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Reduced-Complexity Controllers for LPV Systems: Towards Incremental Synthesis

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***Abstract*—Existing synthesis methods for LPV systems often result in controllers of high complexity. So far, there is no efficient and systematic remedy to this issue as there exists no convex formulation of the problem of finding a solution of reduced complexity to the general case LPV synthesis problem. In this paper, the specific case is considered when parameter-dependent signals are measured. It is proven that these measures can be exploited so that the problem of reduced-complexity controller synthesis can be written as an LMI optimization problem. A complete procedure for the controller construction is provided. The interest of the result is discussed in relation with nonlinear methods. First, an interpretation of the controller strategy is proposed with regard to the feedback linearization method. Second, it is proven that a nonlinear controller ensuring the closed loop incremental properties can be constructed.**

I. Introduction

A. Motivation

To deal with nonlinear systems, gain-scheduling methods are popular [1]. However, although widely and often successfully used, traditional gain-scheduling procedures are in fact heuristic [2]. A rigorous alternative was proposed by the linear parameter-varying (LPV) methods [1], which were developed by extending the H_∞ problem to the LPV context. LPV methods proved to be promising tools for the design of nonlinear systems. Indeed, not only can they result in nonlinear controllers but also, in contrast with traditional nonlinear methods [3], they can take explicitly into account uncertainties, thanks to the H_∞-like framework.

Yet, despite significant achievements, the LPV practice remains limited. A major disappointment is caused by the fact that LPV controllers seem to vary little with respect to the parameters so that their strategy does not seem to correspond to classical nonlinear strategies. This phenomenon was sometimes attributed to the intrinsic conservatism of LPV methods [4], but some recent encouraging results suggest that the choice of the information structure plays a key role [5]. In this paper is shown that with an adequate information structure, the LPV controllers strategy can be interpreted in terms of nonlinear methods.

Another critical issue is the actual obtention of the nonlinear controller. Recall that the real purpose of LPV methods is the obtention of incremental controllers [2]. A rigorous utilisation involves the design of an LPV controller from an LPV system corresponding to non stationary linearizations of the nonlinear system along a trajectory, and not to an embedding of the nonlinear system itself (quasi-LPV). The next step, which consists in constructing the nonlinear controller by integration of the LPV controller, turns out to be difficult. To our best knowledge, there is currently no systematic procedure to solve the incremental synthesis problem. In this paper, a practical method is proposed for the first time to get a nonlinear controller ensuring incremental properties.

Another main criticism addressed to LPV methods is that they usually lead to controllers of high complexity: even for the simplest methods, relying on invariant Lyapunov matrices, the controller complexity typically equals the plant complexity, see [6], [7] (polytopic systems), [8], [9], [10], [11] (rational systems). For methods involving parameter-dependent Lyapunov functions [12], [4], [13], the controller complexity is even higher. Unfortunately, in general, there exist only non convex formulations of the problem of seeking directly a controller of reduced complexity [9], [10]. Here is proven that there are realistic cases when the problem of finding directly a controller of reduced complexity can be formulated as a convex problem and moreover, the existence of a controller is equivalent to the existence of a reduced-complexity controller.

B. Proposed approach

The discussion relies on a technical result concerned with the LPV reduced-complexity synthesis problem. It is proven that the information structure can be exploited to turn the generally non convex reduced-complexity synthesis problem into a convex problem. The considered LPV system state-space matrices are supposed to depend rationally on the parameters, thus admitting a linear fractional transform (LFT) representation:

$$\begin{bmatrix} \dot{x}(t) \\ p(t) \\ z(t) \\ y(t) \end{bmatrix} = M \begin{bmatrix} x(t) \\ q(t) \\ w(t) \\ u(t) \end{bmatrix},\ p(t) = \Delta(t)q(t),$$

where M is a constant matrix, $\Delta(t)$ is a repeated scalar parameter diagonal structure and $p(t) \in \mathbb{R}^k$ is the vector of parameter-dependent signals. The main technical result can be stated as follows: if $l \leq k$ components of $p(t)$ are measured, then the synthesis problem with the extra condition that the controller parameter block should be of dimension $k-l$ can be written as an LMI optimization problem. This particular controller structure has interesting implications in the context of nonlinear control. First, this simplified structure can be interpreted as involving a cancellation of the parameter-dependent terms, comparable to a linearizing feedback [3]. A second remarkable interest is the ability to provide an elegant answer to the incremental synthesis problem.

C. Relation to previous work

In a different context, a sub case of the technical result presented here can be found in [14], [15]. There, the special case is considered when all parameter block outputs are supposed measured and the class of parameters is particular. The interpretation of the result differs from ours as there, the practical interest is only to provide a case in which either [14] the robust LTI synthesis problem is convex or [15] the optimal LPV l_1 synthesis problem is convex. In another context, a comparable technique has also been used to obtain directly reduced-order solutions to LTI or LPV synthesis problems involving full or partial state-feedback in [16], [9], [17]. The resolution of our problem is more challenging due to the general parameter characterization used to reduce the conservatism.

D. Structure of the paper

The paper is structured as follows. Section II defines the problem. The main technical result is stated in Section III. Section IV discusses how the method can be used to solve the incremental synthesis problem and proposes an interpretation of the controller strategy with regard to the linearizing feedback method.

