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Semidefinite Relaxation of a Robust Static Attitude Determination Problem

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Abstract— **This paper presents a tractable method of solving a non-convex, nonlinear optimization problem formulated for robust static attitude determination based on a least squares approach with nonlinear constraints. Considering infinity-norm bounded uncertainties, this robust min-max problem is approximated with a minimization problem, although the objective function and constraints are still nonlinear. We propose an additional regularization term to improve the robust performance. We then use semidefinite relaxation to convert the approximate nonlinear optimization problem into a tractable semidefinite program with a linear objective and linear matrix inequality constraints. We show how to extract the solution of the nonlinear optimization problem from the solution of the semidefinite relaxation. Numerical simulations suggest that the gap between the considered problem and its relaxation is zero.**

I. INTRODUCTION

Semidefinite relaxation (SDR) is a powerful and computationally efficient approximation technique for difficult optimization problems [1]. In particular, it can be applied to many non-convex quadratically constrained quadratic programs (QCQPs) in an almost mechanical fashion. In this work, which is an extension of previous work on robust static attitude determination with norm-bounded uncertainties [2], we use SDR to efficiently solve a nonlinear and non-convex optimization problem formulated for robust static attitude determination.

Static attitude determination has been widely used in satellites and other aerospace systems, such as aircrafts and helicopters, since many years [3], [4]. It is also very useful for a variety of other applications, such as marine systems and automotive, etc. Static approach only needs information of some vector quantities in two coordinate frames, such as the earth magnetic field, sun or star vectors, position, etc., and does not depend on system dynamics. This technique can also be used to initialize a dynamic estimator for highly nonlinear systems, thus reducing likelihood of divergence.

The attitude of a rigid body, formally defined as a coordinate transformation from one frame to the other [5], requires vector information in two coordinate frames. Normally, one of these coordinate frames is fixed in the body of the system, known as the body frame, while the other is called the reference frame, whose selection normally depends upon the control system requirements. This transformation is obtained through a proper orthogonal transformation matrix $C \in \mathbb{R}^{3\times3}$, also known as a *direction cosine matrix*. To determine the attitude, a weighted least squares approach, based on the Wahba problem [6] is often used. The matrix C imposes some constraints, such as the orthogonality constraint $C^TC = I$ and the constraint for matrix C to be proper to preserve the orientation in a rotation, i.e. $\det(C) = +1$. Many efficient solutions of this constrained least squares problem can be found in the literature, mostly developed for satellite applications [3], [7]–[10]. Most of these algorithms are based on a quaternion transformation [11], which transforms the Wahba problem into an eigenvalue problem [3].

Since the vector information for attitude determination is obtained from some sensor or a mathematical model, an error or uncertainty is always present in the values. Although a sensitivity analysis is generally presented with analytical expressions of the maximum error covariance under stochastic variation, the above-mentioned algorithms do not directly address the issue of uncertainty in the measured and model vectors. Some discussions considering uncertainty in the input measurements can be found in [12], [13], but modeling errors are generally not considered. However, these errors could be significant; for example, in the case of the earth magnetic field, which is one of the most common sensors used for attitude determination in many applications such as satellites, aircrafts, etc., errors between sophisticated models and the actual field can be around 20% [14], [15]. The use of simple models, such as the low order IGRF model [15], which are normally preferred due to lower computational cost, result in a less accurate earth magnetic field vector in the reference frame, leading to errors in the attitude estimate. Attitude inaccuracy is further increased due to sensor errors, which are mainly due to noise and installation issues. The magnetic field sensing in the post launch tumbling phase of a satellite is another example of a big source of uncertainty. All such errors can be considered as ∞-norm bounded uncertainties in the input information vectors.

A discussion on considering uncertainty in both sensor measurements and model vectors is given in [2], where a robust optimization (RO) problem is formulated using an affine parameterization of $\infty-$norm bounded uncertainties. In this work, we use a more generalized uncertainty structure to formulate the robust problem. The robust min-max problem is then approximated with a minimization problem using an analytical upper bound. We also propose a new type of regularization to further improve the robust performance. However, this approximate formulation is still non-convex

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with a nonlinear objective and constraints. Our main contribution is to propose a tractable method for solving the associated nonlinear, non-convex quadratically constrained quadratic optimization problem using semidefinite relaxation. The relaxed formulation, which is convex with a linear objective and linear matrix inequality constraints, can be solved efficiently in polynomial time [1]. We also show how we can extract an optimal quaternion from the SDR solution. Further, numerical simulations are presented, which suggest that the obtained quaternion vector results in no gap between the actual problem and its semidefinite relaxation.

The paper is organized as follows. In Section II, a robust estimation problem is presented and then simplified to a form suitable for applying semidefinite relaxation. The application of SDR is discussed in Section III. Section IV gives simulation results, while conclusions and future directions are given in last section.

