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Delay identification for nonlinear time-delay systems with unknown inputs

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***Abstract*— By using the theory of non-commutative rings, this paper studies the delay identification of nonlinear time-delay systems with unknown inputs. Necessary and sufficient conditions are given for both the dependent and independent outputs cases.**

I. INTRODUCTION

Time-delay systems are widely used to model concrete systems in engineering sciences, such as biology, chemistry, mechanics and so on [11], [14], [18]. Many results have been reported for the purpose of stability analysis, by assuming that the time delay of the studied systems is known. And it makes the delay identification one of the most important topics in the field of time-delay systems.

Up to now, various techniques have been proposed for the delay identification problem, such as identification by using variable structure observers [5], by a modified least squares technique [17], by convolution approach [2], by using the fast identification technique proposed in [6] to deal with online identification of continuous-time systems with structured entries [3] and so on.

Recently, authors in [1] proposed to analyze the delay identification for nonlinear control systems with a single unknown constant time delay by using the non-commutative rings theory, which has been applied to analyze nonlinear time-delay systems firstly by [13] for the disturbance decoupling problem of nonlinear time-delay system, and for observability of nonlinear time-delay systems with known inputs by [19]. Inspired by the work of [1], this paper investigates the delay identification problem for nonlinear time-delay systems with unknown inputs.

This paper is organized as follows. Section II recalls the algebraic framework proposed in [19]. Notations and preliminary result are given in Section III where necessary and sufficient conditions are discussed for identifying the delay from only the outputs of systems. Main theorem for the case where the outputs are independent over the non-commutative rings is stated in Section IV, and several illustrative examples are given in order to highlight the proposed results in Section V.

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II. ALGEBRAIC FRAMEWORK

Denote τ the basic time delay, and assume that the times delays are multiple times of τ. Consider the following nonlinear time-delay system:

$$\left\{\begin{array}{l} \dot{x} = f(x(t-i\tau)) + \sum_{j=0}^{s} g^j(x(t-i\tau))u(t-j\tau) \\ y = h(x(t-i\tau)) = [h_1(x(t-i\tau)), \ldots, h_p(x(t-i\tau))]^T \\ x(t) = \psi(t), u(t) = \varphi(t), t \in [-s\tau, 0] \end{array}\right. \tag{1}$$

where $x \in W \subset R^n$ denotes the state variables, $u = [u_1, \ldots, u_m]^T \in R^m$ is the unknown admissible input, $y \in R^p$ is the measurable output. Without loss of generality, we assume that $p \geq m$. And $i \in S_- = \{0, 1, \ldots, s\}$ is a finite set of constant time-delays, f, g^j and h are meromorphic functions[1], $f(x(t-i\tau)) = f(x, x(t-\tau), \ldots, x(t-s\tau))$ and $\psi : [-s\tau, 0] \to R^n$ and $\varphi : [-s\tau, 0] \to R^m$ denote unknown continuous functions of initial conditions. In this work, it is assumed for initial conditions ψ and φ, (1) admits a unique solution.

Based on the algebraic framework introduced in [19], let $\mathcal{K}$ be the field of meromorphic functions of a finite number of the variables from $\{x_j(t-i\tau), j \in [1, n], i \in S_-\}$. With the standard differential operator d, define the vector space $\mathcal{E}$ over $\mathcal{K}$:

$$\mathcal{E} = span_{\mathcal{K}}\{d\xi : \xi \in \mathcal{K}\}$$

which is the set of linear combinations of a finite number of one-forms from $dx_j(t-i\tau)$ with row vector coefficients in $\mathcal{K}$. For the sake of simplicity, we introduce backward time-shift operator δ, which means

$$\delta^i \xi(t) = \xi(t-i\tau), \xi(t) \in \mathcal{K}, \text{ for } i \in Z^+ \tag{2}$$

and

$$\begin{array}{rcl} \delta^i\left(a(t)d\xi(t)\right) & = & \delta^i a(t)\delta^i d\xi(t) \\ & = & a(t-i\tau)d\xi(t-i\tau) \end{array} \tag{3}$$

for $a(t)d\xi(t) \in \mathcal{E}$, and $i \in Z^+$.

Let $\mathcal{K}(\delta]$ denote the set of polynomials of the form

$$a(\delta] = a_0(t) + a_1(t)\delta + \cdots + a_{r_a}(t)\delta^{r_a} \tag{4}$$

where $a_i(t) \in \mathcal{K}$. The addition in $\mathcal{K}(\delta]$ is defined as usual, but the multiplication is given as

$$a(\delta]b(\delta] = \sum_{k=0}^{r_a+r_b} \sum_{i+j=k}^{i \leq r_a, j \leq r_b} a_i(t)b_j(t-i\tau)\delta^k \tag{5}$$

[1]means quotients of convergent power series with real coefficients [4], [19].