E. Notations

The identity matrix of $\mathbb{R}^{n\times n}$ is denoted I_n and the zero matrix of $\mathbb{R}^{n\times m}$ is denoted $0_{n\times m}$. The subscripts are omitted when obvious from context. For two operators A and B, $\mathbf{diag}(A,B)$ denotes the operator $\left[\begin{smallmatrix} A & 0 \\ 0 & B\end{smallmatrix}\right]$. For a full-rank matrix U, $U^\perp$ denotes an orthogonal complement of U, *i.e.*, $UU^\perp = 0$ and $\left[\, U^T \; U^\perp \,\right]$ is of maximal rank, while U^+ denotes the Moore-Penrose inverse of U. For $X\in\mathbb{R}^{n\times m}$ and $k\leq l\leq n$, $r\leq s\leq m$ $X_{[k:l][r:s]}$ denotes the matrix extracted from X made of its lines from k to l and columns from r to s. For a square matrix M, $M>0$ and $M\geq 0$ mean respectively positive and semi-positive definiteness. The symbol $\int$ denotes the integration operation. For a signal $w(t)$, its $\mathcal{L}_2$ norm is denoted $||w||_2$ whenever it is defined. For a matrix M partitioned as $\left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{smallmatrix}\right]$ and an operator Δ, the notation $\mathcal{F}_u(M,\Delta)$ stands for $M_{22}+M_{21}\Delta(I-\Delta M_{11})^{-1}M_{12}$ and $\mathcal{F}_l(M,\Delta)$ stands for $M_{11}+M_{12}\Delta(I-\Delta M_{22})^{-1}M_{21}$ whenever they exist.

II. Setup and preliminary results

Here, the general (or full-complexity) LPV synthesis problem is defined for the considered system and sufficient existence conditions of a solution existence are given as an LMI feasibility problem. The LPV reduced-complexity problem is then introduced, for which sufficient existence conditions of a solution are given as a non convex optimization problem given by LMI conditions coupled with a rank constraint.

A. Considered system

General LPV systems can be defined by the following state-space equations:

$$\begin{cases} \dot{x}(t) = A(\delta(t))x(t) + B_1(\delta(t))w(t) + B_2(\delta(t))u(t) \\ z(t) = C_1(\delta(t))x(t) + D_{11}(\delta(t))w(t) + D_{12}(\delta(t))u(t) \\ y(t) = C_2(\delta(t))x(t) + D_{21}(\delta(t))w(t) + D_{22}(\delta(t))u(t), \end{cases} \quad (1)$$

where $x(t)\in\mathbb{R}^n$ is the state, $w(t)\in\mathbb{R}^{n_w}$ the external input, $z(t)\in\mathbb{R}^{n_z}$ the external output, $u(t)\in\mathbb{R}^{n_u}$ the controlled input, $y(t)\in\mathbb{R}^{n_y}$ the measure and $\delta(t) = \left[\,\delta_1(t)\ \cdots \delta_r(t)\,\right]^T\in\mathbb{R}^r$ is called the parameter vector. For any $i\in\{1,\cdots,r\}$, the parameter $\delta_i(t)$ is supposed to be a real time-varying scalar measured in real time and belonging to a given interval that can be either closed: $\delta_i(t)\in[\underline{\delta}_i,\overline{\delta}_i]$ or open: $\delta_i(t)\in[\underline{\delta}_i,+\infty[$. Without loss of generality, every interval is supposed to contain 0. This paper focuses on LPV systems described by the state-space equations (1) where the matrices are rational functions of the parameters. Such systems can be modeled by an LFT on a parameter block structure [10]:

$$\begin{bmatrix} \dot{x}(t) \\ q(t) \\ z(t) \\ \hline y(t)\end{bmatrix} = \left[\begin{array}{c|c} M & M_u \\ \hline M_y & 0\end{array}\right]\begin{bmatrix} x(t) \\ p(t) \\ w(t) \\ \hline u(t)\end{bmatrix},\ p(t) = \Delta(t)q(t), \quad (2)$$

and $M\in\mathbb{R}^{(n+k+n_z)\times(n+k+n_w)}$, $M_u\in\mathbb{R}^{(n+k+n_z)\times n_u}$, $M_y\in\mathbb{R}^{n_y\times(n+k+n_w)}$ are constant matrices, $\Delta(t)$ is called the parameter block and $q(t)$ and $p(t)\in\mathbb{R}^k$ are respectively the input and the output of the parameter block where $p(t)$ is also called the vector of parameter-dependent signals. The following notation is used:

$$\begin{bmatrix} z(t) \\ y(t)\end{bmatrix} = \mathcal{F}_u\left(\mathcal{F}_u\left(\begin{bmatrix} M & M_u \\ M_y & 0\end{bmatrix}, \int I_n\right), \Delta(t)\right)\begin{bmatrix} w(t) \\ u(t)\end{bmatrix} \quad (3)$$

and the state-space matrices are defined as:

$$\left[\begin{array}{c|c} M & M_u \\ \hline M_y & 0\end{array}\right] = \left[\begin{array}{ccc|c} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \\ \hline C_2 & D_{20} & D_{21} & 0\end{array}\right].$$