Notation: For a vector x, its 2-norm is $\|x\|_2 := \sqrt{x^T x}$, while the infinity-norm is $\|x\|_\infty := \max_{1\le i\le n} |x_i|$. The cross product for vectors x and y is represented as $x \times y$. For a vector, $x \geq 0$ means that each element of x is non-negative. For a matrix, $A \succeq 0$ means that A is positive semidefinite. I_n denotes the identity matrix of size n, while $0_{n\times m}$ represents a matrix of n rows and m columns with all zero entries. The minimum (maximum) eigenvalue of a symmetric matrix A is represented by $\lambda_{\min(\max)}(A)$. Operator $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ represents a matrix of size $n \times n$, having only diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n$.

II. ROBUST ATTITUDE DETERMINATION PROBLEM

This section will briefly introduce the robust attitude determination problem and is an extension of [2] with a generalized uncertainty parameterization. A robust static attitude determination problem using weighted least squares is defined as

$$\begin{aligned} \min_{C} \quad & \max_{\substack{\bar{b}_i \in \mathscr{B}(b_i), \bar{r}_i \in \mathscr{R}(r_i),\\ i=1,\ldots,n}} \quad \frac{1}{2}\sum_{i=1}^{n} w_i \left\|\bar{b}_i - C\bar{r}_i\right\|_2^2 \\ \text{subject to} \quad & C^T C = I, \ \det(C) = 1. \end{aligned} \tag{1}$$

In this equation $\bar{b}_i \in \mathscr{B}(b_i)$ and $\bar{r}_i \in \mathscr{R}(r_i)$, $i = 1, \ldots, n$, n represents number of measurements, where $\mathscr{B}(b_i)$ and $\mathscr{R}(r_i)$ are bounded sets of uncertain vectors $\bar{b}_i$ and $\bar{r}_i$ having uncertainty bounds γ_{bi} and γ_{ri} for each vector in the body and reference frame, w_i represent weights and $\det(\cdot)$ is the determinant of a matrix. With an appropriate definition of the uncertainty sets (see Appendix I) and using a quaternion for coordinate transformation, the optimization problem (1) can be written as

$$\begin{aligned} \hat{q}^* := \arg\min_{q} \quad & \{-q^T K q + \max_{\|\delta\|_\infty \le 1} (p(q)^T \delta + \delta^T Q(q)\delta)\} \\ \text{subject to} \quad & q^T q = 1, \end{aligned} \tag{2}$$

where $\delta \in \mathbb{R}^{6n} := \begin{bmatrix}\delta_{b1} & \delta_{r1}, \ldots, \delta_{bn} & \delta_{rn}\end{bmatrix}$ is a vector of uncertainty parameterization, $p(q) \in \mathbb{R}^{6n}$ and $Q(q) \in \mathbb{R}^{6n\times 6n}$ are given as

$$p(q) := \begin{bmatrix} w_1\gamma_{b1}(b_1 - k_b(q, r_1)) \\ w_1\gamma_{r1}(r_1 - k_r(q, b_1)) \\ \vdots \\ w_n\gamma_{bn}(b_n - k_b(q, r_n)) \\ w_n\gamma_{rn}(r_n - k_r(q, b_n)) \end{bmatrix},$$

where the vector $k_b(q, r_i) := \begin{bmatrix} q^T K_{r_i}^1 q & q^T K_{r_i}^2 q & q^T K_{r_i}^3 q \end{bmatrix}^T$ and $k_r(q, b_i) := \begin{bmatrix} q^T K_{b_i}^1 q & q^T K_{b_i}^2 q & q^T K_{b_i}^3 q \end{bmatrix}^T$. The definition of these matrices is given in Appendix II.

$$Q(q) := \begin{bmatrix} \frac{1}{2}w_1\gamma_{b1}^2 I_3 & -\frac{1}{2}w_1\gamma_{b1}\gamma_{r1}C & \ldots & 0_{3\times3} & 0_{3\times3} \\ -\frac{1}{2}w_1\gamma_{b1}\gamma_{r1}C^T & \frac{1}{2}w_1\gamma_{r1}^2 I_3 & \ldots & 0_{3\times3} & 0_{3\times3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{3\times3} & 0_{3\times3} & \ldots & \frac{1}{2}w_1\gamma_{bn}^2 I_3 & -\frac{1}{2}w_1\gamma_{bn}\gamma_{rn}C \\ 0_{3\times3} & 0_{3\times3} & \ldots & -\frac{1}{2}w_1\gamma_{bn}\gamma_{rn}C^T & \frac{1}{2}w_1\gamma_{rn}^2 I_3 \end{bmatrix},$$

where transformation matrix C is a function of q. The robust problem is non-convex in q, because the matrix K is indefinite, while the maximization term $p(q)^T\delta + \delta^T Q(q)\delta$ in (2) is convex in δ, because Q is positive semidefinite. Both of these characteristics make the optimization problem difficult to solve. To simplify the problem, an upper bound on the maximum of $p(q)^T\delta + \delta^T Q(q)\delta$ over δ is used, presented in the following proposition.

Proposition 1. [2] An upper bound on the maximization term appearing in (2) is

$$0 \le \max_{\|\delta\|_\infty \le 1} (p(q)^T\delta + \delta^T Q(q)\delta) \le \ \|p(q)\|_1 + 6n\lambda_{\max}(Q(q)). \tag{3}$$

A comparison of the analytical upper bound with a tighter bound obtained using SDR of the maximization term is given in Section IV. However, while determining the SDR bound on maximization term, it is assumed that q is known, which is not the case in actual. The plot shows that the average relative error between two bounds is small. Thus, use of the analytical bound can give computational simplification, but at the cost of some loss in accuracy of the true solution of (2), although the loss is not much.