Note that $\mathcal{K}(\delta]$ satisfies the associative law and it is a non-commutative ring (see [19]). However, it is proved that the ring $\mathcal{K}(\delta]$ is a left Ore ring [10], [19], which enables to define the rank of a module over this ring. Let $\mathcal{M}$ denote the left module over $\mathcal{K}(\delta]$: $\mathcal{M} = span_{\mathcal{K}(\delta]}\{d\xi, \xi \in \mathcal{K}\}$, where $\mathcal{K}(\delta]$ acts on $d\xi$ according to (2) and (3).

With the definition of $\mathcal{K}(\delta]$, (1) can be rewritten in a more compact form as follows:

$$\begin{cases} \dot{x} = f(x,\delta) + \sum_{i=1}^{m} G_i u_i(t) \\ y = h(x,\delta) \\ x(t) = \psi(t), u(t) = \varphi(t), t \in [-s\tau, 0] \end{cases} \tag{6}$$

where $f(x,\delta) = f(x(t-i\tau))$ and $h(x,\delta) = h(x(t-i\tau))$ with entries belonging to $\mathcal{K}$, $G_i = \sum_{j=0}^{s} g_i^j \delta^j$ with entries belonging to $\mathcal{K}(\delta]$.

III. Notations and preliminary result

Note that the derivative and Lie derivative for nonlinear systems without delays is well defined (see [9]), then many efforts have been done to extend the classical Lie derivative for nonlinear time-delay systems. Several researchers tried to extend the Lie derivative to nonlinear time-delay systems (see [7], [8], [16], [15]) in the framework of commutative rings. In what follows we define the derivative and Lie derivative for nonlinear time-delay from non-commutative rings point of view.

Let $f(x(t-j\tau))$ and $h(x(t-j\tau))$ for $0 \leq j \leq s$ respectively be an n and p dimensional vector with entries $f_r \in \mathcal{K}$ for $1 \leq r \leq n$ and $h_i \in \mathcal{K}$ for $1 \leq i \leq p$. Let

$$\frac{\partial h_i}{\partial x} = \left[\frac{\partial h_i}{\partial x_1}, \cdots, \frac{\partial h_i}{\partial x_n}\right] \in \mathcal{K}^{1\times n}(\delta] \tag{7}$$

where for $1 \leq r \leq n$:

$$\frac{\partial h_i}{\partial x_r} = \sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t-j\tau)} \delta^j \in \mathcal{K}(\delta]$$

then the Lie derivative for nonlinear systems without delays can be extended to nonlinear time-delay systems in the framework of [19] as follows

$$L_f h_i = \frac{\partial h_i}{\partial x}(f) = \sum_{r=1}^{n}\sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t-j\tau)} \delta^j (f_r) \in \mathcal{K} \tag{8}$$

For $j = 0$, (8) is the classical definition of the Lie derivative of h along f. For $h_i \in \mathcal{K}$, define

$$L_{G_i} h_i = \frac{\partial h_i}{\partial x}(G_i) \in \mathcal{K}(\delta]$$

After having defined the derivative of function belonging to $\mathcal{K}(\delta]$, let study the time delay identification for system (6).

Definition 1: [1] An output equation $\alpha(h, \dot{h}\dots, h^{(k)}, \delta) = 0$ is said to involve δ in an essential way if it cannot be written as $\alpha(h, \dot{h}\dots, h^{(k)}, \delta) = a(\delta]\tilde{\alpha}(h, \dot{h}\dots, h^{(k)})$ with $a(\delta] \in \mathcal{K}(\delta]$.

As stated in [1], if there exists a function for (6) containing only the output, its derivatives and delays in an essential way, then the delay can be identified by numerically finding zeros of such a function. Thus delay identification for (6) becomes to seek such a function.

Let consider the simplest case for identifying the delay for (6), i.e., from only the outputs of (6), which is stated in the following preliminary result.

Theorem 1: There exists a function containing only the output and its delays in an essential way which enables to identify the delay of (6) if and only if

$$rank_{\mathcal{K}(\delta]} \frac{\partial h}{\partial x} < rank_{\mathcal{K}} \frac{\partial h}{\partial x} \tag{9}$$

Proof: **Necessity:**

Suppose that there exists a function containing only the output and its delays in an essential way, let show (9) is satisfied. For this, let consider the opposite, i.e. assume

$$rank_{\mathcal{K}(\delta]} \frac{\partial h}{\partial x} = rank_{\mathcal{K}} \frac{\partial h}{\partial x},$$

and this implies that h does not contain any delay, otherwise we have

$$rank_{\mathcal{K}(\delta]} \frac{\partial h}{\partial x} < rank_{\mathcal{K}} \frac{\partial h}{\partial x}.$$

No involvement of delay in h implies that it is not possible to identify the delay from the output, thus it contradicts the assumption that there exists a function of the output and its delays to identify the delay, and we prove the necessity by contradiction.

