Convex Conditions for Model Reduction of Linear Parameter Varying Systems

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Abstract—**Complexity being one of the main limitations of LPV methods, the need for efficient model reduction techniques is highly motivated. Yet, so far, there exists no convex formulation of the general problem of finding a reduced model of any given complexity. In this paper, we focus on the case when the reduced model is supposed to have a special structure and we then derive convex conditions. Thus, for a system modeled by an LFT on a repeated scalar parameter structure, we prove that the problem can be formulated as an LMI optimization problem in the case when the reduced model is supposed to depend only on some parameters of the original system in the same manner as the plant whereas the dependence on the other parameters has been removed. The method is applicable to quadratically stable systems. A complete construction procedure is provided and a measure of the associated model reduction error is given. The method is illustrated in the context of missile control.**

I. INTRODUCTION

A. Context and problem

LPV synthesis methods have emerged as powerful tools in designing controllers for nonlinear or time-varying systems [1]. A large amount of research has been devoted to their refining: while the oldest and simplest methods make use of a constant Lyapunov matrix to obtain convex conditions [2], [3], [4], the efforts for reducing the conservatism have led to consider more complex parameter-dependent Lyapunov functions [5], [6]. The methods are now considered to have reached a theoretically mature state. However, despite promising features, their use in practice remains limited. One of the main criticisms is the fact that they often result in controllers of high complexity, thus requiring expensive implementation. Indeed, LPV methods typically lead to controllers whose structure mimics the plant structure so that even in the simplest methods using a constant Lyapunov function, the controllers have at least the same complexity as the plant, see *e.g.*, methods to deal with polytopic systems [7], [8] or rational systems [2], [3], [4]. The need for efficient model reduction techniques is thus highly motivated. In this paper, we focus on the model reduction problem and we obtain a convex formulation in the case when the dependence of the reduced model on the parameters is chosen in an appropriate fashion.

B. Previous work

The literature [9], [4], [10] investigates the problem of finding a reduced model of any given complexity. Most of these papers provide methods based on a generalization of balanced truncation model reduction methods. Unfortunately, they fail to lead to a convex formulation of the problem. The method in [9] for example consists in solving an optimization problem expressed by LMI Lyapunov inequalities coupled with a nonconvex rank constraint.

C. Proposed approach

This paper considers a particular case of a reduced model structure: it is supposed to depend on some parameters in the same manner as the plant and no longer at all on the other parameters. The studied problem is to find such a reduced model minimizing the $\mathcal{L}_2$-gain of the difference system. Conditions are derived directly by exploiting the parallel with an LPV synthesis problem [4]. It is then proved that this particular problem can be expressed as a convex optimization problem. A practical procedure for constructing the reduced complexity model is given, based on the resolution of another LMI optimization problem. The result applies to quadratically stable systems and can easily be extended to quadratically stabilizable and detectable systems using the coprime factorization approach proposed in [11].

D. Interests of the result

The problem considered naturally finds an interest when dealing with the case of LPV systems depending on both slowly and fast-varying parameters, see *e.g.*, in the missile model of Reichert [12], [7], [13]. To reduce the complexity, such a model would usually be simplified by arbitrarily freezing the slowly-varying parameters. An interesting question is whether an optimal reduced model can be rather obtained: our method can lead to a reduced model where the dependence on some parameters has been optimally removed while the dependence on the other parameters has been preserved.

The method is also well suited to *a posteriori* simplify controllers that vary little although they are of high complexity. A noticed phenomenon is indeed the fact that LPV methods may lead to controllers that seem not to vary much. The method proposed here makes it possible to find a "best" model reduction of the controller, easier to implement and expected to give similar performance.

The problem can furthermore be transposed to the nonlinear context. A similar issue is considered in [14] where

the focus is on "mildly" nonlinear systems in the scope of controlling them linearly: the problem there is to approximate the nonlinear system by a "best" linear model and to find a corresponding linear controller which is then ensured to work also on the original nonlinear system. Our method allows to deal efficiently with this problem: actually, since nonlinear systems can be modeled as LPV systems by embedding the nonlinearities in newly defined parameters, our method applies directly to nonlinear systems. It is then an alternative to the procedure of [14], having moreover the advantages that it relies on a convex formulation of the problem and that it leads to a reduced system depending only on some of the parameters (or in the nonlinear context, only on some of the nonlinearities).

E. Structure of the paper

The paper is structured as follows. In Section II the considered system is introduced and a general statement of the model reduction problem is proposed. A convex test of existence of a model reduction for quadratically stable systems and a construction methodology are provided in Section III. Section IV illustrates the method in the context of a missile control.