The parameter block is a block-diagonal matrix: $\Delta(t) = \mathbf{diag}(\Delta_1(t),\cdots,\Delta_r(t))$, where each sub-block is $\Delta_i(t) = \delta_i(t)I_{k_i}$ such that $k=\sum_{i=1}^r k_i$. The dimension k of the parameter block is also called the system complexity (with respect to the parameters). The following notation is used: $\underline{S}(\Delta_i) = \{S_i\in\mathbb{R}^{k_i\times k_i} | S_i = S_i^T > 0\}$, $\underline{G}(\Delta_i) = \{G_i\in\mathbb{R}^{k_i\times k_i} | G_i = -G_i^T\}$ and $\underline{S}(\Delta) = \{S | S = \mathbf{diag}(S_1,\cdots,S_r)\}$, $\underline{G}(\Delta) = \{G | G = \mathbf{diag}(G_1,\cdots,G_r)\}$. It can be shown [10] that the parameter block Δ satisfies a family of quadratic constraints of the form:

$$\begin{bmatrix} q(t) \\ p(t)\end{bmatrix}^T \Phi \begin{bmatrix} q(t) \\ p(t)\end{bmatrix} \geq 0, \quad (4)$$

where:

$$\Phi = \begin{bmatrix} aS & bS + G \\ bS + G^T & cS \end{bmatrix} \tag{5}$$

with $S \in \underline{S}(\Delta)$, $G \in \underline{G}(\Delta)$, $a = \mathbf{diag}(a_1, \cdots, a_r)$, $b = \mathbf{diag}(b_1, \cdots, b_r)$, $c = \mathbf{diag}(c_1, \cdots, c_r)$ and:

- if $\delta_i(t) \in [\underline{\delta}_i, \overline{\delta}_i]$, $a_i = -2\underline{\delta}_i\overline{\delta}_i,\ b_i = \underline{\delta}_i + \overline{\delta}_i,\ c_i = -2$;
- if $\delta_i(t) \in [\underline{\delta}_i, +\infty[$, $a_i = -2\underline{\delta}_i,\ b_i = 1,\ c_i = 0$.

For $a,\ b,\ c$ as above, let $\tilde{a},\ \tilde{b},\ \tilde{c}$ be such that: $\left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right] \left[\begin{smallmatrix} -\tilde{c} & \tilde{b} \\ \tilde{b} & -\tilde{a} \end{smallmatrix}\right] = I$ and define for $i \in \{1, \cdots, r\}$ $\Gamma_i = \sqrt{b_i^2 - a_i c_i}$ and $\Gamma = \mathbf{diag}(\Gamma_1, \cdots, \Gamma_r)$.

B. *Preliminary result: the LPV synthesis problem as an LMI existence test*

First, let us state the standard LPV control problem. As the plant admits an LFT representation, the controller is supposed to be represented by an LFT of same complexity.

Problem 2.1 (General LPV control problem): Given $\gamma > 0$ and the LPV plant defined by $\left[\begin{smallmatrix} z(t) \\ y(t) \end{smallmatrix}\right] = \mathcal{F}_u\left(\mathcal{F}_u\left(\left[\begin{smallmatrix} M & M_u \\ M_y & 0 \end{smallmatrix}\right], \int I_n\right), \Delta(t)\right) \left[\begin{smallmatrix} w(t) \\ u(t) \end{smallmatrix}\right]$ (3), find, if they exist, $k_K \leq k$, $n_K \leq n$ and a constant matrix $K \in \mathbb{R}^{(n_K + k_K + n_u) \times (n_K + k_K + n_y)}$ such that there exists $\Delta_K(t) = \mathbf{diag}(\delta_1(t) I_{k_{K_1}}, \cdots, \delta_r(t) I_{k_{K_r}}) \in \mathbb{R}^{k_K \times k_K}$ ensuring that with the controller defined by:

$$u(t) = \mathcal{F}_l\left(\mathcal{F}_u\left(M_K, \int I_{n_K}\right), \Delta_K(t)\right) y(t) \tag{6}$$

the closed loop system is stable and $||z||_2 < \gamma ||w||_2$. ◦

In this paper, a method using a constant Lyapunov matrix is considered to solve this problem. Thus, adapted from [10], the following theorem holds.

Theorem 2.2: There exists a solution to the general LPV control problem 2.1 if there exists a solution to the LMI feasibility problem 2.3 stated below. •

Opt. problem 2.3 (Full-complexity existence test): Find, if they exist, matrices $P = P^T$, $Q = Q^T \in \mathbb{R}^{n \times n}$, $S,\ T \in \underline{S}(\Delta)$ and $G,\ H \in \underline{G}(\Delta)$ such that:

$$M_y^{\perp^T} \left[\frac{M}{I_{n+n_w}}\right]^T \mathcal{M} \left[\frac{M}{I_{n+n_w}}\right] M_y^{\perp} < 0, \tag{7}$$

$$M_u^{T^{\perp^T}} \left[\frac{M^T}{I_{n+n_z}}\right]^T \mathcal{N} \left[\frac{M^T}{I_{n+n_z}}\right] M_u^{T\perp} < 0, \tag{8}$$

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} > 0, \tag{9}$$

where:

$$\mathcal{M} = \left[\begin{array}{ccc|ccc} 0 & & & P & & \\ & aS & & & bS + G & \\ & & \gamma^{-1} I_{n_z} & & & 0 \\ \hline P & & & 0 & & \\ & bS + G^T & & & cS & \\ & & 0 & & & -\gamma I_{n_w} \end{array}\right],$$

$$\mathcal{N} = \left[\begin{array}{ccc|ccc} 0 & & & Q & & \\ & \tilde{a}T & & & \tilde{b}T + H & \\ & & \gamma^{-1} I_{n_w} & & & 0 \\ \hline Q & & & 0 & & \\ & \tilde{b}T + H^T & & & \tilde{c}T & \\ & & 0 & & & -\gamma I_{n_z} \end{array}\right].$$

◦

The skew-symmetric matrices G and H can be enforced to be zero to simplify the conditions. The LMI opt. problem 2.3 becomes then the following one.

Opt. problem 2.4 (Simplified full-complexity existence test): Find, if they exist, matrices $P = P^T > 0,\ Q = Q^T > 0 \in \mathbb{R}^{n \times n}$ and $S,\ T \in \underline{S}(\Delta)$ such that (7), (8), (9) and:

$$\begin{bmatrix} S & I \\ I & T \end{bmatrix} \geq 0. \tag{10}$$

◦

Notice that under this assumption, the problem is simplified at the expense of conservatism since decision variables are set to zero. Next, this assumption is made in one case to obtain convex conditions for the reduced-complexity problem.