Sufficiency:

Suppose that

$$rank_{\mathcal{K}(\delta]} \frac{\partial h}{\partial x} < rank_{\mathcal{K}} \frac{\partial h}{\partial x} \leq p,$$

then it can be interpreted as follows:

$$\frac{\partial h_p}{\partial x} \in span_{\mathcal{K}(\delta]}\{\frac{\partial h_1}{\partial x}, \dots, \frac{\partial h_{p-1}}{\partial x}\}$$

Thus there exists $a_i \in \mathcal{K}(\delta]$ for $1 \leq i \leq p-1$, such that

$$dh_p = a_1 dh_1 + \cdots + a_{p-1} dh_{p-1}$$

The differentiation of the above equation for both sides is equal to zero, implying the two-form for each side is null. Thus each side of the above equation is a closed one-form. By applying Poincaré Theorem, there always exists a function α such that

$$\alpha(h_1, \dots, h_p, \delta) = 0$$

In addition, the function α should involve δ in an essential way, because if it is not the case, then δ can be taken out from α, yielding

$$rank_{\mathcal{K}(\delta]} \frac{\partial h}{\partial x} = rank_{\mathcal{K}} \frac{\partial h}{\partial x},$$

and this contradicts the inequality (9). ∎

Remark 1: Inequality (9) implies that the outputs of (6) are dependent over $\mathcal{K}(\delta]$. Theorem 1 can be seen as a special case of Theorem 2 in [1]. However as we will show in the next section that this condition is not necessary for the case where the output of (6) is independent over $\mathcal{K}(\delta]$.

Example 1: Consider the following dynamical system:

$$\begin{cases} \dot{x} = f(x,u,\delta) \\ y_1 = x_1 \\ y_2 = x_1\delta x_1 + x_1^2 \end{cases} \quad (10)$$

It can be seen that

$$\frac{\partial h}{\partial x} = \begin{pmatrix} 1 & 0 \\ \delta x_1 + 2x_1 + x_1\delta & 0 \end{pmatrix}$$

which yields $rank_{\mathcal{K}(\delta]}\frac{\partial h}{\partial x} = 1$ and $rank_{\mathcal{K}}\frac{\partial h}{\partial x} = 2$. Thus Theorem 1 is satisfied, and the time delay of system (10) can be identified.

In fact, a straightforward calculation gives

$$y_2 = y_1\delta y_1 + y_1^2$$

which permits to identify the time delay δ by applying an algorithm to detect zero-crossing when varying δ.

IV. Main result

Identification of time delay for (6) from its output is analyzed in the last section, this section considers the case where the output of (6) is independent over $\mathcal{K}(\delta]$, i.e., $rank_{\mathcal{K}(\delta]}\frac{\partial h}{\partial x} = p$.

Based on the notations of derivative and Lie derivation of the functions belonging to $\mathcal{K}(\delta]$ introduced in the last section, we can define the relative degree in the following way:

Definition 2: (Relative degree) System (6) has relative degree $(\nu_1, \cdots, \nu_p)$ in an open set $W \subseteq R^n$ if, for $1 \leq i \leq p$, the following conditions are satisfied :

1) for all $x \in W$, $L_{G_j}L_f^r h_i = 0$, for all $1 \leq j \leq m$ and $0 \leq r < \nu_i - 1$;
2) there exists $x \in W$ such that $\exists j \in [1, m]$, $L_{G_j}L_f^{\nu_i - 1}h_i \neq 0$.

If for $1 \leq i \leq p$, (1) is satisfied for all $r \geq 0$, then we set $\nu_i = \infty$.

Then, for (6), define the so-called observability indices introduced in [12]. Note

$$\mathcal{F}_k := span_{\mathcal{K}(\delta]}\left\{dh, dL_f h, \cdots, dL_f^{k-1}h\right\}$$

for $1 \leq k \leq n$, and it was shown that the filtration of $\mathcal{K}(\delta]$-module satisfies $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n$, then define $d_1 = rank_{\mathcal{K}(\delta]}\mathcal{F}_1$, and $d_k = rank_{\mathcal{K}(\delta]}\mathcal{F}_k - rank_{\mathcal{K}(\delta]}\mathcal{F}_{k-1}$ for $2 \leq k \leq n$. Let $k_i = card\{d_k \geq i, 1 \leq k \leq n\}$, then $(k_1, \cdots, k_p)$ are the observability indices[2]. Reorder, if necessary, the output components of (6), such that

$$rank_{\mathcal{K}(\delta]}\{\frac{\partial h_1}{\partial x}, \cdots, \frac{\partial L_f^{k_1-1}h_1}{\partial x}, \cdots, \frac{\partial h_p}{\partial x}, \cdots, \frac{\partial L_f^{k_p-1}h_p}{\partial x}\} = k_1 + \cdots + k_p$$

