain of a positive system evaluated with fixed weighting vectors is characterized by linear scalar inequalities. This is a slight, but still meaningful extension of known results in [9], [10] where the standard L_1 gain is characterized by linear inequalities as well. Then, as a main contribution of this paper, we show that the L_1 gain evaluated with weightings plays an essential role in the stability analysis of interconnected positive systems. Here, we consider the interconnection among more than one positive subsystem requiring that the positivity property is still preserved under the interconnection (we call this property *admissible*, whose precise definition is given later). Then, we prove that an interconnected positive system is admissible and stable if and only if there exists a set of weighting vectors that renders the L_1 gain of each positive subsystem less than unity. Namely, the stability condition is separated into the L_1 gain condition for each subsystem, where they are correlated through the weighting vectors. As such, using terminology in the literature, we could say that the weighting vectors work as *separators*, which has played an important role in the stability analysis of general linear systems [4], [5], [7]. Thus, we establish solid separator-based conditions for the stability of interconnected positive systems. We emphasize that, in contrast with the case of general linear system analysis, the separator-based results for the interconnected positive system hold true irrespective of the number of the subsystems. These results surely bring new insights for the stability of linear positive systems.

We finally show that, as expected from [4], [5], [7], the separator-based conditions are effective particularly when we deal with robust stability analysis of positive systems against uncertainties. In the case where the set of uncertainties is characterized by L_1 gain boundedness with known weightings (i.e., separators), we derive a necessary and sufficient condition for the robust stability in terms of linear scalar inequalities (linear programming problems). On the other hand, in the case where the uncertainties are parametric, we derive sufficient conditions for the robust stability in which we seek for appropriate separators. Nevertheless, it is still possible to ensure their necessity under additional assumptions on the structure of the uncertainty. The effectiveness of these approaches is illustrated by an academic numerical example.

We use the following notations. For given two matrices A and B of the same size, we write $A > B$ ($A \geq B$) if $A_{ij} > B_{ij}$ ($A_{ij} \geq B_{ij}$) holds for all (i, j), where A_{ij} (B_{ij}) stands for the (i, j)-entry of A (B). In relation to this notation, we also define

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$$\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n,\ x > 0\}, \quad \mathbb{R}^n_{+} := \{x \in \mathbb{R}^n,\ x \geq 0\}.$$

We also define $\mathbb{R}^{n\times m}_{++}$ and $\mathbb{R}^{n\times m}_{+}$ with obvious modifications. We denote by e_i $(i = 1, \cdots, n)$ the i-th standard basis of $\mathbb{R}^n$. Finally, the all-ones vector in $\mathbb{R}^n$ is denoted by $\mathbf{1}_n$.

II. Preliminaries

In this section, we gather basic definitions and fundamental results for the positive system analysis.

Definition 1 (Positive Linear System): [2] A linear system is said to be *positive* if its state and output are both nonnegative for any nonnegative initial state and nonnegative input.

Definition 2 (Metzler Martrix): [2] A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Metzler* if its off-diagonal entries are all nonnegative, i.e., $A_{ij} \geq 0$ $(i \neq j)$.

Theorem 1: [2] Let us consider the continuous-time LTI system described by

$$G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{1}$$

Then, this system is positive if and only if A is Metzler, $B \geq 0$, $C \geq 0$, and $D \geq 0$.

In the sequel, we denote by $\mathbb{M}^n$ the set of the Metzler matrices of the size n. Next theorem summarizes known conditions for the Hurwitz stability of Metzler matrices.

Theorem 2: [2], [6] For given $A \in \mathbb{M}^n$, the following conditions are equivalent.

(i) The matrix A is Hurwitz stable.
(ii) For any $h \in \mathbb{R}^n_{+} \setminus \{0\}$, the row vector $h^T A$ has at least one strictly negative entry.
(iii) The inverse of A satisfies $A^{-1} \leq 0$.
(iv) There exists $h \in \mathbb{R}^n_{++}$ such that $h^T A < 0$.

In addition to the conditions in the above theorem, the following simple lemma on the stability of block Metzler matrices plays an important role in this paper. We omit the proof due to limited space.

Lemma 1: For given $P \in \mathbb{M}^n$, $Q \in \mathbb{R}^{n\times m}_{+}$, $R \in \mathbb{R}^{m\times n}_{+}$, and $S \in \mathbb{M}^m$, the following conditions are equivalent.

(i) The Metzler matrix
$$\Pi := \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$
is Hurwitz stable.
(ii) The Metzler matrix P is Hurwitz stable and $S - RP^{-1}Q$ is Metzler and Hurwitz stable.
(iii) The Metzler matrix S is Hurwitz stable and $P - QS^{-1}R$ is Metzler and Hurwitz stable.

III. L_1 Gain Analysis

Let us consider the positive system described by

$$G : \left\{ \begin{array}{rcrcl} \dot{x} & = & Ax & + & Bw, \qquad x(0) = 0, \\ z & = & Cx & + & Dw \end{array} \right. \tag{2}$$

where $A \in \mathbb{M}^n$, $B \in \mathbb{R}^{n\times n_w}_{+}$, $C \in \mathbb{R}^{n_z\times n}_{+}$, $D \in \mathbb{R}^{n_z\times n_w}_{+}$.