C. *Problem formulation*

Here, the problem of reduced-complexity synthesis is stated.

Remark 2.5: Let $\{P,\ Q,\ S,\ G,\ T,\ H\}$ be a solution of the LMI problem 2.3. Then there exists a solution to the general LPV synthesis problem 2.1 such that:

- the controller order (number of states) n_K equals the rank of $I - PQ$;
- the controller complexity (parameter block dimension) k_K equals the rank of $I - (S + \Gamma^{-1}G)(T + \Gamma H)$. ▹

See [10] for a proof. Clearly, a controller of reduced complexity is a particular solution of the general LPV control problem 2.1. The test of existence of a reduced-complexity controller can thus be stated as the following optimization problem.

Opt. problem 2.6 (Reduced-complexity existence test): Let l be an integer such that $l \leq k$. Find, if they exist, matrices $P = P^T$, $Q = Q^T \in \mathbb{R}^{n \times n}$, $S,\ T \in \underline{S}(\Delta)$ and $G,\ H \in \underline{G}(\Delta)$ such that the LMIs (7), (8) and (9) hold, along with the additional rank constraint:

$$\mathrm{rank}\Big((I - (S + \Gamma^{-1}G)(T + \Gamma H)\Big) \leq k - l. \tag{11}$$

◦

If the skew-symmetric matrices G and H are enforced to be zero, the opt. problem 2.6 becomes the following one:

Opt. problem 2.7 (Simplified red.-complexity existence test): Let l be an integer such that $l \leq k$. Find, if they exist, matrices $P = P^T$, $Q = Q^T \in \mathbb{R}^{n \times n}$, $S,\ T \in \underline{S}(\Delta)$ and $G,\ H \in \underline{G}(\Delta)$ such that the LMIs (7), (8), (9) and (10) hold, along with the additional rank constraint:

$$\mathrm{rank}(I - ST) \leq k - l. \tag{12}$$

◦

Yet in general, the opt. problem 2.6, resp. 2.7, is not convex because of the rank constraint (11), resp. (12). Moreover, the existence of a solution to the (full-complexity) opt. problem 2.3, resp. 2.4, does not imply the existence of a solution to the (reduced-complexity) opt. problem 2.6, resp. 2.7.

III. Solution of Reduced Complexity

In this section, cases are studied when the non convex rank constraints associated with reduced-complexity synthesis problems can be removed so that the opt. problems 2.6, resp. 2.7, reduce to LMI optimization problems. It is proven that in these cases, the existence of a solution to the full-complexity opt. problem 2.3, resp. 2.4, is in fact equivalent to the existence of a solution to the reduced-complexity opt. problem 2.6, resp. 2.7. Construction methods for the corresponding reduced-complexity controllers are provided.

A. Convex test of existence of a reduced-complexity solution

Two particular classes of parameters are studied, characterized by the quadratic constraint (4) where either (i) or (ii):

(i) $\Phi = \left[\begin{smallmatrix} S & 0 \\ 0 & -S \end{smallmatrix}\right]$ where $S = S^T > 0 \in \mathbb{R}^{k\times k}$, corresponding to the case in which the parameters vary between -1 and 1. The description does not take into account the fact that the parameters are real, which would require the addition of skew-symmetric scalings [10]. Nevertheless it is a simple and common description [9].

(ii) $\Phi = \left[\begin{smallmatrix} 0 & X \\ X^T & 0 \end{smallmatrix}\right]$ where $X \in \mathbb{R}^{k\times k}$ is such that $X + X^T > 0$, corresponding to the case in which the parameters are real and positive.

The main result is given in the following theorem.

Theorem 3.1: Given $\gamma > 0$, consider the LPV system (3) with parameters characterized by a quadratic constraint (4) with either (i) or (ii). Suppose that besides $y(t)$, $l \leq k$ outputs of the plant parameter block are measured. There exists a solution to the problem 2.3, resp. 2.4, if and only if there exists a solution to the problem 2.6, resp. 2.7. •

It remains to give a method to find a reduced-complexity controller. Without loss of generality, we assume the measured parameter block outputs are the $l \leq k$ last ones *i.e.*, the measures are: $y(t),\ p_{k-l+1}(t), \cdots, p_k(t)$, and note: $\hat{D}_{00} = D_{00_{[1:k][1:k-l]}}$, $\hat{D}_{10} = D_{10_{[1:n_z][1:k-l]}}$, $\hat{B}_0 = B_{0_{[1:n][1:k-l]}}$, $\hat{M}_y = \left[C_2\ D_{20_{[1:n_y][1:k-l]}}\ D_{21} \right]$, $\hat{\overline{M}}_y^{\perp} = \left[\begin{array}{c|c} \hat{M}_y^{\perp} & 0 \\ \hline 0 & I_{n_z+k} \end{array}\right]$.