Since we assume that the output of system (6) is independent over $\mathcal{K}(\delta]$, i.e.

$$rank_{\mathcal{K}(\delta]}\frac{\partial h}{\partial x} = p$$

then observability indices $(k_1, \cdots, k_p)$ for $(h_1, \cdots, h_p)$ are well defined, but the order may be not unique.

After having defined the relative degree and observability indices via the extended Lie derivative for nonlinear time-delay systems in the framework of non-commutative rings, let recall the following theorems presented in [20].

Theorem 2: [20] For $1 \leq i \leq p$, denote k_i the observability indices and ν_i the relative degree index for y_i of (6), and note $\rho_i = \min\{\nu_i, k_i\}$. Then there exists a change of coordinate $\phi(x,\delta) \in \mathcal{K}^{n\times 1}$, such that (6) can be transformed into the following form:

$$\dot{z}_{i,j} = z_{i,j+1} \quad (11)$$

$$\dot{z}_{i,\rho_i} = V_i = L_f^{\rho_i}h_i(x,\delta) + \sum_{j=1}^{m} L_{G_j}L_f^{\rho_i-1}h_i(x,\delta)u_j \quad (12)$$

$$y_i = C_i z_i = z_{i,1} \quad (13)$$

$$\dot{\xi} = \alpha(z,\xi,\delta) + \beta(z,\xi,\delta)u \quad (14)$$

where

$$z_i = \left(h_i, \cdots, L_f^{\rho_i-1}h_i\right)^T \in \mathcal{K}^{\rho_i\times 1}$$

$$\alpha \in \mathcal{K}^{l\times 1}, \beta \in \mathcal{K}^{l\times 1}(\delta] \text{ with } l = n - \sum_{j=1}^{p}\rho_j$$

$$C_i = (1, 0, \cdots, 0) \in R^{1\times\rho_i}$$

Moreover if $k_i < \nu_i$, one has $V_i = L_f^{\rho_i}h_i = L_f^{k_i}h_i$. ■

For (11), note

$$\mathcal{H}(x,\delta) = \Psi(x,\delta) + \Gamma(x,\delta)u \quad (15)$$

with

$$\mathcal{H}(x,\delta) = \left(h_1^{(\rho_1)}, \cdots, h_p^{(\rho_p)}\right)^T$$

$$\Psi(x,\delta) = \left(L_f^{\rho_1}h_1, \cdots, L_f^{\rho_p}h_p\right)^T$$

and

$$\Gamma(x,\delta) = \begin{pmatrix} L_{G_1}L_f^{\rho_1-1}h_1 & \cdots & L_{G_m}L_f^{\rho_1-1}h_1 \\ \vdots & \ddots & \vdots \\ L_{G_1}L_f^{\rho_p-1}h_p & \cdots & L_{G_m}L_f^{\rho_p-1}h_p \end{pmatrix} \quad (16)$$

where $\mathcal{H}(x,\delta) \in \mathcal{K}^{p\times 1}$, $\Psi(x,\delta) \in \mathcal{K}^{p\times 1}$ and $\Gamma(x,\delta) \in \mathcal{K}^{p\times m}(\delta]$. And for (6), let denote Φ the observable space from its outputs:

$$\Phi = \{dh_1, \cdots, dL_f^{\rho_1-1}h_1, \cdots, dh_p, \cdots, dL_f^{\rho_p-1}h_p\} \quad (17)$$

If $rank_{\mathcal{K}(\delta]}\Phi = j$, then without loss of generality, we can select j linearly independent vector over $\mathcal{K}(\delta]$ from Φ, noted as $\Phi = \{dz_1, \cdots, dz_j\}$. Note

$$\pounds = span_{R[\delta]}\{z_1, \cdots, z_j\}$$

where $R[\delta]$ is the commutative ring of polynomials of δ with coefficients belonging to the field R, and let $\pounds(\delta]$ be the set of polynomials of δ with coefficients over $\pounds$, define the module spanned by element of Φ over $\pounds(\delta]$ as follows

$$\Omega = span_{\pounds(\delta]}\{\xi, \xi \in \Phi\} \quad (18)$$

[2]If $\sum_{i=1}^{p} k_i = n$, then (6) is observable with $u = 0$.