A. Addition of a Regularization Terms

The main reason of using the analytical upper bound is computational efficiency and the fact that exact solution of the maximization term may not be guaranteed. However, due to this approximation, the algorithm may not always give good results in terms of robustness. To improve performance, we introduce a new type of regularization by adding a term $-\eta q_4^2$ in the objective function. This regularization is motivated from the definition of quaternion given as

$$q := \begin{bmatrix} \mathbf{q} \\ q_4 \end{bmatrix} = \begin{bmatrix} \hat{e}_1 \sin(\alpha/2) \\ \hat{e}_2 \sin(\alpha/2) \\ \hat{e}_3 \sin(\alpha/2) \\ \cos(\alpha/2) \end{bmatrix}, \tag{4}$$

where $\hat{e} := \begin{bmatrix}\hat{e}_1 & \hat{e}_2 & \hat{e}_3\end{bmatrix}^T$ is the axis of rotation and α is the angle of rotation. The proposed regularization corresponds to minimization of the quaternion angle of rotation i.e. now the optimal solution minimizes both the primary cost as well as the quaternion angle of rotation. Here, $\eta > 0$ is a tunning parameter. In the current work, we used $\eta = 0.5$.

Using this upper bound along with the regularization term, we approximate the original min-max problem (2) with the following non-convex minimization problem

$$\begin{aligned} (q^*, u^*) := \arg\min_{q,u} \quad & -q^T K q + u^T e + 6n\lambda_{\max}(Q) - \eta q_4^2 \\ \text{subject to} \quad & q^T q = 1, \\ & -u_i \leq p_i \leq u_i, i = 1, \ldots, 6n, \end{aligned} \tag{5}$$

where $u := \begin{bmatrix} u_1 & u_2 & \ldots & u_{6n} \end{bmatrix}^T$, $p(q) = \begin{bmatrix} p_1 & p_2 & \ldots & p_{6n} \end{bmatrix}^T$ and $e \in \mathbb{R}^{6n}$ is a vector of ones.

B. Formulation for SDR

Before applying semidefinite relaxation on (5), we first do a further simplification by removing the nonlinear term $6n\lambda_{\max}(Q)$.

Lemma 1. The maximum eigenvalue of the block diagonal matrix $Q(q)$ is a constant.

Proof: To find the eigenvalues of the block diagonal matrix $Q(q) = \text{diag}(Q_1, Q_2, \ldots, Q_n)$, we need to solve n equations i.e. $\det(Q_i - \lambda I_6) = 0, i = 1, \ldots, n$. Consider $i = 1$ case, where we can write

$$\det(Q_1 - \lambda I_6) = \det \begin{bmatrix} \lambda_1(\lambda_2 - a) & 0 & 0 \\ 0 & \lambda_3(\lambda_4 - a) & \\ 0 & 0 & \lambda_5(\lambda_6 - a) \end{bmatrix} = 0,$$

where $a := \frac{1}{2} w_1(\gamma_{b1}^2 + \gamma_{r1}^2)$. The above equation implies that $\lambda_1 = \lambda_3 = \lambda_5 = 0$, and $\lambda_2 = \lambda_4 = \lambda_6 = \frac{1}{2} w_1(\gamma_{b1}^2 + \gamma_{r1}^2)$. Similarly we can find eigenvalues for $Q_i, i = 2, \ldots, n$. Finally, $\lambda_{max}(Q) = \max_i \frac{1}{2}(w_i \gamma_{bi}^2 + w_i \gamma_{ri}^2)$. However, the maximum eigenvalue is independent of q. ■

Finally, using Lemma 1 and representing $q_4^2 = q^T S q$, where $S \in \mathbb{R}^{4\times 4}$ is matrix of zeros with only $S(4,4) = 1$, we can write

$$\begin{aligned} (q^*, u^*) = \arg\max_{q,u} \quad & q^T K_r q - u^T e - \tilde{c} \\ \text{subject to} \quad & q^T q = 1, \\ & -u_i \leq p_i \leq u_i, i = 1, \ldots, 6n \end{aligned} \tag{6}$$

where $K_r := K + \eta S$ and $\tilde{c}$ include all constants. Note that the maximization form is suitable to apply semidefinite relaxation for finding an upper bound.