F. Notations and definitions

The identity matrix of $\mathbb{R}^{n \times n}$ is denoted I_n and the zero matrix of $\mathbb{R}^{n \times m}$ is denoted $0_{n \times m}$. The subscripts are omitted when obvious from context. For two operators A and B, $\mathbf{diag}(A, B)$ denotes the operator $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix} \right]$. For a full-rank matrix U, $U^{\perp}$ denotes an orthogonal complement of U, i.e., $U U^{\perp} = 0$ and $\left[\begin{smallmatrix} U^T & U^{\perp} \end{smallmatrix} \right]$ is of maximal rank, while U^{+} denotes the Moore-Penrose inverse of U. For $X \in \mathbb{R}^{n \times m}$ and $k \leq l \leq n, r \leq s \leq m$ $X_{[k:l][r:s]}$ denotes the matrix extracted from X made of its lines from k to l and columns from r to s. For a square matrix M, $M > 0$ and $M \geq 0$ mean respectively positive and semi-positive definiteness. The symbol $\int$ denotes the integration operation. For a matrix M partitioned as $\left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix} \right]$ and an operator Δ, the notation $\mathcal{F}_u(M, \Delta)$ stands for $M_{22} + M_{21}\Delta(I - \Delta M_{11})^{-1}M_{12}$ and $\mathcal{F}_l(M, \Delta)$ stands for $M_{11} + M_{12}\Delta(I - \Delta M_{22})^{-1}M_{21}$ whenever they exist. The $\mathcal{L}_2$ norm of a signal w from $\mathbb{R}^+$ in $\mathbb{R}^n$, if it exists, is defined as the integral: $||w||_2 = \sqrt{\int_0^{+\infty} w(t)^T w(t) dt}$ and the set of signals for which the $\mathcal{L}_2$ norm is defined is noted $\mathcal{L}_2$. The extended space $\mathcal{L}_2^e$ is then defined as $\mathcal{L}_2^e = \{ w : \mathbb{R}^+ \to \mathbb{R}^n | \forall T > 0, \ P_T(w) \in \mathcal{L}_2 \}$ where for a given signal w and a $T > 0$, the causal truncation operator P_T is such that $\forall t \leq T, \ P_T(w(t)) = w(t)$ and $\forall t > T, \ P_T(w(t)) = 0$. The $\mathcal{L}_2$-gain of an operator H is defined as $||H||_2 = \sup_{w \in \mathcal{L}_2, u \neq 0} ||Hw||_2 / ||w||_2$. For matrices $Z_1, \cdots, Z_r$ where for all $i \in \{1 \cdots r\}$ $Z_i \in \mathbb{R}^{k_i \times k_i}$, the following notation is used: for a given integer $s \leq r$, $\hat{Z}_{1,s} = \mathbf{diag}(Z_1, \cdots, Z_s)$ and $\hat{Z}_{s+1,r} = \mathbf{diag}(Z_{s+1}, \cdots, Z_r)$.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. The considered system

General LPV systems can be described as follows:

$$\begin{cases} \dot{x}(t) & = & A(\delta(t))x(t) + B(\delta(t))w(t) \\ z(t) & = & C(\delta(t))x(t) + D(\delta(t))w(t), \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^{n_w}$ the input and $z(t) \in \mathbb{R}^{n_z}$ the output. The vector $\delta(t) = \left[\begin{array}{ccc} \delta_1(t) & \cdots \delta_r(t) \end{array} \right]^T \in \mathbb{R}^r$ is called the parameter vector, where for $i \in \{1, \cdots, r\}$ $\delta_i(t)$ is a real time-varying scalar parameter measured in real time and belonging to an interval. With no loss of generality, here it is considered that $\delta_i(t) \in [-1, 1]$. This paper is concerned with LPV systems whose state-space matrices are rational functions of the parameters. Such systems can be represented by an LFT on a parameter block structure [4]:

$$\left[\begin{array}{c} \dot{x}(t) \\ q(t) \\ z(t) \end{array} \right] = M \left[\begin{array}{c} x(t) \\ p(t) \\ w(t) \end{array} \right], \quad p(t) = \Delta(t)q(t), \quad (2)$$

where M is a constant matrix and $\Delta(t)$ is called the parameter block. The signals $q(t)$ and $p(t) \in \mathbb{R}^k$ are called respectively the input and the output of the parameter block. The system matrices are defined as:

$$M = \left[\begin{array}{ccc} A & B_0 & B_1 \\ C_0 & D_{00} & D_{01} \\ C_1 & D_{10} & D_{11} \end{array} \right]$$

and the system (2) is described in LFT notation by:

$$z(t) = \mathcal{F}_u \left(\mathcal{F}_u \left(M, \int I_n \right), \Delta(t) \right) w(t). \quad (3)$$

The parameter block Δ is a block diagonal matrix: $\Delta(t) = \mathbf{diag}(\Delta_1(t), \cdots, \Delta_r(t))$, where each sub-block is $\Delta_i(t) = \delta_i(t)I_{k_i}$. The dimension of $\Delta(t)$, also referred to as the LPV system complexity, is then $k = \sum_{i=1}^r k_i$. The following notation is used: $\underline{S}(\Delta_i) = \{S_i \in \mathbb{R}^{k_i \times k_i} | S_i = S_i^T > 0\}$, $\underline{G}(\Delta_i) = \{G_i \in \mathbb{R}^{k_i \times k_i} | G_i = -G_i^T\}$, $\underline{S}(\Delta) = \{S | S = \mathbf{diag}(S_1, \cdots, S_r)\}$, $\underline{G}(\Delta) = \{G | G = \mathbf{diag}(G_1, \cdots, G_r)\}$.

B. The general model reduction problem

In this section, the general definition of a model reduction for a LPV system is given and the general model reduction problem is stated. Next, the particular reduced model structure considered in this paper is introduced and interpreted.

Definition 2.1 (Reduced-complexity model):
Let $\Delta(t) = \mathbf{diag}(\delta_1(t)I_{k_1}, \cdots, \delta_r(t)I_{k_r})$, $\Delta_R(t) = \mathbf{diag}(\delta_1(t)I_{k_{R_1}}, \cdots, \delta_r(t)I_{k_{R_r}})$, $M \in \mathbb{R}^{(n+k+n_z) \times (n+k+n_w)}$, $M_R \in \mathbb{R}^{(n_R+k_R+n_{z_R}) \times (n_R+k_R+n_w)}$ where $n_R \leq n$ and $k_R = \sum_{i=1}^r k_{R_i}$. The system:

$$\mathcal{F}_u \left(\mathcal{F}_u \left(M_R, \int I_{n_R} \right), \Delta_R(t) \right) \quad (4)$$

is a reduced-complexity model of the system $\mathcal{F}_u \left(\mathcal{F}_u \left(M, \int I_n \right), \Delta(t) \right)$ if $n_{z_R} = n_z$ and for all $i \in \{1, \cdots, r\}$ $N_{R_i} \leq N_i$. ◇

Here, a reduced-complexity model is also referred to as a "reduced model". For a given system (3), the model reduction problem is then to find a reduced model (4) that approximates (3) "in some sense". In order to evaluate this approximation, a measure needs to be introduced: it is called the model reduction error and it can be defined as a difference system $\mathcal{L}_2$-gain. For this general problem, no convex formulation is available.