Define $\mathcal{G} = span_{R[\delta]}\{G_1, \ldots, G_m\}$ and its *left* annihilator

$$\mathcal{G}^{\perp} = span_{R[\delta]}\{\omega \in \Omega \mid \omega g = 0, \forall g \in \mathcal{G}\}$$

Based on the above definitions, we are ready to state our main result.

Theorem 3: There exists a function containing the output, its derivative and delays for (11 - 14), if there exists $\omega \in \mathcal{G}^{\perp} \cap \Omega$ such that $\omega f \in \pounds$. ■

Proof: Denote $Q = \{q_1, \cdots, q_p\}$ be $1 \times p$ vector with $q_j \in \mathcal{K}(\delta]$ for $1 \leq j \leq p$. One has

$$\begin{aligned} Q\Gamma &= Q \begin{pmatrix} L_{G_1} L_f^{\rho_1 - 1} h_1 & \cdots & L_{G_m} L_f^{\rho_1 - 1} h_1 \\ \vdots & \ddots & \vdots \\ L_{G_1} L_f^{\rho_p - 1} h_p & \cdots & L_{G_m} L_f^{\rho_p - 1} h_p \end{pmatrix} \\ &= \left(Q \begin{bmatrix} \frac{\partial L_f^{\rho_1 - 1} h_1}{\partial x} \\ \vdots \\ \frac{\partial L_f^{\rho_p - 1} h_p}{\partial x} \end{bmatrix} \right) [G_1, \cdots, G_m] \end{aligned}$$

because of the associativity law over $\mathcal{K}(\delta]$. Then according to the definition (7), one gets

$$Q\Gamma = \omega \left[G_1, \cdots, G_m\right] = \omega G$$

where $\omega = \sum_{c=1}^{n} \sum_{j=1}^{p} q_j \frac{\partial L_f^{\rho_j - 1} h_j}{\partial x_c} dx_c$.

Moreover, it is easy to check that

$$\begin{aligned} \omega f &= \left(Q \begin{bmatrix} \frac{\partial L_f^{\rho_1 - 1} h_1}{\partial x} \\ \vdots \\ \frac{\partial L_f^{\rho_p - 1} h_p}{\partial x} \end{bmatrix} \right) f = Q \left(\begin{bmatrix} \frac{\partial L_f^{\rho_1 - 1} h_1}{\partial x} \\ \vdots \\ \frac{\partial L_f^{\rho_p - 1} h_p}{\partial x} \end{bmatrix} f \right) \\ &= Q \begin{bmatrix} L_f^{\rho_1} h_1 \\ \vdots \\ L_f^{\rho_p} h_p \end{bmatrix} = Q\Psi \end{aligned}$$

According to (15), one has

$$Q\mathcal{H} = Q(\Psi + \Gamma u) = \omega f + \omega G u \tag{19}$$

where $\mathcal{H} = \left[y_1^{(\rho_1)}, \cdots, y_p^{(\rho_p)}\right]^T$ is a vector which can be estimated in finite time.

Suppose that there exists $\omega \in \mathcal{G}^{\perp} \cap \Omega$ such that $\omega f \in \pounds$, which implies there exists Q with entries belonging to $\pounds(\delta]$, such that

$$Q\Gamma = \omega G = 0$$

and

$$Q\mathcal{H} = \omega f \in \pounds$$

which implies that one obtains the following relation:

$$Q(\mathcal{H} - \Psi) = 0 \tag{20}$$

which contains only the output, its derivatives and delays. ■

Remark 2: If one can find a $\omega \in \mathcal{G}^{\perp} \cap \Omega$ such that $\omega f \notin \pounds$, which means that Theorem 3 is not satisfied, then $\omega f \mod \pounds$ gives new but observable variables not belonging to $\pounds$, which in fact can be seen as extended output $\bar{y}$. Combining y and $\bar{y}$, a new canonical form of (11-14) can then be deduced, in such a way that it is still possible to obtain (20) by checking Theorem 3.

Obviously, if (20) contains the delay δ in an essential way, then the time delay of (11-14) can be identified by detecting zero-crossing of (20). Before to give necessary and sufficient conditions guaranteeing the essential involvement of δ in (20), define

$$\mathcal{Y} = \left(h_1, \ldots, L_f^{\rho_1 - 1} h_1, \ldots, h_p, \ldots, L_f^{\rho_p - 1} h_p\right)^T$$

and denote $\mathcal{K}_0 \subset \mathcal{K}$ the field of meromorphic functions of x, which will be used in the following theorem.