For given *weighting vectors* $q_z \in \mathbf{R}^{n_z}_{++}$ and $q_w \in \mathbf{R}^{n_w}_{++}$, we are interested in computing a variant of the L_1 gain of the system G defined by

$$\|G_{q_z,q_w}\|_{1+} := \sup_{\|q_w^T w\|_1 = 1,\ w \in L_{1+}} \|q_z^T z\|_1. \tag{3}$$

Here, for $v(t) : \mathbb{R} \to \mathbb{R}$, we define

$$\|v\|_1 := \int_0^\infty |v(t)| dt$$

and L_{1+} is the set of element-wise positive and L_1 bounded signals as in

$$L_{1+} := \{v(t) : \|v_i\|_1 < \infty,\ v_i(t) \geq 0\ \forall t \in [0, \infty)\}.$$

If we let $q_z = \mathbf{1}_{n_z}$ and $q_w = \mathbf{1}_{n_w}$, the definition (3) reduces to the standard L_1 gain and this is employed as a performance index in recent studies on switched positive systems [9], [10]. The main contribution of this paper is to show that the extension of q_z and q_w to general positive vectors is surely meaningful. As clarified later on, this extension leads us to fruitful results, such as separator-based conditions for stability of interconnected positive systems.

The next theorem shows that the L_1 gain $\|G_{q_z,q_w}\|_{1+}$ is characterized by linear inequalities. The proof is omitted due to limited space.

Theorem 3: Let us consider the positive system G described by (2). Then, for given $q_z \in \mathbb{R}^{n_z}_{++}$, $q_w \in \mathbb{R}^{n_w}_{++}$, and $\gamma > 0$, the following conditions are equivalent.

(i) The matrix $A \in \mathbb{M}^n$ is Hurwitz stable and $\|G_{q_z,q_w}\|_{1+} < \gamma$.
(ii) There exists $h \in \mathbb{R}^n_{++}$ such that
$$\begin{bmatrix} h^T A + q_z^T C & h^T B + q_z^T D - \gamma q_w^T \end{bmatrix} < 0. \tag{4}$$
(iii) The matrix $A \in \mathbb{M}^n$ is Hurwitz stable and the following inequality holds:
$$q_z^T G(0) < \gamma q_w^T. \tag{5}$$
Here, $G(s)$ is the transfer matrix of the system G.

The following two corollaries are direct consequences of the condition (iii) in Theorem 3.

Corollary 1: For the positive system G described by (2) with $A \in \mathbb{M}^n$ being Hurwitz stable, the L_1 gain $\|G_{q_z,q_w}\|_{1+}$ is given by

$$\|G_{q_z,q_w}\|_{1+} = \min\ \gamma \text{ subject to } q_z^T G(0) \leq \gamma q_w^T \tag{6}$$

or equivalently,

$$\|G_{q_z,q_w}\|_{1+} = \max_i \frac{(q_z^T G(0))_i}{q_{w,i}}. \tag{7}$$

Corollary 2: For the positive system G described by (2) with $A \in \mathbb{M}^n$ being Hurwitz stable, the L_1 gain $\|G_{q_z,q_w}\|_{1+}$ is finite for any fixed $q_z > 0$, $q_w > 0$.

In relation to the above corollary, we define $\|G_{q_z,q_w}\|_{1+} = \infty$ for any $q_z > 0$ and $q_w > 0$ if the positive system G is *unstable* (i.e, the matrix $A \in \mathbb{M}^n$ is not Hurwitz stable).

IV. Stability Analysis of Interconnected Positive Systems

In this section, we analyze stability of interconnected positive systems. It turns out that the L_1 gain with weightings introduced in the preceding section plays an important role.

A. Interconnection of Two Positive Systems

Let us consider two positive systems G_1 and G_2 represented by

$$G_1 : \left\{ \begin{array}{rcccl} \dot{x}_1 & = & A_1 x_1 & + & B_1 u_1, \\ y_1 & = & C_1 x_1 & + & D_1 u_1, \end{array} \right.$$
$$G_2 : \left\{ \begin{array}{rcccl} \dot{x}_2 & = & A_2 x_2 & + & B_2 u_2, \\ y_2 & = & C_2 x_2 & + & D_2 u_2. \end{array} \right. \tag{8}$$

We consider the case where the inputs and the outputs have compatible dimensions so that the feedback-connection shown Fig. 1 with $u_1 = y_2$ and $u_2 = y_1$ is well-defined. For conciseness, we denote by $\mathop{\Xi}_{i=1}^{2} G_i$ the interconnected system. In relation to the well-posedness of the feedback-connection, we make the next definition.

Definition 3: The interconnected system $\mathop{\Xi}_{i=1}^{2} G_i$ is said to be *admissible* if the Metzler matrix $D_1 D_2 - I$ is Hurwitz stable.