Here, we consider the case when the reduced model is enforced to have a special structure. Thus, consider the original system (3) where $\Delta(t) = \mathbf{diag}(\delta_1(t)I_{k_1}, \cdots, \delta_r(t)I_{k_r})$. For a given integer $s \leq r$, the reduced model is enforced to be of the form (4) where $\Delta_R(t) = \mathbf{diag}(\delta_1(t)I_{k_{R_1}}, \cdots, \delta_r(t)I_{k_{R_r}})$ is such that for every $i \in \{1, \ldots, r\}$:

$$k_{R_i} = \begin{cases} k_i & \forall i \in \{1, \ldots, s\} \\ 0 & \forall i \in \{s+1, \ldots, r\}. \end{cases} \quad (5)$$

In other words, the parameter block of the reduced model is supposed to be a block diagonal structure formed exclusively from full copies of some of the plant parameter sub-blocks. This can be interpreted by saying that the reduced model is enforced to depend on some of the plant parameters in the same fashion as the plant (that is, through an LFT of same complexity) while the dependence on the other parameters has been removed. With no loss of generality, we assume that the removed parameters are the last ones.

III. CONVEX CONDITIONS FOR MODEL REDUCTION OF QUADRATICALLY STABLE SYSTEMS

In this section, the considered model reduction problem is formally stated. A convex formulation of the problem is then derived and a construction method is presented. Let us recall first the definition of quadratic stability.

Definition 3.1: The LPV system defined by the equations (1) is said to be quadratically stable if it is well-posed and there exists a matrix $P = P^T > 0$, called (constant) Lyapunov matrix such that there exists $\eta > 0$ such that $A(\delta(t))^T P + PA(\delta(t)) < -\eta I$. $\diamond$

Recall that the system (1) is said to be well-posed if for any input $w \in \mathcal{L}_2^e$, the signals x, z are in $\mathcal{L}_2^e$ and uniquely defined.

A. The considered model reduction problem for quadratically stable systems

For quadratically stable systems, the model reduction error is the $\mathcal{L}_2$-gain of the difference between the original model and the reduced model. The problem is then the next one.

Problem 3.1 (Model reduction problem): Let $\epsilon > 0$ and consider the quadratically stable system defined for any input $w(t)$ as $z(t) = \mathcal{F}_u\left(\mathcal{F}_u\left(M, \int I_n\right), \Delta(t)\right)w(t)$ (3) where $\Delta(t) = \mathbf{diag}(\delta_1(t)I_{k_1}, \cdots, \delta_r(t)I_{N_r})$. Let $s \leq r$ and $\Delta_R(t) = \mathbf{diag}(\delta_1(t)I_{k_{R_1}}, \cdots, \delta_r(t)I_{k_{R_r}})$ such that for every $i \in \{1, \ldots, r\}$:

$$k_{R_i} = \begin{cases} k_i & \forall i \in \{1, \ldots, s\} \\ 0 & \forall i \in \{s+1, \ldots, r\} \end{cases}$$

and $k_R = \sum_{i=1}^{s} k_i$. Find $n_R \leq n$ and a constant matrix $M_R \in \mathbb{R}^{(n_R + k_R + n_z) \times (n_R + k_R + n_w)}$ such that

the system defined for any input $w(t)$ as $z_R(t) = \mathcal{F}_u\left(\mathcal{F}_u\left(M_R, \int I_{n_R}\right), \Delta_R(t)\right)w(t)$ (4) is a reduced model of (3) such that $||z - z_R||_2 < \epsilon||w||_2$. $\circ$

B. Existence test as an LMI optimization problem for quadratically stable systems

In this section, an existence test of a solution to the model reduction problem for quadratically stable systems is proposed as a convex optimization problem. The main result is given in Theorem 3.2. Recall that for matrices $Z_1, \cdots, Z_r$, for a given $s \leq r$ we use the notation: $\hat{Z}_{1,s} = \mathbf{diag}(Z_1, \cdots, Z_s)$ and $\hat{Z}_{s+1,r} = \mathbf{diag}(Z_{s+1}, \cdots, Z_r)$.

Theorem 3.2: There exists a solution to the (model reduction) problem 3.1 if there exists a solution to the following LMI feasibility problem: find, if they exist, matrices $P = P^T$, $\widetilde{Q} = \widetilde{Q}^T \in \mathbb{R}^{n \times n}$, $\forall i \in \{1, \cdots, s\}$ $\widetilde{Y}_i \in \mathbb{R}^{k_i \times k_i}$ and $\forall i \in \{1, \cdots, r\}$ $X_i \in \mathbb{R}^{k_i \times k_i}$ such that (6), (7), (8), (9), (10), (11), where:

$$\Psi_1^T \left[\begin{array}{ccc|cc|ccc} 0 & & & & 0 & P & & \\ & 0 & & & 0 & & \hat{X}_{1,s} & \\ & & 0 & & 0 & & & \hat{X}_{s+1,r} \\ \hline 0 & 0 & 0 & \epsilon^{-1}I & 0 & 0 & & 0 \\ \hline P & & & 0 & & 0 & & \\ & \hat{X}_{1,s}^T & & 0 & & & 0 & \\ & & \hat{X}_{s+1,r}^T & 0 & & & & 0 \end{array}\right] \Psi_1 < 0, \quad (6)$$