III. SEMIDEFINITE RELAXATION FOR ROBUST ESTIMATION PROBLEM

In this section we will apply semidefinite relaxation on the formulation presented in (6). Suppose, $\bar{\gamma}$ is an upper bound for the objective function of (6). We obtain the following expression, such that the right hand side is equal to the left hand side.

$$\begin{aligned} q^T K_r q - u^T e - \tilde{c} - \quad \bar{\gamma} \quad & = -\mu_1(1 - q^T q) \\ - \quad & \mu_2(u_1 - p_1) - \mu_3(u_1 + p_1) \\ - \quad & \mu_4(u_2 - p_2) - \mu_5(u_2 + p_2) \\ & \vdots \\ - \quad & \mu_{12n}(u_{6n} - p_{6n}) - \mu_{12n+1}(u_{6n} + p_{6n}) \\ - \quad & x^T \mathscr{L}(\mu) x, \end{aligned} \tag{7}$$

where $x^T := \begin{bmatrix} q^T & u^T & 1 \end{bmatrix}^T$ and $\mu := \begin{bmatrix} \mu_1 & \mu_2 & \ldots & \mu_{12n+1} \end{bmatrix}^T$,

$$\mathscr{L}(\mu) := \begin{bmatrix} \mathscr{L}_{1,1}(\mu) & 0_{4\times 1} & \ldots & 0_{4\times 1} & 0_{4\times 1} \\ 0_{1\times 4} & 0 & \ldots & 0 & \frac{1-\mu_2-\mu_3}{2} \\ 0_{1\times 4} & 0 & \ldots & 0 & \frac{1-\mu_4-\mu_6}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{1\times 4} & 0 & \ldots & 0 & \frac{1-\mu_{12n}-\mu_{12n+1}}{2} \\ 0_{1\times 4} & \frac{1-\mu_2-\mu_3}{2} & \ldots & \frac{1-\mu_{12n}-\mu_{12n+1}}{2} & \ell_{j,j} \end{bmatrix},$$

$$\begin{aligned} \mathscr{L}_{1,1}(\mu) := \quad & \mu_1 I_4 - (\mu_2 - \mu_3) w_1 \gamma_{b1} K_{r1}^1 - (\mu_4 - \mu_5) w_1 \gamma_{b1} K_{r1}^2 \\ & -(\mu_6 - \mu_7) w_1 \gamma_{b1} K_{r1}^3 - (\mu_8 - \mu_9) w_1 \gamma_{r1} K_{b1}^1 - \ldots \\ & -(\mu_{12n} - \mu_{12n+1}) w_n \gamma_{r1} K_{bn}^3 - K_r, \end{aligned}$$

$$\ell_{j,j} := \bar{\gamma} - \mu_1 + \sum_{l=1}^{6n} (\mu_{2l} - \mu_{2l+1}) c_l + \tilde{c},$$

where j is the size of x and $c := \begin{bmatrix} w_1 \gamma_{b1} b_1^T & w_1 \gamma_{r1} r_1^T & \ldots & w_n \gamma_{bn} b_n^T & w_n \gamma_{rn} r_n^T \end{bmatrix}^T$.

Now if all the terms on the right hand side are either zero or negative, we can say that $\bar{\gamma}$ is an upper bound on the cost of (6). Using this relaxation, we can write an optimization problem to find the minimum value of this upper bound ensuring the right hand side is either zero or negative.

$$(\bar{\gamma}^*, \mu^*) := \arg\min_{\bar{\gamma},\mu} \{\bar{\gamma} \mid \mathscr{L}(\mu) \succeq 0, \mu_i \geq 0, i = 2, 3, \ldots, 12n+1\}. \tag{8}$$

Problem (8) can further be simplified using a reduced set of optimization variables.

Theorem 1. Using a reduced set of optimization variables $\mu_r := \begin{bmatrix} \mu_1 & \mu_2 & \mu_4 & \ldots & \mu_{12n} \end{bmatrix}^T$, an equivalent formulation to (8) is

$$\begin{aligned} \mu_r^* = \arg\min_{\mu_r} \quad & \mu_1 - \sum_{l=1}^{6n} (2\mu_{2l} - 1) c_l - \tilde{c} \\ \text{subject to} \quad & \mu_2, \mu_4, \ldots, \mu_{12n} \geq 0, \\ & 1 - \mu_2 \geq 0, 1 - \mu_4 \geq 0, \ldots, 1 - \mu_{12n} \geq 0, \\ & \mathscr{L}_{1,1}(\mu_r) \succeq 0, \end{aligned} \tag{9}$$

where $\mathscr{L}_{1,1}(\mu_r)$ is given as

$$\begin{aligned} \mathscr{L}_{1,1}(\mu_r) \quad := \quad & \mu_1 I_4 - 2\mu_2 w_1 \gamma_{b_1} K_{r_1}^1 - 2\mu_4 w_1 \gamma_{b_1} K_{r_1}^2 \\ - \quad & 2\mu_6 w_1 \gamma_{b_1} K_{r_1}^3 - 2\mu_8 w_1 \gamma_{r_1} K_{b_1}^1 - \ldots \\ - \quad & 2\mu_{12n} w_n \gamma_{b_n} K_{r_n}^3 + w_1 \gamma_{b1} K_{r_1}^1 + w_1 \gamma_{b_1} K_{r_1}^2 \\ + \quad & w_1 \gamma_{b_1} K_{r_1}^3 + w_1 \gamma_{r_1} K_{b_1}^1 + \cdots + w_{12n} \gamma_{b_n} K_{r_n}^3 \\ - \quad & K_r, \end{aligned}$$