Theorem 3.2: For the setup of Theorem 3.1 and the case (i), consider the LMI optimization problem of finding, if they exist, matrices $P = P^T,\ Q = Q^T \in \mathbb{R}^{n\times n}$, $T = T^T > 0 \in \mathbb{R}^{k\times k}$, $S_1 = S_1^T > 0 \in \mathbb{R}^{(k-l)\times(k-l)}$ such that the LMIs (13), (8), (9), (14) hold, where:

$$\hat{\overline{M}}_y^{\perp T} \left[\begin{array}{cc|c} A^TP+PA & P\left[\,\hat{B}_0\ \ B_1\,\right] & \left[\,C_0^T\ \ C_1^T\,\right] \\ \left[\begin{smallmatrix}\hat{B}_0^T \\ B_1^T\end{smallmatrix}\right]P & -\left[\begin{smallmatrix} S_1 & 0 \\ 0 & \gamma I\end{smallmatrix}\right] & \left[\begin{smallmatrix}\hat{D}_{00}^T & \hat{D}_{10}^T \\ D_{01}^T & D_{11}^T\end{smallmatrix}\right] \\ \hline \left[\begin{smallmatrix} C_0 \\ C_1\end{smallmatrix}\right] & \left[\begin{smallmatrix}\hat{D}_{00} & D_{01} \\ \hat{D}_{10} & D_{11}\end{smallmatrix}\right] & -\left[\begin{smallmatrix} T & 0 \\ 0 & \gamma I\end{smallmatrix}\right] \end{array}\right] \hat{\overline{M}}_y^{\perp} < 0 \quad (13)$$

$$\left[\begin{array}{cc} S_1 & \left[\,I_{k-l}\ \ 0\,\right] \\ \left[\begin{smallmatrix} I_{k-l} \\ 0\end{smallmatrix}\right] & T \end{array}\right] > 0. \quad (14)$$

If there exists a solution $\{P, Q, S_1, T\}$ to this LMI optimization problem then there exist matrices S_2 and S_3 such that $\{P, Q, S, T\}$ is a solution to the non convex opt. problem 2.7, where $S = \left[\begin{smallmatrix} S_1 & S_2 \\ S_2^T & S_3\end{smallmatrix}\right]$. •

Theorem 3.3: For the setup of Theorem 3.1 and the case (ii), consider the LMI optimization problem of finding, if they exist, matrices $P = P^T,\ Q = Q^T \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{k\times k}$ with $Y + Y^T > 0$, $X_1 \in \mathbb{R}^{(k-l)\times(k-l)}$ with $X_1 + X_1^T > 0$ and $X_3 \in \mathbb{R}^{l\times(k-l)}$ such that the LMIs (15), (8), (9) hold, where:

$$\begin{aligned} &\hat{M}_y^{\perp T} \left(\begin{bmatrix} A^TP+PA & P\hat{B}_0 & PB_1 \\ \hat{B}_0^TP & 0 & 0 \\ B_1^TP & 0 & -\gamma I \end{bmatrix} + \begin{bmatrix} C_0^T \\ \hat{D}_{00}^T \\ D_{01}^T \end{bmatrix} \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}^T + \right. \\ &\left. + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}^T \begin{bmatrix} C_0^T \\ \hat{D}_{00}^T \\ D_{01}^T \end{bmatrix}^T + \begin{bmatrix} C_1^T \\ \hat{D}_{10}^T \\ D_{11}^T \end{bmatrix} \gamma^{-1} \begin{bmatrix} C_1^T \\ \hat{D}_{10}^T \\ D_{11}^T \end{bmatrix}^T \right) \hat{M}_y^{\perp} < 0. \end{aligned} \quad (15)$$

If there exists a solution $\{P, Q, X_1, X_3, Y\}$ to this LMI optimization problem then there exist matrices X_2 and X_4 such that $\{P, Q, S, G, T, H\}$ is a solution of the non convex opt. problem 2.6, where $S = \frac{1}{2}(X + X^T)$, $G = \frac{1}{2}(X - X^T)$, $T = \frac{1}{2}(Y + Y^T)$, $H = \frac{1}{2}(Y - Y^T)$ and $X = \left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4\end{smallmatrix}\right]$. •

B. Proofs

1) Preliminary completion lemmas: Before proving the main result on reduced-complexity synthesis involving parameter block outputs measurements, some preliminary results on matrix completion are stated.

Lemma 3.1: Let two matrices $R = R^T > 0 \in \mathbb{R}^{k\times k}$ and $S_3 = S_3^T > 0 \in \mathbb{R}^{(k-l)\times(k-l)}$ be such that : $\left[\begin{smallmatrix} R & \left[\begin{smallmatrix} 0 \\ I_{k-l}\end{smallmatrix}\right] \\ \left[\,0\ I_{k-l}\,\right] & S_3 \end{smallmatrix}\right] \geq 0$. Then for any matrix $S_2 \in \mathbb{R}^{l\times(k-l)}$ there exists a matrix $S_1 = S_1^T > 0 \in \mathbb{R}^{l\times l}$ such that the matrix S defined by $S = \left[\begin{smallmatrix} S_1 & S_2 \\ S_2^T & S_3\end{smallmatrix}\right]$ satisfies (a) and (b):

(a) $\text{rank}(I - RS) = k - l$,

(b) $\left[\begin{smallmatrix} R & I \\ I & S\end{smallmatrix}\right] \geq 0$. •

Lemma 3.2: Let matrices $X_1 \in \mathbb{R}^{(k-l)\times(k-l)}$ and $Y \in \mathbb{R}^{k\times k}$ be such that $X_1 + X_1^T > 0$ and $Y + Y^T > 0$. Then for any matrix $X_3 \in \mathbb{R}^{l\times(k-l)}$ there exist matrices $X_2 \in \mathbb{R}^{(k-l)\times l}$ and $X_4 \in \mathbb{R}^{l\times l}$ such that the matrix X defined by $X = \left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4\end{smallmatrix}\right]$ satisfies (a) and (b):