In the sequel, we require the admissibility of the interconnected system $\mathop{\Xi}_{i=1}^{2} G_i$ whenever we analyze its stability. The meaning of this presupposition, and its rationality as well, can be explained as follows. If $\det(D_1 D_2 - I) \neq 0$, then the interconnection in Fig. 1 is well-posed, and the state-space description of the interconnected system is represented by (9) given at the top of the next page. Thus, if the admissibility is ensured, we see that

(i) the interconnection in Fig. 1 is well-posed;
(ii) in addition to $D_1 D_2 - I$, the Metzler matrix $D_2 D_1 - I$ is Hurwitz as well, and hence $(I - D_1 D_2)^{-1} \geq 0$ and $(I - D_2 D_1)^{-1} \geq 0$ hold. Therefore the matrix $A_{\rm cl}$ in (9) is Metzler. It follows that the positive nature of G_1 and G_2, i.e., the positivity of the states x_1 and x_2, is still preserved under the interconnection.

Now, we are ready to state the main result of this paper. In the next theorem, we will show that the admissibility and stability of $\mathop{\Xi}_{i=1}^{2} G_i$ can be fully characterized by the L_1 gain defined in the preceding subsection.

Theorem 4: Let us consider the positive systems G_1 and G_2 described by (8). Then, the following conditions are equivalent:

(i) The interconnected system $\mathop{\Xi}_{i=1}^{2} G_i$ is admissible and Hurwitz stable.
(ii) The Metzler matrices A_1 and A_2 are Hurwitz stable, and there exist $\widetilde{q}_1 > 0$ and $\widetilde{q}_2 > 0$ such that $\|G_{1,\widetilde{q}_1,\widetilde{q}_2}\|_{1+} \|G_{2,\widetilde{q}_2,\widetilde{q}_1}\|_{1+} < 1$.
(iii) There exist $q_1 > 0$ and $q_2 > 0$ such that $\|G_{1,q_1,q_2}\|_{1+} < 1, \quad \|G_{2,q_2,q_1}\|_{1+} < 1$.
(iv) There exist $h_1 > 0,\ h_2 > 0$ and $q_1 > 0,\ q_2 > 0$ such that
$$\left[\begin{array}{cc} h_1^T A_1 + q_1^T C_1 & h_1^T B_1 + q_1^T D_1 - q_2^T \end{array}\right] < 0,$$
$$\left[\begin{array}{cc} h_2^T A_2 + q_2^T C_2 & h_2^T B_2 + q_2^T D_2 - q_1^T \end{array}\right] < 0.$$
(v) The Metzler matrices A_1, A_2, and $G_1(0)G_2(0) - I$ (or equivalently, $G_2(0)G_1(0) - I$) are all Hurwitz stable.
(vi) The Metzler matrix
$$\Pi := \left[\begin{array}{cccc} A_1 & 0 & 0 & B_1 \\ 0 & A_2 & B_2 & 0 \\ C_1 & 0 & -I & D_1 \\ 0 & C_2 & D_2 & -I \end{array}\right]$$
is Hurwitz stable.

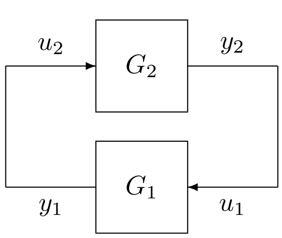

Fig. 1. Interconnected system $\mathop{\Xi}_{i=1}^{2} G_i$.

In this theorem, the equivalence of (ii) and (iii) can be seen straightforwardly. Indeed, the implication (iii) $\Rightarrow$ (ii) is obvious. On the other hand, if (ii) holds, then there exist $\gamma > 0$ such that

$$\|G_{1,\widetilde{q}_1,\widetilde{q}_2}\|_{1+} < \gamma, \quad \|G_{2,\widetilde{q}_2,\widetilde{q}_1}\|_{1+} < \frac{1}{\gamma}. \tag{10}$$

Here we used Corollary 2 implicitly. From (10), we see that the condition (ii) surely holds with, ex., $(q_1, q_2) = (\frac{1}{\gamma}\widetilde{q}_1, \widetilde{q}_2)$ or $(q_1, q_2) = (\widetilde{q}_1, \gamma\widetilde{q}_2)$. The equivalence of (iii) and (iv) is obvious from (ii) of Theorem 3. The equivalence of (iv) and (v) is a direct consequence of (iii) in Theorem 3. Finally, the equivalence of (iv) and (vi) also follows immediately from (iv) of Theorem 2 if we note that the inequalities in (iv) can be restated equivalently as

$$\left[\begin{array}{c} h_1 \\ h_2 \\ q_1 \\ q_2 \end{array}\right]^T \left[\begin{array}{cccc} A_1 & 0 & 0 & B_1 \\ 0 & A_2 & B_2 & 0 \\ C_1 & 0 & -I & D_1 \\ 0 & C_2 & D_2 & -I \end{array}\right] < 0.$$

Therefore, Theorem 4 is verified if we prove (i) $\Leftrightarrow$ (vi).