Proof. In (8), the symmetric matrix $\mathscr{L}(\mu)$ has zero diagonal elements. For $\mathscr{L}(\mu)$ to be positive semidefinite, as required in (8), all row/column elements corresponding to zero diagonal entries must also be zero [18, Theorem 4.2.6] i.e. $1 - \mu_2 - \mu_3 = 0, 1 - \mu_4 - \mu_5 = 0, 1 - \mu_6 - \mu_7 = 0$ and so on. Using this property, we can force these elements to be zero by eliminating $\mu_3, \mu_5, \ldots, \mu_{12n+1}$ from (8) with additional constraints $1 - \mu_2 \geq 0, 1 - \mu_4 \geq 0, \ldots, 1 - \mu_{12n+1} \geq 0$. Moreover, as we are minimizing the scalar $\bar{\gamma}$ subject to constraints, its minimum possible value, satisfying the constraint $\mathscr{L}(\mu) \succeq 0$, is when $\ell_{j,j} = 0$, giving $\bar{\gamma} = \mu_1 - \sum_{l=1}^{6n} (2\mu_{2l} - 1) c_l - \tilde{c}$. So with these modifications, instead of $\mathscr{L}(\mu) \succeq 0$, we only need $\mathscr{L}_{1,1}(\mu_r) \succeq 0$, hence can write (9) using reduced set of optimization variables, which is equivalent to solving (8) for minimum upper bound on (5). ■

A. Finding q^*

Although the solution of the semidefinite program (9) gives a minimum upper bound on the robust estimation problem (6), our main interest is to find q^* that could maximize the cost (6). Now the question arises, can we find q^* using the solution μ_r^* of (9)? Suppose μ_r^* results in a zero value of the right hand side of (7), then $\bar{\gamma}^*$, i.e. the minimum value of cost (9), is equal to the maximum cost of (6), and the corresponding q will be our required q^*.

In this regard, as a first step, we establish whether there exists a q that can make $q^T \mathscr{L}_{1,1}^* q = 0$, where $\mathscr{L}_{1,1}^* := \mathscr{L}_{1,1}(\mu_r^*)$. If such a q exists, it will further ensure $x^T \mathscr{L}^* x = 0$, where $\mathscr{L}^* := \mathscr{L}(\mu^*)$ and μ^* can be obtained from μ_r^*.

Lemma 2. Let μ_r^* be a minimizer for the SDR problem (9), then $\lambda_{\min}(\mathscr{L}_{1,1}^*) = 0$.

Proof. Using μ_r, the objective function of (9) can be written as $J := \mu_1 - d$, where d is sum of all remaining terms. Now whatever the sign of d is, the cost function J is minimum when μ_1 is minimum. However, at the same time, we need $\mathscr{L}_{1,1}(\mu_r) \succeq 0$. We can also write $\mathscr{L}_{1,1}(\mu_r) = \mu_1 I_4 - K_\mu$, where K_μ is sum of all other terms in the expression. This is a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_4$, and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$. Then, $\mu_1 I_4 - K_\mu$ will have eigenvalues $\mu_1 - \lambda_1, \mu_1 - \lambda_2, \mu_1 - \lambda_3, \mu_1 - \lambda_4$. Now, $\mu_1 = \lambda_1$ is the smallest possible value that can make $\mathscr{L}_{1,1}(\mu_r) \succeq 0$. This optimal value of μ_1, i.e. μ_1^* will also ensure $\lambda_{\min}(\mathscr{L}_{1,1}^*)$ equal to zero. ■

Theorem 2. The matrix $\mathscr{L}_{1,1}^*$ has at least one eigenvalue equal to zero, as stated in Lemma 2, and $\tilde{q}$ is an eigenvector of $\mathscr{L}_{1,1}^*$ corresponding to the zero eigenvalue, then this $\tilde{q}$ will result in $\tilde{q}^T \mathscr{L}_{1,1}^* \tilde{q} = \tilde{x}^T \mathscr{L}^* \tilde{x} = 0$, where $\tilde{x} := \begin{bmatrix} \tilde{q}^T & \tilde{u} & 1 \end{bmatrix}^T$.

Proof. If $\tilde{q}$ is an eigenvector of $\mathscr{L}_{1,1}^*$ corresponding to the zero eigenvalue, then it will belong to $\mathscr{N}(\mathscr{L}_{1,1}^*)$, where $\mathscr{N}(\cdot)$ represents the null space, making both $\mathscr{L}_{1,1}^* \tilde{q} = 0$ and $\tilde{q}^T \mathscr{L}_{1,1}^* \tilde{q} = 0$. Moreover, all elements of matrix $\mathscr{L}^*$ are zero, except sub-matrix $\mathscr{L}_{1,1}^*$ and $\tilde{q}^T \mathscr{L}_{1,1}^* \tilde{q} = 0$ will also result in $\tilde{x}^T \mathscr{L}^* \tilde{x} = 0$. ■

Next, we present an important conjecture, relating vector $\tilde{q}$ determined using Theorem 2 and q^*, the solution of (6).