$$\Psi_2^T \left[\begin{array}{ccc|cc|ccc} 0 & & & \widetilde{Q} & & & 0 & \\ & 0 & & & \hat{\widetilde{Y}}_{1,s} & & 0 & \\ & & 0 & & & \hat{X}_{s+1,r} & 0 \\ \hline \widetilde{Q} & & & 0 & & & 0 & \\ & \hat{\widetilde{Y}}_{1,s}^T & & & 0 & & 0 & \\ & & \hat{X}_{s+1,r}^T & & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon I \end{array}\right] \Psi_2 < 0, \quad (7)$$

$$\widetilde{Q} > 0, \quad (8)$$

$$P - \widetilde{Q} > 0, \quad (9)$$

$$X_i + X_i^T > 0, \ \forall i \in \{1, \cdots, r\}, \quad (10)$$

$$\widetilde{Y}_i + \widetilde{Y}_i^T > 0, \ \forall i \in \{1, \cdots, s\}, \quad (11)$$

where:

$$\Psi_1 = \left[\begin{array}{c|c} A & B_0 \\ \frac{1}{\sqrt{2}}C_0 & \frac{1}{\sqrt{2}}(I + D_{00}) \\ \hline C_1 & D_{10} \\ \hline I & 0 \\ \frac{1}{\sqrt{2}}C_0 & -\frac{1}{\sqrt{2}}(I - D_{00}) \end{array}\right],$$

$$\Psi_2 = \left[\begin{array}{c|c|c} A & B_0 & B_1 \\ -\frac{1}{\sqrt{2}}C_0 & -\frac{1}{\sqrt{2}}(I + D_{00}) & -\frac{1}{\sqrt{2}}D_{01} \\ \hline I & 0 & 0 \\ -\frac{1}{\sqrt{2}}C_0 & \frac{1}{\sqrt{2}}(I - D_{00}) & -\frac{1}{\sqrt{2}}D_{01} \\ 0 & 0 & I \end{array}\right].$$

$\bullet$

C. Proof

Consider the setup of Problem 3.1. The system defined by $z_R(t) = \mathcal{F}_u\left(\mathcal{F}_u\left(M_R, \int I_{n_R}\right), \Delta_R(t)\right)w(t)$ with $\Delta_R(t) = \mathbf{diag}(\Delta_1(t), \cdots, \Delta_s(t))$ is by definition a reduced model of the system defined by $z(t) = \mathcal{F}_u\left(\mathcal{F}_u\left(M, \int I_n\right), \Delta(t)\right)w(t)$ with $\Delta(t) = \mathbf{diag}(\Delta_1(t), \cdots, \Delta_r(t))$. Let us prove then the

convex conditions of Theorem 3.2 for the existence of M_R such that for a given $\epsilon > 0$, $||z - z_R||_2 < \epsilon ||w||_2$.

Let $\Delta_\Sigma(t) = \mathbf{diag}\,(\Delta_1(t), \Delta_1(t), \cdots, \Delta_s(t), \Delta_s(t), \cdots$ $\cdots, \Delta_{s+1}(t), \cdots, \Delta_r(t))$ and let the matrix Σ be such that $z(t) - z_R(t) = \mathcal{F}_u\left(\mathcal{F}_u\left(\Sigma, \int I_{n+n_R}\right), \Delta_\Sigma(t)\right) w(t)$. Observing that $\Sigma = \mathcal{M} + D_{R_z} M_R D_{R_w}$ where $\mathcal{M} = D_z M D_w$ and:

$$D_z = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & I_{n_z} \end{array}\right], \quad D_w = \left[\begin{array}{c|c} D^T & 0 \\ \hline 0 & I_{n_w} \end{array}\right], \quad (12)$$

$$D = \left[\begin{array}{ccccc|c} \begin{matrix} I_n \\ 0_{n\times n} \end{matrix} & & & & & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ & \begin{matrix} I_{N_1} \\ 0_{N_1\times N_1} \end{matrix} & & & & \\ 0 & & \ddots & & & 0 \\ & & & \begin{matrix} I_{N_s} \\ 0_{N_s\times N_s} \end{matrix} & & \\ & & & & I_{N_{s+1}} & \\ 0 & & 0 & & & \ddots \\ & & & & & I_{N_r} \end{array}\right],$$

$$D_{R_z} = \left[\begin{array}{c|c} D_R & 0 \\ \hline 0 & I_{n_z} \end{array}\right], \quad D_{R_w} = \left[\begin{array}{c|c} D_R^T & 0 \\ \hline 0 & -I_{n_w} \end{array}\right],$$
$$(13)$$

with:

$$D_R = \left[\begin{array}{ccccc} \begin{matrix} 0_{n\times n} \\ I_n \end{matrix} & & & & \begin{matrix} 0 \\ 0 \end{matrix} \\ & \begin{matrix} 0_{N_1\times N_1} \\ I_{N_1} \end{matrix} & & & \\ 0 & & \ddots & & \\ & & & \begin{matrix} 0_{N_s\times N_s} \\ I_{N_s} \end{matrix} & \\ & & & & 0_{N_{s+1}\times(N_1+\ldots+N_s)} \\ 0 & & & & \vdots \\ & & & & 0_{N_r\times(N_1+\ldots+N_s)} \end{array}\right],$$