Before moving onto the proof, it should be noted that Theorem 4 implies that the interconnected system $\mathop{\Xi}_{i=1}^{2} G_i$ is stable *only if* G_1 and G_2 are both stable. This is consistent with the well-known fact that, under positivity-preserving interconnection, the system $\mathop{\Xi}_{i=1}^{2} G_i$ is stable only if G_1 and G_2 are both stable [2], [1].

Proof of (i) $\Leftrightarrow$ (vi) in Theorem 4: It is an elementary fact that the Metzler matrix $\left[\begin{array}{cc} -I & D_1 \\ D_2 & -I \end{array}\right]$ is Hurwitz stable if and only if $D_1 D_2 - I$ is. This is a sub-case of Lemma 1 with $-I$ on the diagonal as well. Therefore the condition (i) can be restated equivalently as the Metzler matrices

$$\left[\begin{array}{cc} -I & D_1 \\ D_2 & -I \end{array}\right]$$

and

$$A_{\rm cl} = \left[\begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array}\right] - \left[\begin{array}{cc} 0 & B_1 \\ B_2 & 0 \end{array}\right] \left[\begin{array}{cc} -I & D_1 \\ D_2 & -I \end{array}\right]^{-1} \left[\begin{array}{cc} C_1 & 0 \\ 0 & C_2 \end{array}\right]$$

are both Hurwitz stable. Thus, the equivalence of (i) and (vi) follows directly from Lemma 1. ■

We have several remarks on Theorem 4. First of all, the condition (ii) can be interpreted as a sort of *small gain condition* that is quite popular in the community of control theory. In the literature, the *gain* is usually defined via the

$$\left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array}\right] = A_{\rm cl}\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right], \quad A_{\rm cl} := \left[\begin{array}{cc} A_1 + B_1(I-D_2D_1)^{-1}D_2C_1 & B_1(I-D_2D_1)^{-1}C_2 \\ B_2(I-D_1D_2)^{-1}C_1 & A_2 + B_2(I-D_1D_2)^{-1}D_1C_2 \end{array}\right]. \tag{9}$$

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L_2 induced norm and plenty of results have been obtained for stability analysis of interconnected systems [8]. We have shown that, if we focus on the interconnection of positive systems, the small-gain-type condition can be obtained even if we replace the common L_2 gain by the L_1 gain with weightings. On the other hand, if we rewrite (ii) as in (iii) or (iv), we see that the positive vectors (q_1, q_2) work as if they are *separators*, which again have played a crucial role in the stability analysis of interconnected systems [4], [5], [7]. We emphasize that the separator-based conditions (ii), (iii) and (iv) are necessary and sufficient, and this strong result is far from being easily achievable for general linear system analysis.

The conditions in Theorem 3 with separators (q_1, q_2) are of course of little interest for stability analysis of exactly known systems. However, they are surely effective for robust stability analysis, particularly when the uncertain system of interest is composed of an exactly known positive stable system G and an uncertain system Δ as shown in Fig. 2. As commonly done in the literature discussing separator-based conditions for general linear systems, there are basically two strategies for the use of (iv) in Theorem 4:

1. We fix the separators $q_1 > 0$ and $q_2 > 0$ by taking typical properties of the uncertain component Δ into consideration. In this case, robust stability analysis amounts to examining only the L_1 gain condition for exactly known system, with fixed $q_1 > 0$ and $q_2 > 0$.
2. We jointly seek for the separators $q_1 > 0$, $q_2 > 0$ as well as for $h_1 > 0$, $h_2 > 0$. Even for those robust stability analysis problems where direct analysis on the closed-loop system is difficult, it is often the case that we can obtain numerically tractable conditions by the separator-based results in Theorem 4.

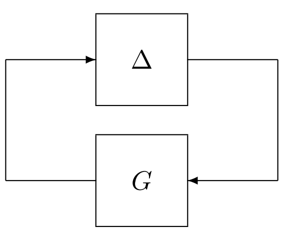

Fig. 2. Interconnection between exactly known G and uncertain Δ.

In the rest of this subsection, let us focus on the robust stability analysis along the first line stated above. To this end, we define the following two sets of uncertainties.

Definition 4: For given $q_1 \in \mathbb{R}^l_{++}$ and $q_2 \in \mathbb{R}^m_{++}$, we define $\boldsymbol{\Delta}^{\rm dy}_{q_2,q_1}$ and $\boldsymbol{\Delta}^{\rm st}_{q_2,q_1}$ by

$$\boldsymbol{\Delta}^{\rm dy}_{q_2,q_1} := \{\Delta(s):\ \text{positive,}\ \ \text{stable, and LTI with}\ \|\Delta_{q_2,q_1}\|_{1+} \le 1\},$$
$$\boldsymbol{\Delta}^{\rm st}_{q_2,q_1} := \left\{\Delta \in \mathbb{R}^{m\times l}_+ :\ \|\Delta_{q_2,q_1}\|_{1+} \le 1\right\}.$$

By this definition, we characterized uncertainties in terms of their L_1 gain with weighting vectors q_1 and q_2. In the following, we analyze robust stability of the closed-loop system Fig. 2 against these uncertainties. We assume that the exactly known component G is a positive, stable and LTI system with coefficient matrices $A \in \mathbb{M}^n$, $B \in \mathbb{R}^{n\times m}_+$, $C \in \mathbb{R}^{l\times n}_+$, $D \in \mathbb{R}^{l\times m}_+$. Under these preliminaries, we first show that the following strong theorem holds.