Conjecture 1. If only one eigenvalue of $\mathscr{L}_{1,1}^*$ is zero, then the vector $\tilde{q}$, that makes $\tilde{q}^T \mathscr{L}_{1,1}^* \tilde{q} = 0$, also ensures the relaxation gap between the approximate problem (6) and its semidefinite relaxation (9) is zero, making $\tilde{q} = q^*$.

It is evident from the right hand side of (7), that the first and last terms are zero. If the terms relating with inequality constraints in (6) are zero, then the gap will also be zero. The mathematical proof of this claim will be discussed elsewhere, however we will present numerical simulation results in support of this claim.

IV. SIMULATION RESULTS

We consider the problem of attitude determination for a low cost CubeSat [19], a pico-class of satellite moving in a circular orbit of radius 650 km, using two measurements only, namely the earth magnetic field and the sun vector. For the earth magnetic field, two magnetometers are installed, one inside the satellite, which is mainly used in the post-launch phase when the satellite is recovering from launch disturbances, while the second magnetometer is installed on an extended boom, which is deployed once the satellite has de-tumbled and achieved an equilibrium. The sun vector is sensed by a pair of sun sensors installed on the satellite body. Both of these measurements are in the body frame. For the earth magnetic field vector in the reference frame, we used the first order IGRF model [15], while the reference sun vector is obtained using a simplified sun model based on the sun ephemeris. Both sensor measurements and reference vectors are not accurate. For example, sensor measurements are affected by noise, misalignments, etc. Especially in the post-launch tumbling phase, the measurement errors further increase due to the use of an internal magnetometer installed on-board the satellite, which suffers from interaction with the magnetic field generated by the surrounding electronics. Similarly, the reference vectors are also not exact because they are obtained from mathematical models, which are normally based on low-order approximations for simplification and computational benefits. In this work, we are considering all such errors as ∞-norm bounded uncertainties, and for simulation purpose we set an uncertainty bound of 30% of the 2-norm of body and reference frame vectors.

Firstly, we give a comparison of the analytical upper bound (3) with the bound obtained using SDR. For this comparison, we used two pairs of unit vectors, one in the body frame and second in the reference frame, given as

$$\begin{aligned} b_1 &= \begin{bmatrix} 0.706 & -0.094 & 0.702 \end{bmatrix}^T, \\ r_1 &= \begin{bmatrix} 0.748 & -0.415 & 0.518 \end{bmatrix}^T, \\ b_2 &= \begin{bmatrix} -0.106 & -0.342 & -0.934 \end{bmatrix}^T, \\ r_2 &= \begin{bmatrix} -0.265 & 4.103 \times 10^{-5} & -0.964 \end{bmatrix}^T. \end{aligned} \tag{10}$$

A uniformly distributed random error in the range $\pm\gamma_{bi}$ and $\pm\gamma_{ri}$ is introduced in corresponding vectors for each simulation run, using the uncertainty description given in Appendix I. A comparison of both bounds and their relative errors for 100 simulations is given in Figure 1. The plot shows that the relative error is less than 5% on average. This corresponds to the error in the solution of (2), introduced by the use of the analystical upper bound.

Secondly, we give a quantitative comparison of the optimal quaternion obtained from (6) using `fmincon` (with *interior-point* algorithm, tolerance of 10^{-12} and an initial guess of eigenvector of K matrix corresponding to largest eigenvalue i.e. the quaternion for non-robust solution) and solution of (9) using `mincx` (with same tolerance). For the vector set given in (10), we added a random bounded error and the resulting perturbed vectors are

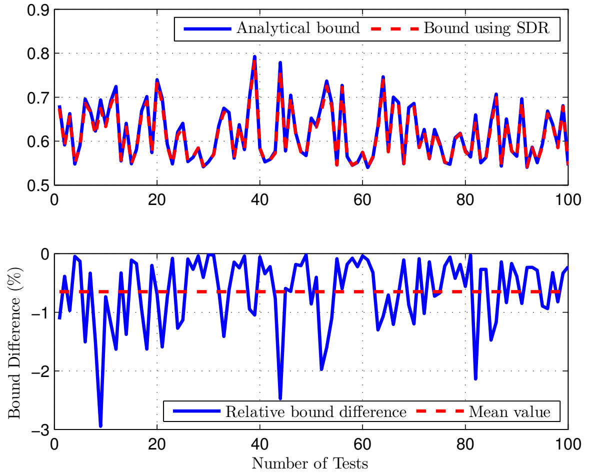

Fig. 1. Comparison of the analytical bound given in (3) and the bound obtained by semidefinite relaxation of the maximization term in (2).