(a) $\text{rank}(I - XY) = k - l$,

(b) $X + X^T > 0$. •

2) Proof of Theorems 3.1, 3.2 and 3.3: Let us suppose that the $l \leq k$ last outputs of the plant parameter block are measured and that the parameters are characterized by the quadratic constraint (4) with either (i) or (ii). The technical results follow from the exploitation of the special nature of the measurements $y(t),\ p_{k-l+1}(t),\ \cdots,\ p_k(t)$. Now $M_y = \left[\begin{array}{c|cc|c} C_2 & D_{20_{[1:n_y][1:k-l]}} & D_{20_{[1:n_y][k-l+1:k]}} & D_{21} \\ 0 & 0 & I_l & 0 \end{array}\right]$ so that $M_y^{\perp} = \left[\, W_1 \,|\, W_2\ \ 0 \,|\, W_3 \,\right]^T$ where $\left[\, W_1\ \ W_2\ \ W_3 \,\right]^T = \left[\, C_2\ \ D_{20_{[1:n_y][1:k-r]}}\ \ D_{21} \,\right]^{\perp}$. Then the conditions of the opt. problem 2.3, resp. 2.4, can be simplified. Some decision variables become degrees of freedom and can therefore be chosen to satisfy the non convex rank constraint (11). Computational details are given below.

• if (i): consider the setup of the problem 2.4. Then because of the zero line in $M_y^{\perp}$, (7) is simplified. Introducing the notation $S = \left[\begin{smallmatrix} S_1 & S_2 \\ S_2^T & S_3\end{smallmatrix}\right]$ where $S_1 \in \mathbb{R}^{(k-l)\times(k-l)}$, (7) thus becomes:

$$\hat{\overline{M}}_y^{T} \left[\begin{array}{ccc|cc} A^TP+PA & P\hat{B}_0 & PB_1 & C_0^T & C_1^T \\ \hat{B}_0^TP & -S_1 & 0 & \hat{D}_{00}^T & \hat{D}_{10}^T \\ B_1^TP & 0 & -\gamma I & D_{01}^T & D_{11}^T \\ \hline C_0 & \hat{D}_{00} & D_{01} & -S^{-1} & 0 \\ C_1 & \hat{D}_{10} & D_{11} & 0 & -\gamma I \end{array}\right] \hat{\overline{M}}_y < 0. \quad (16)$$

On the other hand, according to Packard's Lemma 6.2 [8], (10) implies that there exist matrices $R \in \mathbb{R}^{k\times l}$, $U \in \mathbb{R}^{l\times l}$ such that the matrix $\left[\begin{smallmatrix} T & R \\ R^T & U\end{smallmatrix}\right]$ is symmetric positive definite and $S^{-1} = T - R^TU^{-1}T$. Replacing and applying Schur's Lemma yields (16) if and only if:

$$\Psi + \Phi_1^T R \Phi_2 + \Phi_2^T R^T \Phi_1 < 0 \quad (17)$$

where, with $\overline{\overline{M}}_y^{\perp} = \textbf{diag}(\overline{M}_y^{\perp}, I_k)$,

$$\Psi = \hat{\overline{\overline{M}}}_y^{\perp T} \left[\begin{array}{ccc|ccc} A^TP + PA & P\hat{B}_0 & PB_1 & 0 & C_0^T & C_1^T \\ \hat{B}_0^TP & -S_1 & 0 & 0 & \hat{D}_{00}^T & \hat{D}_{10}^T \\ B_1^TP & 0 & -\gamma I & 0 & D_{01}^T & D_{11}^T \\ \hline 0 & 0 & 0 & -U & 0 & 0 \\ C_0 & \hat{D}_{00} & D_{01} & 0 & -T & 0 \\ C_1 & \hat{D}_{10} & D_{11} & 0 & 0 & -\gamma I \end{array}\right] \hat{\overline{\overline{M}}}_y^{\perp},$$

$$\Phi_1 = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & I_k & 0 \end{array}\right], \; \Phi_2 = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & I_k & 0 & 0 \end{array}\right].$$

The Elimination Lemma [18] implies that there exists R such that (17) holds if and only if:

$$\Phi_1^{\perp T} \Psi \Phi_1^{\perp} < 0, \tag{18}$$

$$\Phi_2^{\perp T} \Psi \Phi_2^{\perp} < 0. \tag{19}$$

Developing, we obtain that (19) implies (18) and reads in fact (13). On the other hand, since S_3 only still appears in (10), the Elimination Lemma implies that there exists S_3 such that (10) holds if and only if (14). Now S_2 and S_3 no longer being constrained in the resulting LMI optimization problem, they can be chosen freely and Lemma 3.1 implies that they exist such that rank$(I - ST) = k - l$ and (10). This proves Theorem 3.1 in the case (i) and Theorem 3.2.

• if (ii): consider the setup of the problem 2.3. The particular structure of $M_y^{\perp}$ is exploited to simplify the LMI (7). With the notation $X = \left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right]$ where $X_1 \in \mathbb{R}^{(k-l)\times(k-l)}$ and $X_4 \in \mathbb{R}^{l\times l}$, (7) becomes (15). Suppose that $\{P, Q, X1, X3, Y\}$ is a solution to the LMI problem (15), (8), (9). Now X_2 and X_4 no longer being constrained in this LMI problem, they can be chosen freely and Lemma 3.2 implies that they exist such that rank$(I - XY) = k - l$ and $X + X^T > 0$. This proves Theorem 3.1 in the case (ii) and Theorem 3.3.

C. Controller construction

A systematic procedure for constructing the state-space matrices of a reduced-complexity controller is proposed next.

Proposition 3.4: Consider an LPV plant (3), $\gamma > 0$ and the parameter block characterized by a quadratic constraint (4) with conditions either (i) or (ii) being satisfied.