Theorem 5: For given $q_1 \in \mathbb{R}^l_{++}$ and $q_2 \in \mathbb{R}^m_{++}$, the closed-loop system in Fig. 2 is admissible and stable for all $\Delta \in \boldsymbol{\Delta}^{\rm dy}_{q_2,q_1}$ *if and only if* $\|G_{q_1,q_2}\|_{1+} < 1$ holds, or equivalently, there exists $h > 0$ such that

$$\left[\begin{array}{cc} h^TA + q_1^TC & h^TB + q_1^TD - q_2^T \end{array}\right] < 0. \tag{11}$$

Proof of Theorem 5: Sufficiency is obvious from (ii) of Theorem 4. To prove the necessity by contradiction, suppose (11) does not hold. Then, from the condition (iii) in Theorem 3, we see that

$$(q_1^TG(0))_{j^\star} \ge q_{2,j^\star} \tag{12}$$

holds for at least one index $j^\star$. If we define

$$\Delta^\star := \frac{1}{q_{2,j^\star}} e_{j^\star} q_1^T \in \mathbb{R}^{m\times l}_+,$$

it is obvious that $\Delta^\star_{q_2,q_1} = 1$ and hence $\Delta^\star \in \boldsymbol{\Delta}^{\rm dy}_{q_1,q_2}$. Furthermore, we obtain from (12) that

$$q_1^TG(0)\Delta^\star = (q_1^TG(0))_{j^\star}\frac{1}{q_{2,j^\star}}q_1^T \ge q_1^T.$$

This clearly shows that that $G(0)\Delta^\star - I$ is *not* Hurwitz stable. From (v) of Theorem 4, this implies that for the closed-loop system with $\Delta = \Delta^\star$, at least one of the admissibility and stability requirements is violated. ■

It is important to note that, in the necessity part of the above proof, we have shown that the worst-case uncertainty that destabilizes the closed-loop system and/or violates the admissibility condition is always chosen as a nonnegative matrix $\Delta^\star \in \mathbb{R}^{m\times l}_+$ (rather than a dynamical positive system). The next corollary readily follows from this fact.

Corollary 3: For given $q_1 \in \mathbb{R}^l_{++}$ and $q_2 \in \mathbb{R}^m_{++}$, the closed-loop system in Fig. 2 is admissible and stable for all $\Delta \in \boldsymbol{\Delta}^{\rm st}_{q_2,q_1}$ *if and only if* $\|G_{q_1,q_2}\|_{1+} < 1$.

We note that the sufficiency of this corollary is obvious from Theorem 5. What is important is that the necessity still holds even if we restrict the class of the uncertainty from $\Delta \in \boldsymbol{\Delta}^{\rm dy}_{q_2,q_1}$ to $\Delta \in \boldsymbol{\Delta}^{\rm st}_{q_2,q_1}$.

B. Interconnection of N Positive Systems

We next consider the interconnection of $N(\ge 3)$ positive systems. Surprisingly enough, it turns out that we can still derive a stability condition corresponding to (iii) (and hence (iv)) in Theorem 4. This is in sharp contrast with the case where we deal with general (non-positive) linear systems.

In order to deal with general interconnections among N subsystems while to facilitate our notations, let us assume that the i-th subsystem G_i is given by

$$G_i : \begin{cases} \dot{x}_i &= A_i x_i + \sum_{j=1}^{N} B_{ij} u_{ij}, \\ y_{ji} &= C_{ji} x_i \end{cases} \tag{13}$$

where $A_i \in \mathbb{M}^{n_i}$, $B_{ij} \in \mathbb{R}_+^{n_i \times n_{u_{ij}}}$, and $C_{ji} \in \mathbb{R}_+^{n_{y_{ji}} \times n_i}$. We consider the case where these N subsystems are interconnected by $u_{ij} = y_{ij}$ $(i, j = 1, \cdots, N)$. This implies that $n_{u_{ij}} = n_{y_{ji}}$ hold for the dimension of the signals u_{ij} and y_{ji} $(i, j = 1, \cdots, N)$.

For example, in the case where $N = 3$, the state space realization of G_1 is given by

$$\dot{x}_1 = A_1 x_1 + \begin{bmatrix} B_{11} & B_{12} & B_{13} \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \end{bmatrix},$$
$$\begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix} x_1$$

and the overall interconnected system is represented by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_1 + B_{11}C_{11} & B_{12}C_{12} & B_{13}C_{13} \\ B_{21}C_{21} & A_2 + B_{22}C_{22} & B_{23}C_{23} \\ B_{31}C_{31} & B_{32}C_{32} & A_3 + B_{33}C_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \tag{14}$$

It should be noted that, in (13), we assume that each subsystem has its own minor feedback. This assumption has been made just for notational simplicity and hence in practice we can let $B_{ii} = 0$ and $C_{ii} = 0$ $(i = 1, \cdots, N)$. We also note that the admissibility issue does not appear here since we assume that the direct-feedthrough term of each subsystem is zero in (13).