TABLE I

COMPARISON OF THE QUATERNION OBTAINED FROM (6) AND (9) FOR THE VECTOR SET (11)

q^*	$\tilde{q}$	$\lvert q^* - \tilde{q}\rvert$
0.1385731079	0.1385731105	2.5796×10^{-9}
0.0245276544	0.0245276547	3.3654×10^{-10}
-0.0357041309	-0.0357041319	9.3820×10^{-10}
0.9894044183	0.9894044179	4.0349×10^{-10}

$$
\begin{aligned}
b_1 &= \begin{bmatrix}0.794 & -0.073 & 0.603\end{bmatrix}^T,\\
r_1 &= \begin{bmatrix}0.76 & -0.365 & 0.537\end{bmatrix}^T,\\
b_2 &= \begin{bmatrix}-0.083 & -0.347 & -0.934\end{bmatrix}^T,\\
r_2 &= \begin{bmatrix}-0.28 & 5.72\times10^{-5} & -0.96\end{bmatrix}^T.
\end{aligned} \tag{11}
$$

A comparison of the two quaternions is given in Table I. Note that q^* is obtained using Theorem 2. The error between two quaternion is almost zero. Lastly, we present performance comparison of the robust and non-robust approaches in the presence of uncertainties, using in-orbit data obtained from nonlinear closed-loop simulations for the satellite. The ideal data was corrupted by adding uniformly distributed random errors in the range of $\pm\gamma_{bi}$ and $\pm\gamma_{ri}$ in corresponding vectors. We present attitude determination results for 25 minutes of flight data obtained with a sample time of 1 second. The simulation was initialized with roll, pitch and yaw body rates of 0.5, 0.5 and 0.1 deg/s and roll, pitch and yaw angles of 10, 0, 0 deg, respectively. We solved the robust problem formulated in (6) using the nonlinear optimization solver `fmincon` of MATLAB, while the problem formulated using semidefinite relaxation in (9) was solved using the LMI toolbox command `mincx`. First, the performance benefit of the robust approach is given in Figure 2, showing an improvement over the non-robust approach in the presence of uncertainties. It can be observed that due to the uncertainties in the input information, the non-robust approach can give large errors in the estimated attitude angles, while the robust approach gives much better performance, limiting the maximum attitude error to a smaller band. Next, a comparison of the difference between the robust problem and its semidefinite relaxation is presented. In this regard, the eigenvalues of $\mathscr{L}^*_{1,1}$ are presented in Figure 3, showing that the smallest eigenvalue is zero for all cases, validating Theorem 2. Figure 4 support Conjecture 1. Figure 4 shows the relaxation gap between the robust problem and its semidefinite relaxation i.e. $q^T K_r q - u^T e - \bar{\gamma}$, where the first part $q^T K_r q - u^T e$ is calculated using the results obtained from `fmincon` and $\bar{\gamma}$ is obtained from `mincx`. It can be observed that the gap is zero for all time instances of the simulation.

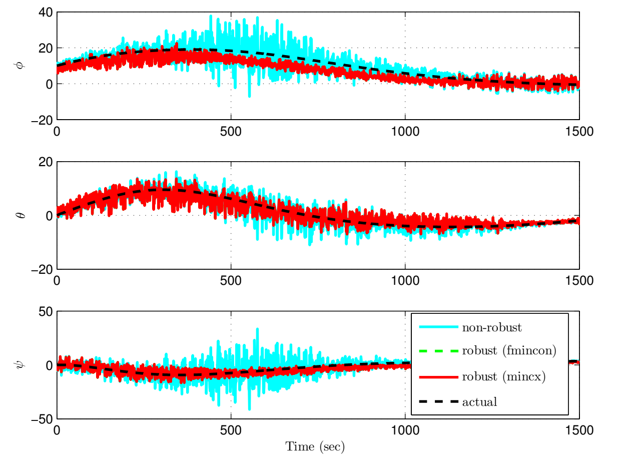

Fig. 2. A comparison of attitude angles obtained using non-robust and robust algorithms. The dotted line shows the original data without errors while the other two cases include errors within the chosen uncertainty bound.

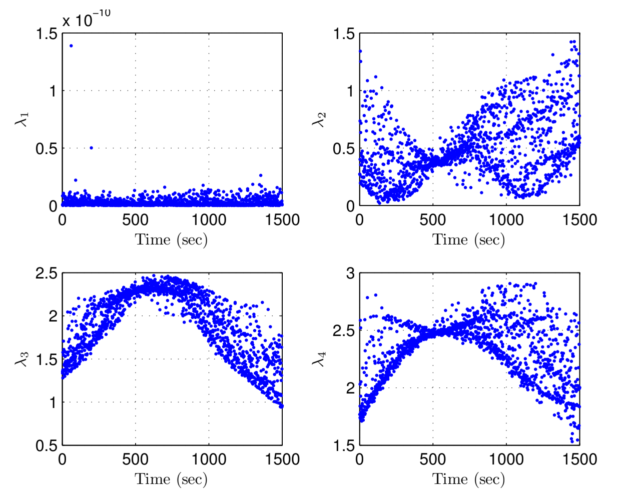

Fig. 3. Eigenvalues of $\mathscr{L}^*_{1,1}$ matrix.