Lemma 3.2 in [4] implies that M_R is such that $||z - z_R||_2 < \epsilon||w||_2$ if there exist matrices $\overline{P} = \overline{P}^T > 0 \in \mathbb{R}^{(n+n_R)\times(n+n_R)}$, $\overline{S} = \mathbf{diag}\left(\overline{S}_1, \cdots, \overline{S}_s, S_s, S_{s+1}, \cdots, S_r\right)$ and $\overline{G} = \mathbf{diag}\left(\overline{G}_1, \cdots, \overline{G}_s, G_{s+1}, \cdots, G_r\right)$ where $\forall i \in \{1, \cdots, s\}$ $\overline{S}_i \in \underline{S}(\Delta_i, \Delta_i)$, $\overline{G}_i \in \underline{G}(\Delta_i, \Delta_i)$ and $\forall i \in \{s+1, \cdots, r\}$ $S_i \in \underline{S}(\Delta_i)$ and $G_i \in \underline{G}(\Delta_i)$, such that:

$$\left[\frac{\Sigma}{I}\right]^T \left[\begin{array}{cc|cc} 0 & & \overline{P} & \\ & \overline{S} & & \overline{G} \\ & & \epsilon^{-1}I_{n_z} & \\ \hline \overline{P} & & 0 & \\ & \overline{G}^T & & -\overline{S} \\ & & 0 & -\epsilon I_{n_w} \end{array}\right] \left[\frac{\Sigma}{I}\right] < 0.$$
$$(14)$$

Introducing the partition:

$$\left[\begin{array}{cc|cc} X_1 & & Y_1 & \\ & X_2 & & Y_2 \\ \hline Y_1^T & & Z_1 & \\ & Y_2^T & & Z_2 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & & \overline{P} & \\ & \overline{S} & & \overline{G} \\ & & \epsilon^{-1}I_{n_z} & \\ \hline \overline{P} & & 0 & \\ & \overline{G}^T & & -\overline{S} \\ & & 0 & -\epsilon I_{n_w} \end{array}\right],$$
$$(15)$$

the problem can be rewritten [4] so that (14) holds if and only if:

$$\mathcal{G} + \mathcal{U}^T M_R \mathcal{V} + \mathcal{V}^T M_R^T \mathcal{U} < 0, \qquad (16)$$

where:

$$\mathcal{G} = \left[\begin{array}{ccc} \mathcal{M}_{1,1}^T Y_1 + Y_1^T \mathcal{M}_{1,1} + Z_1 & Y_1^T \mathcal{M}_{1,2} & \mathcal{M}_{2,1}^T \\ \mathcal{M}_{1,2}^T Y_1 & Z_2 - Y_2^T X_2^{-1} Y_2 & (\mathcal{M}_{2,2} + X_2^{-1} Y_2)^T \\ \mathcal{M}_{2,1} & \mathcal{M}_{2,2} + X_2^{-1} Y_2 & -X_2^{-1} \end{array}\right],$$
$$(17)$$

$$\mathcal{U} = \left[\begin{array}{ccc} D_{R_{z1}}^T Y_1 & 0 & D_{R_{z2}}^T \end{array}\right], \quad \mathcal{V} = \left[\begin{array}{cc} D_{R_w} & 0 \end{array}\right] \quad (18)$$

and $\mathcal{M}_{i,j}$ and $D_{R_{z_i}}$ are sub-matrices of $\mathcal{M}$ and D_{R_z} whose dimensions are deductible from context.

The Elimination Lemma [15] implies that M_R exists such that there exist $\overline{P}, \overline{S}, \overline{G}$ such that (16) holds if and only if there exist $\overline{P}, \overline{S}, \overline{G}$ such that:

$$\begin{cases} \mathcal{U}^{\perp\,T} \mathcal{G} \mathcal{U}^\perp < 0 \\ \mathcal{V}^{\perp\,T} \mathcal{G} \mathcal{V}^\perp < 0. \end{cases} \tag{19}$$

Exploiting the particular structure of D and D_R and proceeding to some manipulations leads to the conclusion that there exist matrices $\overline{P}, \overline{S}, \overline{G}$ verifying (19) if and only if there exist matrices $P = P^T, Q = Q^T \in \mathbb{R}^{n\times n}$ and $\forall i \in \{1, \cdots, r\}$ matrices $S_i, T_i \in \underline{S}(\Delta_i)$ and $G_i, H_i \in \underline{G}(\Delta_i)$ such that (20), (21), (22), (23) hold, where:

$$\Phi_1^T \left[\begin{array}{cccc|cccc} 0 & & & & & 0 & P & \\ & \hat{S}_{1,s} & & & & 0 & & \hat{G}_{1,s} \\ & & \hat{S}_{s+1,r} & & & 0 & & & \hat{G}_{s+1,r} \\ \hline 0 & 0 & 0 & \epsilon^{-1}I_{n_z} & 0 & 0 & 0 \\ P & & & & 0 & & \\ & \hat{G}_{1,s}^T & & & 0 & & -\hat{S}_{1,s} \\ & & \hat{G}_{s+1,r}^T & & 0 & & & -\hat{S}_{s+1,r} \end{array}\right] \Phi_1 < 0,$$
$$(20)$$

$$\Phi_2^T \left[\begin{array}{cccc|cccc} 0 & & & & & 0 & Q & \\ & \hat{T}_{1,s} & & & & 0 & & \hat{H}_{1,s} \\ & & \hat{T}_{s+1,r} & & & 0 & & & \hat{H}_{s+1,r} \\ \hline 0 & 0 & 0 & \epsilon^{-1}I_{n_w} & 0 & 0 & 0 \\ Q & & & & 0 & & \\ & \hat{H}_{1,s}^T & & & 0 & & -\hat{T}_{1,s} \\ & & \hat{H}_{s+1,r}^T & & 0 & & & -\hat{T}_{s+1,r} \end{array}\right] \Phi_2 < 0,$$
$$(21)$$