If we denote by $\overset{N}{\underset{i=1}{\Xi}} G_i$ the interconnected positive system of interest, we can prove that the next theorem holds.

Theorem 6: Let us consider the N positive systems G_i given in (13). Then, the following conditions are equivalent:

(i) The interconnected positive system $\overset{N}{\underset{i=1}{\Xi}} G_i$ is stable.

(ii) There exists $q_{ij} \in \mathbb{R}_{++}^{n_{u_{ij}}}$ $(i, j = 1, \cdots, N)$ such that $\|G_{i,q_{i,z},q_{i,w}}\|_{1+} < 1 \quad (i = 1, \cdots, N)$ where
$$q_{i,z} = \begin{bmatrix} q_{1i} \\ \vdots \\ q_{Ni} \end{bmatrix}, \quad q_{i,w} = \begin{bmatrix} q_{i1} \\ \vdots \\ q_{iN} \end{bmatrix}.$$

(iii) There exists $h_i \in \mathbb{R}_{++}^{n_i}$ $(i = 1, \cdots, N)$ and $q_{ij} \in \mathbb{R}_{++}^{n_{u_{ij}}}$ $(i, j = 1, \cdots, N)$ such that
$$\begin{aligned} & h_i^T A_i + \sum_{j=1}^{N} q_{ji}^T C_{ji} < 0, \\ & h_i^T B_{ij} - q_{ij}^T < 0 \quad (i, j = 1, \cdots, N). \end{aligned} \tag{15}$$

To see the conditions in this theorem more concretely, let us consider the case $N = 3$ for example. Then, the conditions in (ii) can be written as

$$\begin{aligned} &\|G_{1,[\, q_{11}^T \; q_{21}^T \; q_{31}^T \,]^T,[\, q_{11}^T \; q_{12}^T \; q_{13}^T \,]^T}\|_{1+} < 1, \\ &\|G_{2,[\, q_{12}^T \; q_{22}^T \; q_{32}^T \,]^T,[\, q_{21}^T \; q_{22}^T \; q_{23}^T \,]^T}\|_{1+} < 1, \\ &\|G_{3,[\, q_{13}^T \; q_{23}^T \; q_{33}^T \,]^T,[\, q_{31}^T \; q_{32}^T \; q_{33}^T \,]^T}\|_{1+} < 1. \end{aligned}$$

On the other hand, the conditions in (iii) become

$$\begin{aligned} & h_1^T A_1 + \begin{bmatrix} q_{11} \\ q_{21} \\ q_{31} \end{bmatrix}^T \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix} < 0, \ h_1^T \begin{bmatrix} B_{11} & B_{12} & B_{13} \end{bmatrix} - \begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \end{bmatrix}^T < 0, \\ & h_2^T A_2 + \begin{bmatrix} q_{12} \\ q_{22} \\ q_{32} \end{bmatrix}^T \begin{bmatrix} C_{12} \\ C_{22} \\ C_{32} \end{bmatrix} < 0, \ h_2^T \begin{bmatrix} B_{21} & B_{22} & B_{23} \end{bmatrix} - \begin{bmatrix} q_{21} \\ q_{22} \\ q_{23} \end{bmatrix}^T < 0, \\ & h_3^T A_3 + \begin{bmatrix} q_{13} \\ q_{23} \\ q_{33} \end{bmatrix}^T \begin{bmatrix} C_{13} \\ C_{23} \\ C_{33} \end{bmatrix} < 0, \ h_3^T \begin{bmatrix} B_{31} & B_{32} & B_{33} \end{bmatrix} - \begin{bmatrix} q_{31} \\ q_{32} \\ q_{33} \end{bmatrix}^T < 0. \end{aligned}$$

As noted around (14), if $B_{ii} = 0$ and $C_{ii} = 0$ $(i = 1, 2, 3)$ as usual, the above condition can be simplified as

$$\begin{aligned} & h_1^T A_1 + \begin{bmatrix} q_{21} \\ q_{31} \end{bmatrix}^T \begin{bmatrix} C_{21} \\ C_{31} \end{bmatrix} < 0, \ h_1^T \begin{bmatrix} B_{12} & B_{13} \end{bmatrix} - \begin{bmatrix} q_{12} \\ q_{13} \end{bmatrix}^T < 0, \\ & h_2^T A_2 + \begin{bmatrix} q_{12} \\ q_{32} \end{bmatrix}^T \begin{bmatrix} C_{12} \\ C_{32} \end{bmatrix} < 0, \ h_2^T \begin{bmatrix} B_{21} & B_{23} \end{bmatrix} - \begin{bmatrix} q_{21} \\ q_{23} \end{bmatrix}^T < 0, \\ & h_3^T A_3 + \begin{bmatrix} q_{13} \\ q_{23} \end{bmatrix}^T \begin{bmatrix} C_{13} \\ C_{23} \end{bmatrix} < 0, \ h_3^T \begin{bmatrix} B_{31} & B_{32} \end{bmatrix} - \begin{bmatrix} q_{31} \\ q_{32} \end{bmatrix}^T < 0. \end{aligned}$$