V. CONCLUSIONS AND FUTURE WORK

We presented a semidefinite relaxation based approach to efficiently solve a non-convex nonlinear optimization problem, which is an approximation of a robust attitude determination problem with norm-bounded uncertainties. We approximated this problem into a non-convex quadratic program with quadratic constraints, and then used semidefinite

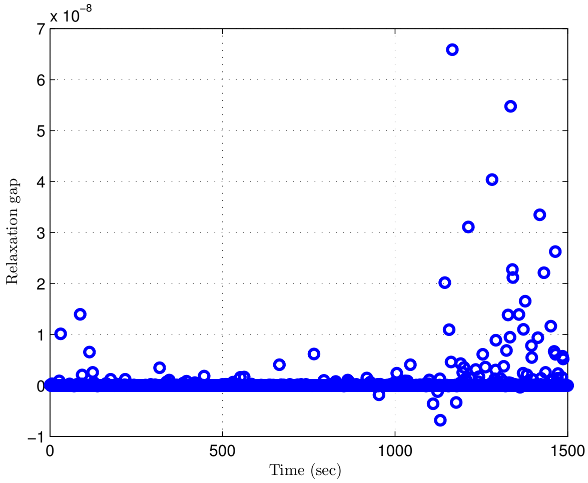

Fig. 4. Gap between the maximum cost of (6) and its upper bound $\bar{\gamma}^*$ obtained using the solution of (9).

relaxation to transform into a semidefinite program with linear cost and linear matrix inequality constraints. It was also shown how to extract the attitude information from the relaxed formulation. Further, the numerical results showed that the gap between (6) and its relaxation (9) is practically zero, showing that the extracted quaternion is the solution to the nonlinear optimization problem (6).

The presented approach can be strengthened by doing further work on some issues. For example, Conjecture 1 needs mathematical justification to guarantee a zero relaxation gap. Moreover, a detailed quantitative analysis of the computational benefit obtained over the solution of the original nonlinear problem will also be helpful in this regard. These tasks could be a possible direction for future work.

Appendix I
Uncertainty Description

A general description of bounded sets $\mathscr{B}(b)$ and $\mathscr{R}(r)$ of uncertain vectors $\bar{b}$ and $\bar{r}$ is given here along with an affine uncertainty parameterization. Let $\beta, \rho \in \mathbb{R}^3$ are vectors of perturbation variables for uncertainty parameterization and $\gamma_b, \gamma_r \in \mathbb{R}$ are bounds on uncertainty for input vectors in the body and reference frame, respectively. We are considering that each input vector can have different error bounds. This type of uncertainty is called an interval uncertainty and corresponding perturbation set represents a box. Further, we normalize each perturbation vector in the body and reference frame with the corresponding uncertainty bound and denote it as $\delta_b = \beta/\gamma_b$ and $\delta_r = \rho/\gamma_r$. Using these normalized perturbation vectors, we describe the uncertainty sets in the body and reference frame as

$$\mathscr{B}(b) = \left\{ b + \sum_{l=1}^{3} \delta_{bl}\tilde{b}_l \mid \|\delta_b\|_\infty \leq 1 \right\},$$

$$\mathscr{R}(r) = \left\{ r + \sum_{l=1}^{3} \delta_{rl}\tilde{r}_l \mid \|\delta_r\|_\infty \leq 1 \right\},$$

where $\tilde{b}_l := \gamma_b e_l$ and $\tilde{r}_l := \gamma_r e_l$ are fixed vectors for a given problem with e_l being +ve l^{th} standard basis vector in $\mathbb{R}^3$.

Appendix II
Definition of Few Matrices

$$K^1_{r_i} = \begin{bmatrix} r_{i1} & r_{i2} & r_{i3} & 0 \\ r_{i2} & -r_{i1} & 0 & -r_{i3} \\ r_{i3} & 0 & -r_{i1} & r_{i2} \\ 0 & -r_{i3} & r_{i2} & r_{i1} \end{bmatrix}, K^2_{r_i} = \begin{bmatrix} -r_{i2} & r_{i1} & 0 & r_{i3} \\ r_{i1} & r_{i2} & r_{i3} & 0 \\ 0 & r_{i3} & -r_{i2} & -r_{i1} \\ r_{i3} & 0 & -r_{i1} & r_{i2} \end{bmatrix},$$

$$K^3_{r_i} = \begin{bmatrix} -r_{i3} & 0 & r_{i1} & -r_{i2} \\ 0 & -r_{i3} & r_{i2} & r_{i1} \\ r_{i1} & r_{i2} & r_{i3} & 0 \\ -r_{i2} & r_{i1} & 0 & r_{i3} \end{bmatrix}, K^1_{b_i} = \begin{bmatrix} b_{i1} & b_{i2} & b_{i3} & 0 \\ b_{i2} & -b_{i1} & 0 & r_{i3} \\ b_{i3} & 0 & -b_{i1} & -b_{i2} \\ 0 & b_{i3} & -b_{i2} & b_{i1} \end{bmatrix},$$

$$K^2_{b_i} = \begin{bmatrix} -b_{i2} & b_{i1} & 0 & -b_{i3} \\ b_{i1} & b_{i2} & b_{i3} & 0 \\ 0 & b_{i3} & -r_{i2} & r_{i1} \\ -b_{i3} & 0 & b_{i1} & b_{i2} \end{bmatrix}, K^3_{b_i} = \begin{bmatrix} -b_{i3} & 0 & b_{i1} & b_{i2} \\ 0 & -b_{i3} & b_{i2} & -b_{i1} \\ b_{i1} & b_{i2} & b_{i3} & 0 \\ b_{i2} & -b_{i1} & 0 & b_{i3} \end{bmatrix}.$$

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