• – if (i): find a solution $\{P, Q, S_1, T\}$ to the LMI optimization problem (13), (8), (9), (14) and find S_2, S_3 such that rank$(I - ST) = k - l$ and $S = S^T > 0$ where $S = \left[\begin{smallmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{smallmatrix}\right]$. Construct $\overline{S} \in \mathbb{R}^{(2k-l)\times(2k-l)}$ such that $\overline{S} = \left[\begin{smallmatrix} S & R_S \\ R_S^T & I \end{smallmatrix}\right]$, where $R_S \in \mathbb{R}^{k\times k_R}$ is such that $S - T^{-1} = R_S R_S^T$. Let $\overline{T} = \overline{S}^{-1}$.

– if (ii): find a solution $\{P, Q, X_1, X_3, Y\}$ to the LMI optimization problem (15), (8), (9) and X_2, X_4 such that rank$(I - XY) = k - l$ and $X + X^T > 0$, where $X = \left[\begin{smallmatrix} X_1 & X_2 \\ X_3 & X_4 \end{smallmatrix}\right]$. Construct $\overline{X} \in \mathbb{R}^{(2k-l)\times(2k-l)}$ such that $\overline{X} = \left[\begin{smallmatrix} X & NB \\ M & B \end{smallmatrix}\right]$, where $N \in \mathbb{R}^{k\times k_R}$, $M \in \mathbb{R}^{k_R\times k}$ are such that $X - Y^{-1} = NM$ and $B = 4\left(N^T(X + X^T)^{-1}N\right)^{-1}\left(I - N^T(X + X^T)^{-1}M^T\right)$. Let $\overline{Y} = \overline{X}^{-1}$ and define $\overline{S} = \frac{1}{2}(\overline{X} + \overline{X}^T)$ and $\overline{T} = \frac{1}{2}(\overline{Y} + \overline{Y}^T)$.

• Define $n_K = $ rank$(I - PQ)$, if (i): $k_K = $ rank$(I - ST)$ and if (ii): $k_K = $ rank$(I - XY)$.

• Find $\overline{P} = \left[\begin{smallmatrix} P & R_P \\ R_P^T & I \end{smallmatrix}\right]$, where $R_P \in \mathbb{R}^{n\times n_R}$ is such that $P - Q^{-1} = R_P R_P^T$.

• Solve the reduced-complexity Bounded Real Lemma LMI problem for matrix $M_K \in \mathbb{R}^{(n_K+n_u+k_K)\times(n_K+n_y+k_K)}$: $\Psi + D_u M_K D_y + D_y^T M_K^T D_u < 0$, where:

$$\Psi = \left[\begin{array}{c|c|c} \mathbf{A}^T\overline{P} + \overline{P}\mathbf{A} & \overline{P}\mathbf{B} & \mathbf{C}^T \\ \hline \mathbf{B}^T\overline{P} & -\left[\begin{smallmatrix} \overline{S} & 0 \\ 0 & \gamma I \end{smallmatrix}\right] & \mathbf{D}^T \\ \hline \mathbf{C} & \mathbf{D} & -\left[\begin{smallmatrix} \overline{T} & 0 \\ 0 & \gamma I \end{smallmatrix}\right] \end{array}\right], \; D_u = \left[\begin{array}{c} \overline{P}\mathbf{B}_2 \\ \hline 0 \\ \mathbf{D}_{12} \end{array}\right], \; D_y^T = \left[\begin{array}{c} \mathbf{C}_2^T \\ \mathbf{D}_{21}^T \\ \hline 0 \end{array}\right]$$

and the bold matrices are defined by:

$$\left[\begin{array}{c|c|c} \mathbf{A} & \mathbf{B} & \mathbf{B}_2 \\ \hline \mathbf{C} & \mathbf{D} & \mathbf{D}_{12} \\ \hline \mathbf{C}_2 & \mathbf{D}_{21} & \end{array}\right] = \left[\begin{array}{cc|ccc|ccc} A & 0 & 0 & B_0 & B_1 & I_{n_K} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{k_K} \\ C_0 & 0 & 0 & D_{00} & D_{01} & 0 & D_{02} & 0 \\ C_1 & 0 & 0 & D_{10} & D_{11} & 0 & D_{12} & 0 \\ \hline 0 & I_{n_K} & 0 & 0 & 0 & & & \\ C_2 & 0 & 0 & D_{20} & D_{21} & & & \\ 0 & 0 & I_{k_K} & 0 & 0 & & & \end{array}\right].$$

Partition $\Delta(t) = \textbf{diag}(\Delta_K(t), \Delta_r(t))$ where $\Delta_K(t) \in \mathbb{R}^{k_K\times k_K}$. Then $u(t) = \mathcal{F}_l\left(\mathcal{F}_u\left(M_K, \int I_{n_K}\right), \Delta_K(t)\right) y(t)$ is a solution of the general LPV synthesis problem 2.3.

The procedure implemented in Matlab was completed in a few seconds for the studied examples.