Similarly to the conditions in Theorem 4, the separator-based conditions in Theorem 6 will be effective when we deal with robustness issues. In addition to that, we have a prospect that the conditions in Theorem 6 can be used for LP-based L_1 control system synthesis for large-scale positive systems. This topic is currently under investigation.

V. ROBUST STABILITY ANALYSIS AGAINST PARAMETRIC UNCERTAINTIES

In Subsection IV-A, we consider the robust stability of uncertain system shown in Fig. 2 and derived Theorem 5 and Corollary 3. There, we assumed that the uncertainties are L_1 gain bounded with known weightings q_1 and q_2, and derived necessary and sufficient conditions for the stability against these types of uncertainties. In this section, we consider the case where the uncertainties are *parametric*. More precisely, we focus on the following two classes of uncertainties:

Dynamic Uncertainty $\Delta_\alpha^{\mathrm{dy}}$:

$$\begin{aligned} & \Delta_\alpha^{\mathrm{dy}}(s) := \left[\begin{array}{c|c} A(\alpha) & B(\alpha) \\ \hline C(\alpha) & D(\alpha) \end{array} \right], \\ & \left[\begin{array}{c|c} A(\alpha) & B(\alpha) \\ \hline C(\alpha) & D(\alpha) \end{array} \right] = \sum_{i=1}^{L} \alpha_i \left[\begin{array}{c|c} A_{[i]} & B_{[i]} \\ \hline C_{[i]} & D_{[i]} \end{array} \right]. \end{aligned} \tag{16}$$

Static Uncertainty $\Delta_\alpha^{\mathrm{st}}$:

$$\Delta_\alpha^{\mathrm{st}} := \sum_{i=1}^{L} \alpha_i \Delta_{[i]}. \tag{17}$$

Here, $A_{[i]} \in \mathbb{M}^{\widetilde{n}}$, $B_{[i]} \in \mathbb{R}_+^{l \times \widetilde{n}}$, $C_{[i]} \in \mathbb{R}_+^{\widetilde{n} \times m}$, $D_{[i]} \in \mathbb{R}_+^{m \times l}$ $(i = 1, \cdots, L)$ are known matrices. Similarly, $\Delta_{[i]} \in \mathbb{R}_+^{m \times l}$ $(i = 1, \cdots, L)$ are known precisely. On the other hand, the parameter $\alpha \in \mathbb{R}^L$ is uncertain, and assumed to satisfy $\alpha \in \boldsymbol{\alpha}$ where

$$\boldsymbol{\alpha} := \left\{ \alpha \in \mathbb{R}_+^L : \sum_{i=1}^{L} \alpha_i = 1 \right\}.$$

Similarly to Subsection IV-A, we analyze robust stability of the closed-loop system Fig. 2 against these uncertainties. As before, we assume that the exactly known component G is a positive, stable, and LTI system with coefficients $A \in \mathbb{M}^n$,

$B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{l\times n}$, $D \in \mathbb{R}_+^{l\times m}$. If we directly work on the closed-loop system of the form (9), we have to examine robust stability of a matrix that depends rationally on the uncertain parameter α. In this treatment, it is hard, or at least not easy, to obtain computationally tractable and efficient conditions for robust stability analysis. However, by means of the separator-based conditions in Theorem 4, we can easily obtain tractable linear inequality conditions for the robust stability analysis.

Theorem 7: The closed-loop system in Fig. 2 with $\Delta = \Delta_\alpha^{\mathrm{dy}}$ is admissible and stable for all $\alpha \in \boldsymbol{\alpha}$ *if* there exist $h_p > 0$ and $q > 0,\ q_p > 0$ such that

$$\begin{array}{l} q^T G(0) - q_p^T < 0, \\ \left[\begin{array}{cc} h_p^T A_{[i]} + q_p^T C_{[i]} & h_p^T B_{[i]} + q_p^T D_{[i]} - q^T \end{array}\right] < 0 \\ \qquad\qquad\qquad (i = 1, \cdots, L). \end{array} \tag{18}$$

The validity of this theorem is easily confirmed from (iv) of Theorem 4 and (iii) of Theorem 3 with simple convexity arguments. On the other hand, for the static uncertainty $\Delta_\alpha^{\mathrm{st}}$ $(\alpha \in \boldsymbol{\alpha})$, we can obtain much simpler robust stability condition because of the condition (v) in Theorem 4.