$$\left[\begin{array}{cc} P & I \\ I & Q \end{array}\right] > 0, \qquad (22)$$

$$\left[\begin{array}{cc} \hat{S}_{s+1,r} & \hat{G}_{s+1,r} \\ \hat{G}_{s+1,r}^T & -\hat{S}_{s+1,r} \end{array}\right] \left[\begin{array}{cc} \hat{T}_{s+1,r} & \hat{H}_{s+1,r}^T \\ \hat{H}_{s+1,r} & -\hat{T}_{s+1,r} \end{array}\right] = \left[\begin{array}{cc} I & 0 \\ 0 & I \end{array}\right],$$
$$(23)$$

where:

$$\Phi_1 = \left[\begin{array}{cc} A & B_0 \\ C_0 & D_{00} \\ \hline C_1 & D_{10} \\ \hline I & 0 \\ 0 & I \end{array}\right], \quad \Phi_2 = \left[\begin{array}{cc} A & C_0^T \\ B_0^T & D_{00}^T \\ \hline B_1^T & D_{01}^T \\ \hline I & 0 \\ 0 & I \end{array}\right].$$

This problem is not convex because of (23). Next, let us prove that it is in fact equivalent to an LMI optimization problem with respect to some new unknowns. We proceed to a change of variables: $\forall i \in \{1, \cdots, r\}$, by definition of S_i, T_i, G_i and H_i there exist matrices $X_i, Y_i \in \mathbb{R}^{k_i \times k_i}$ such that $X_i + X_i^T > 0$, $Y_i + Y_i^T > 0$ and $S_i = \frac{1}{2}\left(X_i + X_i^T\right)$, $G_i = \frac{1}{2}\left(X_i^T - X_i\right)$, $T_i = \frac{1}{2}\left(Y_i + Y_i^T\right)$ and $H_i = \frac{1}{2}\left(Y_i^T - Y_i\right)$. Let us rewrite now the terms of the problem with respect to these new variables.

• Replacing S_i and G_i by their expression in terms of X_i and rearranging, (20) reads (6).

• Similarly, replacing T_i and H_i by their expression in terms of Y_i, (21) is rewritten:

$$\Pi^T \left[\begin{array}{cccc|cccc} 0 & & & & & 0 & Q & \\ & 0 & & & & 0 & & \hat{Y}_{1,s} \\ & & 0 & & & 0 & & & \hat{Y}_{s+1,r} \\ \hline 0 & 0 & 0 & \epsilon^{-1}I_{n_w} & 0 & 0 & 0 \\ Q & & & & 0 & & \\ & \hat{Y}_{1,s}^T & & & 0 & & 0 \\ & & \hat{Y}_{s+1,r}^T & & 0 & & & 0 \end{array}\right] \Pi < 0, \quad (24)$$

where:

$$\Pi = \left[\begin{array}{cc|c} A^T & & C_0^T \\ \frac{1}{\sqrt{2}}B_0^T & & \frac{1}{\sqrt{2}}(I+D_{00})^T \\ \hline B_1^T & & D_{01}^T \\ \hline I & & 0 \\ \frac{1}{\sqrt{2}}B_0^T & & -\frac{1}{\sqrt{2}}(I-D_{00})^T \end{array}\right].$$

On the other hand, note that (23) is equivalent to $\hat{Y}_{s+1,r} = \hat{X}_{s+1,r}^{-1}$. Replacing in (24), post-multiplying by $\left[\begin{array}{cc} I & 0 \\ \frac{1}{2}B_0^T & -\frac{1}{2}(I-D_{00})^T \end{array}\right]^{-1}$ and pre-multiplying by its transpose yields (24) if and only if:

$$\overline{M}_\Pi^T \left[\begin{array}{ccc|ccc|ccc} 0 & & & & & & \tilde{Q} & & \\ & 0 & & & & & & \hat{Y}_{1,s} & \\ & & 0 & & & & & & \hat{X}_{s+1,r}^{-1} \\ \hline 0 & 0 & 0 & \epsilon^{-1}I_{n_w} & 0 & 0 & 0 & & \\ \hline P & & & & 0 & & & & \\ & \hat{Y}_{1,s}^T & & & & 0 & & & \\ & & \hat{X}_{s+1,r}^{-T} & & & & 0 & & \end{array}\right] \overline{M}_\Pi < 0,$$

(25)

where $\overline{M}_\Pi = \left[\begin{array}{c} M_\Pi \\ \hline I \end{array}\right]$ and M_Π is defined uniquely by $\left[\begin{array}{c} M_\Pi \\ \hline I \end{array}\right] \left[\begin{array}{cc} I & 0 \\ \frac{1}{2}B_0^T & -\frac{1}{2}(I-D_{00})^T \end{array}\right] = \Pi$. Let $\tilde{Q} = Q^{-1}$ and $\hat{\tilde{Y}}_{1,s} = \hat{Y}_{1,s}^{-1}$. Define $\overline{\overline{M}}_\Pi = \left[\begin{array}{c} M_\Pi^T \\ \hline I\ 0 \\ 0\ I \end{array}\right]$. Then, post-multiplying