Theorem 4: (20) involves the delay δ in an essential way if and only if

$$rank_{\mathcal{K}(\delta]} \frac{\partial \mathcal{Y}}{\partial x} < rank_{\mathcal{K}} \frac{\partial \{\mathcal{Y}, \Psi\}}{\partial x} \tag{21}$$

or for any scalar $a(\delta] \in \mathcal{K}(\delta]$,

$$\frac{\partial (Q\Psi)}{\partial x} \notin a(\delta] span_{\mathcal{K}_0} \{\frac{\partial \Psi}{\partial x}\} \tag{22}$$

and

$$rank_{\mathcal{K}(\delta]} \frac{\partial \mathcal{Y}}{\partial x} = rank_{\mathcal{K}} \frac{\partial \{\mathcal{Y}, \Psi\}}{\partial x} \tag{23}$$

■

Proof: **Necessity:**

Suppose that (20) involves the delay δ in an essential way, we will show that either (21) or (22-23) is satisfied. We prove this by contradiction and let consider the opposite of (21-23), i.e.

$$rank_{\mathcal{K}(\delta]} \frac{\partial \mathcal{Y}}{\partial x} = rank_{\mathcal{K}} \frac{\partial \{\mathcal{Y}, \Psi\}}{\partial x} \tag{24}$$

and

$$\frac{\partial (Q\Psi)}{\partial x} \in a(\delta] span_{\mathcal{K}_0} \{\frac{\partial \Psi}{\partial x}\}. \tag{25}$$

It can be seen that (24) implies that Ψ is a function of $\mathcal{Y}$ without δ, and (25) implies that there exists a scalar $a(\delta] \in \mathcal{K}(\delta]$ such that

$$Q = a(\delta] \left(B\right),$$

where B is the associated vector with entries belonging to the field $\mathcal{K}_0$, then

$$Q(\mathcal{H} - \Psi) = a(\delta] \left(B(\mathcal{H} - \Psi)\right) = 0,$$

and this contradicts the assumption that (20) involves the delay δ in an essential way, so we proved the necessity by contradiction.

Sufficiency:

Firstly, suppose that

$$rank_{\mathcal{K}(\delta]} \frac{\partial \mathcal{Y}}{\partial x} < rank_{\mathcal{K}} \frac{\partial \{\mathcal{Y}, \Psi\}}{\partial x},$$

then Ψ should be a function of $\mathcal{Y}$ containing δ in an essential way, since if it is not the case, one can find a function α such that

$$\alpha \left(\Psi, \mathcal{Y}\right) = 0,$$

which implies that

$$rank_{\mathcal{K}} \frac{\partial \{\mathcal{Y}, \Psi\}}{\partial x} \leq rank_{\mathcal{K}} \frac{\partial \{\mathcal{Y}\}}{\partial x} = rank_{\mathcal{K}(\delta]} \frac{\partial \mathcal{Y}}{\partial x},$$

since $\mathcal{Y}$ is linearly independent over $\mathcal{K}(\delta]$. Because Ψ is a function of $\mathcal{Y}$ containing δ in an essential way, thus for any Q with entries belonging to $\pounds(\delta]$, the function $Q(\mathcal{H}-\Psi)=0$ always contains δ in an essential way.

Secondly, since

$$rank_{\mathcal{K}(\delta]}\frac{\partial\mathcal{Y}}{\partial x}\leq rank_{\mathcal{K}}\frac{\partial\{\mathcal{Y}\}}{\partial x}\leq rank_{\mathcal{K}}\frac{\partial\{\mathcal{Y},\Psi\}}{\partial x}$$

and

$$rank_{\mathcal{K}(\delta]}\frac{\partial\mathcal{Y}}{\partial x}\leq rank_{\mathcal{K}(\delta]}\frac{\partial\{\mathcal{Y},\Psi\}}{\partial x}\leq rank_{\mathcal{K}}\frac{\partial\{\mathcal{Y},\Psi\}}{\partial x},$$

so if

$$rank_{\mathcal{K}(\delta]}\frac{\partial\mathcal{Y}}{\partial x}=rank_{\mathcal{K}}\frac{\partial\{\mathcal{Y},\Psi\}}{\partial x},$$

then one obtains

$$rank_{\mathcal{K}}\frac{\partial\{\mathcal{Y}\}}{\partial x}=rank_{\mathcal{K}}\frac{\partial\{\mathcal{Y},\Psi\}}{\partial x}$$

and

$$rank_{\mathcal{K}(\delta]}\frac{\partial\mathcal{Y}}{\partial x}=rank_{\mathcal{K}(\delta]}\frac{\partial\{\mathcal{Y},\Psi\}}{\partial x}.$$

The above two equalities imply that Ψ should be a function of $\mathcal{Y}$ without delays, otherwise one has

$$rank_{\mathcal{K}(\delta]}\frac{\partial\mathcal{Y}}{\partial x}\neq rank_{\mathcal{K}}\frac{\partial\{\mathcal{Y},\Psi\}}{\partial x},$$

and this contradicts the assumption of the satisfactory of (23).