Theorem 8: The closed-loop system in Fig. 2 with $\Delta = \Delta_\alpha^{\mathrm{st}}$ is admissible and stable for all $\alpha \in \boldsymbol{\alpha}$ *if* there exists $q > 0$ such that

$$q^T(\Delta_{[i]} G(0) - I) < 0 \ (i = 1, \cdots, L). \tag{19}$$

Again, this theorem readily follows from simple convexity arguments. In general, the linear inequality condition (19) is conservative and far from necessary. However, if the input of G is scalar, then the term $\Delta_{[i]}G(0)$ in (19) is a scalar as well, and from simple convexity arguments, we see that the closed-loop system in Fig. 2 with $\Delta = \Delta_\alpha^{\mathrm{st}}$ is admissible and stable for all $\alpha \in \boldsymbol{\alpha}$ if and only if (19) holds.

VI. Numerical Example

In this section, we illustrate the effectiveness of the results in this paper via a simple academic example. Let us consider the interconnection shown Fig. 2. The nominal system G is a positive and stable system with coefficient matrices

$$\left[\begin{array}{cc} A & B \\ C & D \end{array}\right] = \left[\begin{array}{ccc:cc} -3.5 & 0.2 & 1.0 & 0.7 & 0.5 \\ 0.5 & -3.3 & 0.8 & 0.4 & 0.4 \\ 0.9 & 0.7 & -3.3 & 0.2 & 0 \\ \hdashline 0.7 & 0.2 & 0.9 & 0 & 0.8 \\ 0.5 & 0.3 & 0.9 & 0.7 & 0 \end{array}\right].$$

We assume that the uncertainty component Δ is given in the form of (16) where $L = 2$ and

$$\left[\begin{array}{cc} A_{[1]} & B_{[1]} \\ C_{[1]} & D_{[1]} \end{array}\right] = \left[\begin{array}{ccc:cc} -2.5 & 0.1 & 0.8 & 0 & 0.2 \\ 0.6 & -2.8 & 0.5 & 0 & 0.4 \\ 0.3 & 0.1 & -2.9 & 0.2 & 0.4 \\ \hdashline 0.5 & 0.1 & 0.5 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.3 & 0.1 & 0.4 \end{array}\right],$$

$$\left[\begin{array}{cc} A_{[2]} & B_{[2]} \\ C_{[2]} & D_{[2]} \end{array}\right] = \left[\begin{array}{ccc:cc} -2.7 & 0.6 & 0.4 & 0.1 & 0.3 \\ 0.5 & -2.7 & 0.6 & 0.3 & 0.2 \\ 0 & 0.2 & -3.0 & 0.4 & 0.3 \\ \hdashline 0.1 & 0.2 & 0.3 & 0.1 & 0.3 \\ 0.1 & 0.4 & 0.3 & 0 & 0.2 \end{array}\right].$$

For robust admissibility and stability analysis of the interconnected system, we examined the feasibility of (18). Then, it turns out to be feasible with

$$h = \begin{bmatrix} 1.07 \\ 1.00 \\ 1.47 \end{bmatrix},\ h_p = \begin{bmatrix} 1.12 \\ 1.00 \\ 1.18 \end{bmatrix},\ q = \begin{bmatrix} 1.18 \\ 2.15 \end{bmatrix},\ q_p = \begin{bmatrix} 2.95 \\ 1.88 \end{bmatrix}.$$

Therefore, we can conclude that the interconnected system is robustly admissible and stable.

In this numerical example, the *standard* L_1 gain for each system is computed as

$$\|G_{\mathbf{1}_2,\mathbf{1}_2}\|_{1+} = 1.43,\ \|G_{[1],\mathbf{1}_2,\mathbf{1}_2}\|_{1+} = 0.76,\ \|G_{[2],\mathbf{1}_2,\mathbf{1}_2}\|_{1+} = 0.67.$$

Since $\|G_{\mathbf{1}_2,\mathbf{1}_2}\|_{1+}\|G_{[1],\mathbf{1}_2,\mathbf{1}_2}\|_{1+} > 1$, we cannot draw any affirmative conclusions from the outset if we employ the standard L_1 gain. However, by jointly searching for the weighings q and q_p, we indeed succeeded in ensuring robust admissibility and stability. For comparison, the L_1 gain with computed q and q_p are given as follows:

$$\|G_{q,q_p}\|_{1+} = 0.92,\ \|G_{[1],q_p,q}\|_{1+} = 0.73,\ \|G_{[2],q_p,q}\|_{1+} = 0.57.$$

VII. Conclusion

In this paper, we investigated L_1 gain analysis problems of positive systems and applied obtained results to stability of interconnected systems. In particular, we have shown that the stability of interconnected systems can be characterized by L_1 gains of subsystems with appropriately selected weightings. These weightings work as separators, and we clarified that those separator-based conditions are surely effective particularly when we deal with robust stability analysis against L_1 gain bounded and parametric uncertainties.

The L_1 gain condition and separator-based stability conditions in this paper are given in terms of linear inequalities. We have a strong prospect that these conditions lead us to LP-based L_1 controller synthesis even under the presence of uncertainties. This topic is currently under investigation.

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