(25) by $\mathbf{diag}\left(\tilde{Q}, \hat{\tilde{Y}}_{1,s}, \hat{X}_{1,s}\right)$, pre-multiplying by its transpose and proceeding to some manipulations yields (25) if and only if:

$$\overline{\overline{M}}_\Pi^T \left[\begin{array}{ccc|ccc|c} 0 & & & & \tilde{Q} & & 0 \\ & 0 & & & & \hat{\tilde{Y}}_{1,s} & 0 \\ & & 0 & & & \hat{X}_{s+1,r} & 0 \\ \hline \tilde{Q} & & & 0 & & & 0 \\ & \hat{\tilde{Y}}_{1,s}^T & & & 0 & & 0 \\ & & \hat{X}_{s+1,r} & & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\epsilon I_{n_w} \end{array}\right] \overline{\overline{M}}_\Pi < 0. \quad (26)$$

Post-multiplying by $\left[\begin{array}{ccc} I & 0 & 0 \\ -\frac{1}{2}C_0 & \frac{1}{2}(I-D_{00}) & -\frac{1}{2}D_{01} \\ 0 & 0 & I \end{array}\right]^{-1}$ and pre-multiplying by its transpose finally yields (26) holds if and only (7) holds.

• According to Schur's Lemma, (22) is equivalent to $Q^{-1} > 0$ and $P-Q^{-1} > 0$ *i.e.*, by definition of $\tilde{Q}$, to (8) and (9). To summarize, there exists M_R such that $||z-z_R||_2 < \epsilon||w||_2$ if the LMI optimization problem of Theorem 3.2 admits a solution.

D. Construction

In this section, a procedure for constructing a reduced model is proposed based on the resolution of a second LMI optimization problem. For the system defined for any input $w(t)$ as $z(t) = \mathcal{F}_u\left(\mathcal{F}_u\left(M, \int I_n\right), \Delta(t)\right)w(t)$ (3) with $\Delta(t) = \mathbf{diag}(\Delta_1(t), \cdots, \Delta_r(t))$, of complexity k, let us assume that for $s \leq r$ there exists a reduced model defined for any input $w(t)$ as $z_R(t) = \mathcal{F}_u\left(\mathcal{F}_u\left(M_R, \int I_{n_R}\right), \Delta_R(t)\right)w(t)$ (4) with $\Delta_R(t) = \mathbf{diag}(\Delta_1(t), \cdots, \Delta_s(t))$, of complexity k_R. Let $\{P, \tilde{Q}, \tilde{Y}_1, \cdots, \tilde{Y}_s, X_1, \cdots, X_r\}$ be a solution of the LMI existence test of Theorem 3.2. Define $\Delta_\Sigma(t) = \mathbf{diag}\left(\Delta_1(), \Delta_1(t), \cdots, \Delta_s(t), \Delta_s(t), \Delta_{s+1}(t), \cdots, \Delta_r(t)\right)$ and the matrix Σ such that $z(t) - z_R(t) = \mathcal{F}_u\left(\mathcal{F}_u\left(\Sigma, \int I_{n+n_R}\right), \Delta_\Sigma(t)\right)w(t)$ and recall that $\Sigma = \mathcal{M} + D_{R_z}M_R D_{R_w}^T$ where $\mathcal{M} = D_z M D_w^T$ and the matrices are defined by (12), (13). Next, to construct a reduced complexity model, proceed as follows.

1) Define $Q = \tilde{Q}^{-1}$, $\hat{Y}_{1,s} = \tilde{Y}_{1,s}^{-1}$ and $n_R = \mathbf{rank}(I - PQ)$.
2) Find a matrix $\overline{P} \in \mathbb{R}^{(n+n_R)\times(n+n_R)}$ such that $\overline{P} = \left[\begin{array}{cc} P & R_P \\ R_P^T & I \end{array}\right]$, where $R_P \in \mathbb{R}^{n\times n_R}$ is such that $P - Q^{-1} = R_P R_P^T$.
3) Find matrices $\overline{S}, \overline{G} \in \mathbb{R}^{(N+N_R)\times(N+N_R)}$ defined by:

$$\overline{S} = \left[\begin{array}{cc} \overline{S}_{1,s} & 0 \\ 0 & \hat{X}_{s+1,r} + \hat{X}_{s+1,r}^T \end{array}\right],$$

$$\overline{G} = \left[\begin{array}{cc} \overline{G}_{1,s} & 0 \\ 0 & \hat{X}_{s+1,r}^T - \hat{X}_{s+1,r} \end{array}\right],$$

where $\overline{S}_{1,s} = \frac{1}{2}(\overline{X}_{1,s} + \overline{X}_{1,s}^T)$, $\overline{G}_{1,s} = \frac{1}{2}(\overline{X}_{1,s}^T - \overline{X}_{1,s})$ and $\overline{X}_{1,s} \in \mathbb{R}^{2k_R \times 2k_R}$ is such that $\overline{X}_{1,s} = \left[\begin{array}{cc} \hat{X}_{1,s} & VC \\ U & C \end{array}\right]$, where $V \in \mathbb{R}^{k\times k_R}$, $U \in \mathbb{R}^{k_R\times k}$ are such that $\hat{X}_{1,s} - \hat{Y}_{1,s}^{-1} = VU$ and $C = 2\left(V^T\hat{S}_{1,s}^{-1}V\right)^{-1}\left(I - \frac{1}{2}V^T\hat{S}_{1,s}^{-1}U^T\right)$.
4) From $\overline{P}, \overline{S}, \overline{G}$, construct X_1, Y_1, Z_1 according to (15) and then $\mathcal{G}, \mathcal{U}, \mathcal{V}$ according to (17) and (18).
5) Solve for M_R the following LMI feasibility problem:

$$\mathcal{G} + \mathcal{U}^T M_R \mathcal{V} + \mathcal{V}^T M_R^T \mathcal{U} < 0.$$

Then the system defined by $z_R(t) = \mathcal{F}_u\left(\mathcal{F}_u\left(M_R, \int I_{n_R}\right), \Delta_R(t)\right)w(t)$ is a model reduction of the system defined by $z(t) = \mathcal{F}_u\left(\mathcal{F}_u\left(M, \int I_n\right), \Delta(t)\right)w(t)$ such that $||z - z_R||_2 < \epsilon||w||_2$ and the model reduction error is defined by ϵ.