Moreover, if (22) is true, it implies that Q contains delays in an essential way, otherwise one can find a vector B with entries belonging to $\mathcal{K}_0$ and a scalar $a(\delta]\in\mathcal{K}(\delta]$ such that

$$Q=a(\delta]\,(B)\,,$$

which yields

$$\frac{\partial(Q\Psi)}{\partial x}\in a(\delta]span_{\mathcal{K}_0}\{\frac{\partial\Psi}{\partial x}\}$$

and this contradicts the assumption of (22).

Finally the essential involvement of δ in Q means the essential involvement of δ in the function:

$$Q(\mathcal{H}-\Psi)=0$$

even when Ψ does not contain any delay. ■

Remark 3: A similar condition as (21) of Theorem 4 is stated as a necessary and sufficient condition for identifying time delay for nonlinear time-delay systems with known inputs in [1]. However as we proved above that it is only sufficient, but not necessary for the case with unknown inputs.

V. Illustrative examples

In this section, let consider several examples in order to highlight the proposed results.

Example 2: Consider the following dynamical system:

$$\begin{cases}\dot{x}_1=-\delta x_1+\delta x_4u_1\\ \dot{x}_2=-\delta x_3+x_4\\ \dot{x}_3=x_3-\delta x_4u_1\\ \dot{x}_4=u_2\\ y_1=x_1\\ y_2=\delta x_1+x_3\end{cases}\tag{26}$$

then it is easy to check that $k_1=k_2=\nu_1=\nu_2=1$, which gives $\rho_1=\rho_2=1$ and $\Phi=\{dx_1,\delta dx_1+dx_3\}$.

Set $\mathcal{G}=span_{R[\delta]}\{G_1,\ldots,G_m\}$, then one has

$$\mathcal{G}^{\perp}=span_{R[\delta]}\left\{dx_1+dx_3,dx_2\right\}$$

Since $rank_{\mathcal{K}(\delta]}\Phi=2$, it yields $\pounds=span_{R[\delta]}\left\{x_1,\delta x_1+x_3\right\}$ and

$$\Omega=span_{\pounds(\delta]}\left\{\xi,\xi\in\Phi\right\}=span_{\pounds(\delta]}\left\{dx_1,dx_3\right\}$$

then one obtains

$$\begin{aligned}\Omega\cap\mathcal{G}^{\perp}&=span_{\pounds(\delta]}\left\{dx_1,dx_3\right\}\\&\quad\cap span_{R[\delta]}\left\{dx_1+dx_3,dx_2\right\}\\&=span_{\pounds(\delta]}\left\{dx_1+dx_3\right\}\end{aligned}$$

Obviously, one can find the one-form, for example, $\omega=\delta dx_1+\delta dx_3$, satisfying $\omega\in\Omega\cap\mathcal{G}^{\perp}$ and $\omega f=-\delta^2x_1+\delta x_3\in\pounds$. So there exists a function of the output, its derivative and its delays, noted as

$$Q(\mathcal{H}-\Psi)=0\tag{27}$$

where $\mathcal{H}=(\dot{y}_1,\dot{y}_2)$, $\Psi=\left(-\delta x_1,-\delta^2x_1+x_3\right)^T$ and $Q=(\delta-\delta^2,\delta)$ is determined by

$$Q\mathcal{H}=\omega f.$$

According to the definition of $\mathcal{Y}$, one has $\mathcal{Y}=(x_1,x_3)^T$, which gives

$$rank_{\mathcal{K}(\delta]}\frac{\partial\mathcal{Y}}{\partial x}=2<rank_{\mathcal{K}}\frac{\partial\{\mathcal{Y},\Psi\}}{\partial x}=3$$

then Theorem 4 is satisfied and (27) involves δ in an essential way. A straightforward calculation gives

$$(1-\delta)\dot{y}_1+\dot{y}_2=y_2-2\delta y_1$$

which can be used to identify δ.