IV. Interpretation of the controller strategy

Let us assume now that all the parameter-dependent terms (all the components of $p(t)$) are measured. Recall that the method of Section III then yields an LPV controller of complexity zero, *i.e.*, an LTI controller: $u(s) = K_{y\to u}(s)y(s) + K_{p\to u}(s)p(s)$. Being actually LTI, this LPV controller is integrable so a nonlinear controller can be easily obtained. This means that if the LPV system corresponds to the non stationary linearizations of the nonlinear system, then the nonlinear controller ensures the nonlinear closed loop incremental properties. This method provides thus for the first time a solution to the incremental synthesis problem [2]. Moreover, there are cases when the controller strategy can be interpreted in terms of feedback linearization [3]. Next, both aspects are illustrated using an example. Consider the nonlinear system:

$$u = G(y) : \left\{ \begin{array}{rcl} \dot{x} & = & 10\sin x(t) + 10u(t) \\ y(t) & = & x(t) \end{array} \right. \tag{20}$$

and suppose the only specification is that $y(t)$ tracks a reference $y_r(t)$. The nonlinear feedback method [3] yields a controller canceling the nonlinearity: $u(t) = -\sin x(t) + K(y_r(t) - y(t))$ where K is a linear operator chosen for performance, typically an H_∞ controller. Performance can be tuned through a weighting function W_e (see in Figure 1) and here is chosen: $W_e(s) = \dfrac{s + 0.1226}{0.01s + 0.1751}$.

Next, the method of Section III is used to get a nonlinear controller. First, an LPV model corresponding to a non stationary linearization of the nonlinear system is obtained:

$$\left\{ \begin{array}{rcl} \dot{\tilde{x}}(t) & = & 10\tilde{p}(t) + 10\tilde{u}(t) \\ \tilde{q}(t) & = & \tilde{x}(t) \\ \tilde{y}(t) & = & \tilde{x}(t) \end{array} \right. , \; \tilde{p}(t) = \delta(t)\tilde{q}(t) \tag{21}$$

and $\delta(t) \in [-1, 1]$. The trajectory $\delta(t) = \cos x_0(t)$ corresponds to the linearization of (20) in $x_0(t)$. Next, suppose that both the tracking error $\tilde{y}_r(t) - \tilde{y}(t)$ and the parameter-dependent term $\tilde{p}(t)$ are available ($l = 1$). Notice that the assumption that $\tilde{p}(t)$ is known is not stronger that the assumption needed to construct the linearizing feedback. Using the previously defined weighting function W_e leads to the $\mathcal{L}_2$ criterion of Figure 2. The method of Section III is now applied to get an LPV controller of complexity $k_R = k - l = 0$, that is to say, an LTI controller, hence integrable. Recalling that $\tilde{p}(t) = \cos x_0(t)\tilde{x}(t)$, a nonlinear controller for (20) is thus simply of the form:

$$u(t) = K_{(\tilde{y}_r - \tilde{y})\to\tilde{u}}(y_r(t) - y(t)) + K_{\tilde{p}\to\tilde{u}}(\sin x(t)). \quad (22)$$

This nonlinear controller is then a solution to the incremental synthesis problem [2]. Figure 3 (left) shows that $K_{(\tilde{y}_r - \tilde{y})\to\tilde{u}}$ coincides with the linearizing feedback cancellation term -1.

Actually, a realistic design must take into account uncertainties. It was proven that in the LPV context robustness can be achieved by introducing a weighting function W_u as in Figure 2 (dotted). Here is chosen $W_u(s) = \dfrac{s + 1.994 \cdot 10^{-3}}{3.994s + 0.3 \cdot 10^{-6}}$. Running the procedure yields a nonlinear controller having the same structure (22). On the other hand, the feedback linearizing method does not take explicitly into account robustness. To become compatible with W_u (see Figure 1, (dotted)), and therefore, to ensure robustness, the cancellation term can be filtered. The nonlinear feedback thus becomes $u(t) = -F(\sin x(t)) + K(y(t) - y_r(t))$ where F is a filter such that $F(s) \approx 1$ in low frequency and $F(s) \approx W_u^{-1}(s)$ in high frequency. Figure 3 (right) displays the Bode plots of F (full line) and $K_{\tilde{p}\to\tilde{u}}$ (dashed line), emphasizing their similarity. Thus, here, the transfer $K_{\tilde{p}\to\tilde{u}}$ can be interpreted as a filtered cancellation term.

V. Conclusion

In this paper, a practical method is given to obtain directly reduced-complexity controllers for LPV systems in the case in which parameter-dependent signals are supposed measured. The result shows interesting features for nonlinear control. First, there are cases in which the strategy of the reduced-complexity controller can be interpreted as a filtered cancellation of the parameter-dependent terms, suggesting a conciliation of LPV methods and traditional nonlinear methods. Second, it gives a practical method to solve the incremental synthesis problem.

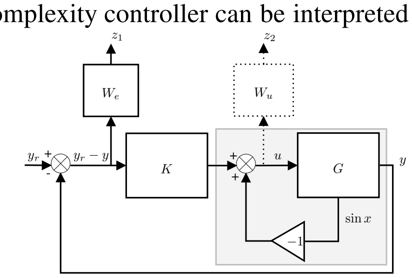


Fig. 1. Example - Criterion for linearizing feedback synthesis

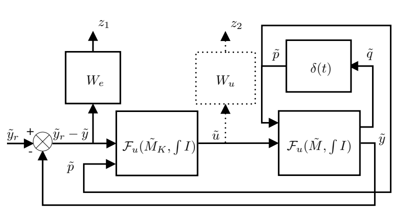
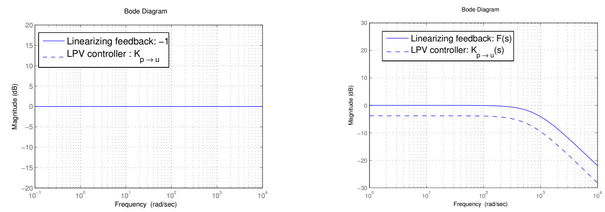


Fig. 2. Example - Criterion for LPV synthesis



Exact cancellation (W_e) Filtered cancellation (W_e, W_u)

Fig. 3. Example - Comparison

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