IV. APPLICATION

To illustrate, we consider the well-known missile benchmark of Reichert [12]. The original model being nonlinear, it is first necessary to build a corresponding LPV system. We consider the simple model represented as an LFT on a single parameter:

$$\left[\begin{array}{c} \dot{\alpha}(t) \\ \dot{q}(t) \\ \hline q_1(t) \\ \hline \eta_c(t) - \eta(t) \\ q(t) \end{array}\right] \left[\begin{array}{c|c|c} A & B_0 & B_1 \\ \hline C_0 & D_{00} & D_{01} \\ \hline C_1 & D_{10} & D_{11} \end{array}\right] \left[\begin{array}{c} \alpha(t) \\ q(t) \\ \hline p_1(t) \\ \hline u(t) \end{array}\right],$$

$$p_1(t) = \delta(t)q_1(t),$$

where $\alpha(t)$ is the angle of attack, $q(t)$ the pitch rate, $\eta(t)$ the acceleration, $\eta_c(t)$ the reference acceleration and $u(t)$ the tail deflection. The matrices are constant and their exact definition can be found in reference [17]. The time-varying parameter $\delta(t)$ is defined as a polynomial in the state $\alpha(t)$. Following the usual LPV methodology, an $\mathcal{L}_2$-gain criterion with suitable weighting functions is obtained (see [17] for details) and the LPV synthesis method with constant Lyapunov matrix of [4] is applied, yielding an LPV controller ensuring the closed loop stability and an $\mathcal{L}_2$-gain less than $\gamma = 1.3$. This LPV controller naturally has the same

complexity as the plant *i.e.*, it admits an LFT representation of the form:

$$\left[\begin{array}{c} \dot{x}_K(t) \\ \hline q_2(t) \\ \hline u(t) \end{array}\right] \left[\begin{array}{c|c|c} A_K & B_{K0} & B_{K1} \\ \hline C_{K0} & D_{K00} & D_{K01} \\ \hline C_{K1} & D_{K10} & D_{K11} \end{array}\right] \left[\begin{array}{c} x_K(t) \\ q_2(t) \\ \hline \eta_c(t) - \eta(t) \\ q(t) \end{array}\right],$$

$$p_2(t) = \delta(t)q_2(t). \tag{27}$$

Yet the frozen Bode plots for different values of the parameter displayed on Figure 1 (dashed lines) suggest that this controller varies weakly with the parameter. In fact, for this system, it is even known [18] that there exists an LTI controller achieving good performance.

The model reduction method described in this paper typically presents an interest in this case. Here, it allows to construct an optimal controller of reduced complexity (where the dependence on the parameter has been removed), that is to say an LTI controller of the form:

$$\left[\begin{array}{c} \dot{x}_{K_R}(t) \\ \hline u_R(t) \end{array}\right] = \left[\begin{array}{c|c} A_{KR} & B_{KR} \\ \hline C_{KR} & D_{KR} \end{array}\right] \left[\begin{array}{c} x_{K_R}(t) \\ \hline \eta_c(t) - \eta(t) \\ q(t) \end{array}\right]. \tag{28}$$

The original controller (27) being quadratically stable, the method of Section III is directly applied to obtain the model reduction. Thus, an LTI controller (28) is obtained such that:

$$||u - u_R||_2 < \epsilon \left|\left|\left[\begin{array}{c} \eta_c - \eta \\ q \end{array}\right]\right|\right|_2,$$

with a model reduction error less than $\epsilon = 0.5$. The Bode plot of this reduced controller is displayed on Figure 1 (full line), superimposed on the original controller frozen Bode plots (dashed lines). Performing an analysis with a method based on a constant Lyapunov matrix [4] proves that the reduced controller also ensures the closed loop stability and a superior bound on the $\mathcal{L}_2$-gain equal to $\gamma_R = 1.6$, *i.e.*, of the same order as with the original controller.

V. CONCLUSION

This paper addresses the problem of model reduction for LPV systems modeled by an LFT on a parameter block diagonal structure. The case is studied when the reduced model depends on some paramaters through an LFT of same complexity as the plant and no longer at all on the other parameters. Then, in contrast to the general case, it is proved that the LPV model reduction problem can be written as an LMI optimization problem. The method proposed in this paper is an original contribution in several directions. First, in contrast with existing procedures for model reduction of LPV systems, our method relies on a convex LMI optimization problem. Moreover, it is naturally suited for a wide range of applications when the plant depends "mildly" on some parameters (*e.g.*, slow-varying parameters) and "strongly" on some others. It allows to obtain an optimal reduced model that no longer depends on some of the parameters. Finally, it applies directly to the nonlinear context using the fact that LPV systems can model nonlinear systems by defining parameters as embeddings of nonlinearities.

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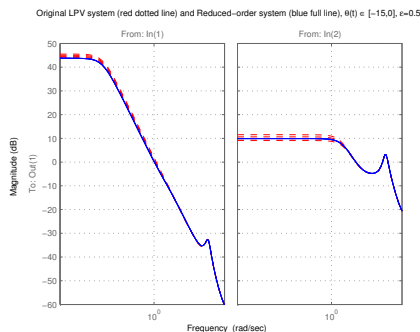
Fig. 1. Frozen Bode plots of (1) original controller: LPV system with LFT on one parameter (dashed) and (2) reduced controller: LTI system (full).