Example 3: Consider the following system:

$$\begin{cases}\dot{x}_1&=x_1x_2+x_2u\\ \dot{x}_2&=x_1^2+x_1\delta u\\ y_1&=x_1\\ y_2&=x_2\end{cases}\tag{28}$$

and one has $k_1=k_2=\nu_1=\nu_2=1$, which gives $\rho_1=\rho_2=1$ and $\Phi=\{dx_1,dx_2\}$. By simple calculations, one obtains

$$\begin{aligned}\mathcal{G}^{\perp}&=span_{R[\delta]}\{x_1\delta dx_1-\delta x_2dx_2\},\\ \pounds&=span_{R[\delta]}\{x_1,x_2\},\end{aligned}$$

and

$$\Omega=span_{\pounds(\delta]}\left\{dx_1,dx_2\right\}$$

which gives

$$\begin{aligned}\Omega\cap\mathcal{G}^{\perp}&=span_{\pounds(\delta]}\left\{dx_1,dx_2\right\}\\&\quad\cap span_{R[\delta]}\{x_1\delta dx_1-\delta x_2dx_2\}\\&=span_{\pounds(\delta]}\{x_1\delta dx_1-\delta x_2dx_2\}\end{aligned}$$

An one-form can be found, such as $\omega = x_1\delta dx_1 - \delta x_2 dx_2 \in \Omega \cap \mathcal{G}^{\perp}$, satisfying $\omega f = x_1\delta x_1\delta x_2 - \delta x_2 x_1^2 \in \pounds$. Thus the following function

$$Q(\mathcal{H} - \Psi) = 0 \tag{29}$$

contains only the output, its derivatives and delays, where $\mathcal{H} = (\dot{y}_1, \dot{y}_2)^T$, $\Psi = \left(x_1 x_2, x_1^2\right)^T$ and $Q = (x_1\delta, -\delta x_2)$ is determined by the equality $Q\mathcal{H} = \omega f$.

Since $\mathcal{Y} = (x_1, x_2)^T$, one has

$$rank_{\mathcal{K}(\delta]} \frac{\partial \mathcal{Y}}{\partial x} = rank_{\mathcal{K}} \frac{\partial \{\mathcal{Y}, \Psi\}}{\partial x} = 2$$

However, since

$$\begin{aligned} \frac{\partial (Q\Psi)}{\partial x} &= \frac{\partial (x_1\delta x_1\delta x_2 - x_1^2\delta x_2)}{\partial x} \\ &= \left(x_1\delta x_2\delta + \delta x_1\delta x_2 - 2x_1\delta x_2, x_1\delta x_1\delta - x_1^2\delta\right) \end{aligned}$$

and

$$\frac{\partial \Psi}{\partial x} = \begin{pmatrix} x_2 & x_1 \\ 2x_1 & 0 \end{pmatrix},$$

then it can be checked that for any scalar $a(\delta] \in \mathcal{K}(\delta]$, the following condition

$$\frac{\partial (Q\Psi)}{\partial x} \notin a(\delta] span_{\mathcal{K}_0}\{\frac{\partial \Psi}{\partial x}\}$$

is always satisfied. Thus Theorem 4 is satisfied, and the equation (29) involves δ in an essential way, and it gives

$$y_1\delta\dot{y}_1 - \delta y_2\dot{y}_2 = y_1\delta y_1\delta y_2 - \delta y_2 y_1^2$$

which permits to identify δ.

Remark 4: If we replace the first equation of (28) by $\dot{x}_1 = x_1x_2 + \delta u$, i.e.

$$\left\{ \begin{array}{ll} \dot{x}_1 & = x_1x_2 + \delta u \\ \dot{x}_2 & = x_1^2 + x_1\delta u \\ y_1 & = x_1 \\ y_2 & = x_2 \end{array} \right. \tag{30}$$

we have the same $\mathcal{Y}$, $\mathcal{H}$ and Ψ as those for (28), but a different $Q = (\delta x_1\delta, -\delta)$. Then we still have

$$rank_{\mathcal{K}(\delta]} \frac{\partial \mathcal{Y}}{\partial x} = rank_{\mathcal{K}} \frac{\partial \{\mathcal{Y}, \Psi\}}{\partial x} = 2.$$

However,

$$\frac{\partial (Q\Psi)}{\partial x} = \frac{\partial (\delta x_1^2\delta x_2 - \delta x_1^2)}{\partial x} = \left(2\left(\delta x_2 - 1\right)\delta x_1\delta, \delta x_1^2\delta\right)$$

then there exists $a(\delta] = \delta x_1\delta$ such that

$$\frac{\partial (Q\Psi)}{\partial x} = a(\delta] span_{\mathcal{K}_0}\{\frac{\partial \Psi}{\partial x}\}$$

thus it is not possible to identify δ for (30). In fact, by simplification, we obtain

$$y_1\dot{y}_1 - \dot{y}_2 = y_1^2 y_2 - y_1^2$$

which does not involve any delay.

VI. Conclusion

This paper deals with the delay identification of time-delay systems with unknown inputs. Necessary and sufficient condition for the simplest case, i.e. identification of the delay from the output, has been studied. For a more generic case, sufficient condition is given in order to guarantee the existence of a function containing only the output, its derivatives and delays. Necessary and sufficient condition is then discussed to check whether the deduced function can be used to identify the delay.

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