2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

A Generalized Fractional-Order Iterative Learning Control

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***Abstract*— In this paper, we discuss in time domain the convergence of the iterative process for fractional-order nonlinear systems. A linear adaptive generalized fractional-order iterative learning control scheme is derived, which guarantees the convergence of the algorithm without the knowledge of the system order. The implementation of generalized fractional-order iterative learning controllers can be achieved by using the idea of fractional-order element networks. Most of the classical fractional-order iterative learning control methods fall into the scheme of this paper. A number of numerical simulations are illustrated to validate the concepts.**

***Index Terms*— Iterative learning control, Fractional calculus, Nonlinear systems, Convergence, Adaptiveness.**

I. Introduction

Iterative learning control (ILC), which belongs to the intelligent control methodology, is an approach for improving the transient performance of systems that operate repetitively over a fixed time interval [1], [2]. The advantages of the ILC algorithm are shown in its applications to the nonlinear systems and the systems with uncertainty or unknown structure information, etc [1], [2], [3], [4], [5], [6]. In recent years, the adaptiveness and robustness of linear or nonlinear ILC schemes become very popular topics, which extended the applications of ILC to more complicated problems [7], [8], [9], [10], [11], [12]. Some other interesting conclusions and surveys can be found in [13], [14], [15], [16], [17], [18]. Moreover, in the past three years, the applications of the ILC technique to medical treatments and engineering are getting more and more popular [5], [6], [19], [20].

The combination of ILC and fractional calculus was first proposed in [21], in which a D^{α} type ILC algorithm was discussed. In the following ten years, many fractional-order ILC problems were presented aiming at enhancing the performance of ILC scheme for linear or nonlinear systems [22], [23], [24], [25], [26]. However, most of these discussions are focused on PD^{α} type ILC updating laws, where $\alpha \in (0,1)$ equals the order of the fractional-order system, i.e. the design of ILC is strongly relied on the accurate value of α. But, in practice, the order of a fractional-order system is usually determined by experimental data or empirical values so that the errors are inevitable. Moreover, it was shown in [23] that the difference between the orders of ILC and system is closely related to the performance of the ILC algorithm. Especially when the order of ILC is higher than the system order, the convergence of the ILC algorithm cannot be guaranteed. It follows that, in reality, a relatively small $\beta \le \alpha$ should be chosen for the PD^{β} cases in spite of the exact value of system order α. But, the lower the β in the ILC algorithm is, the slower the tracking speed of the control system. Particularly, if the system is a fractional-order one, the classical PD type ILC scheme does not work in many cases.

Allow for the application of fractional-order ILC scheme to the control of complex systems and the cancelation of the dependence of ILC scheme to the system orders, a generalized fractional-order (GFO) ILC scheme is derived in this paper. The convergence and adaptiveness problems are discussed. Moreover, the realizations of the GFO-ILC are presented by using the idea of fractional-order element networks [27].

The rest of this paper is organized as follows. Some preliminaries are introduced in Section II. The fractional-order nonlinear systems are presented in Section III. In Sections IV and V, the convergence of GFO-ILC algorithms are derived, which is the main part of this paper. The adaptiveness and implementation of GFO-ILC schemes are shown in Section VI. Numerical simulations are shown in Section VII. Conclusions and future works are summarized in Section VIII.

II. Preliminaries

A. Fractional Calculus

Fractional calculus plays an important role in modern science [26], [28], [31]. In this paper, we use both Riemann-Liouville and Caputo fractional operators as our main tools. The unified formula of a fractional-order integral (Riemann-Liouville fractional-order integral) with order $\alpha \in (0,1)$ is defined as ${}_{t_0}^{RL}\mathscr{D}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau$, where $f(t)$ is an arbitrary integrable function, ${}_{t_0}\mathscr{D}_t^{-\alpha}$ is the fractional integral of order α on $[t_0, t]$, and $\Gamma(\cdot)$ denotes the Gamma function. Especially, when $t_0 = 0$,

$$ {}_0^{RL}\mathscr{D}_t^{-\alpha} f(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau. \quad (1) $$

For an arbitrary real number p, the Riemann-Liouville and Caputo fractional derivatives are defined respectively as ${}_{t_0}^{RL}\mathscr{D}_t^{p} f(t) = \frac{\mathrm{d}^{[p]+1}}{\mathrm{d}t^{[p]+1}} [\ {}_{t_0}\mathscr{D}_t^{-([p]-p+1)} f(t)]$ and

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This work is partially supported by the The Natural Science Foundation of Shandong Province (Grant No. ZR2010AM007) and the Independent Innovation Foundation of Shandong University (Grant No. 2010TB022) to Yan Li.

${}_{t_0}^{C}\mathscr{D}_t^{p} f(t) = {}_{t_0}\mathscr{D}_t^{-([p]-p+1)}[\frac{\mathrm{d}^{[p]+1}}{\mathrm{d}t^{[p]+1}} f(t)]$, where $[p]$ stands for the integer part of p, ${}^{RL}\mathscr{D}$ and ${}^{C}\mathscr{D}$ are Riemann-Liouville and Caputo fractional derivatives, respectively. In this paper $f^{(\alpha)}(t)$ denotes either the Riemann-Liouville or Caputo α_{th} order derivative. Moreover, it can be proved that, if $f(0) = 0$,

$$
{}_0^{RL}\mathscr{D}_t^{-\alpha}\ {}_0^{C}\mathscr{D}_t^{\alpha} f(t) = f(t). \tag{2}
$$

B. Mittag-Leffler Functions

Similar to the exponential function frequently used in the solutions of integer-order systems, a function frequently used in the solutions of fractional order systems is the Mittag-Leffler function defined as $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, where $z \in \mathbb{C}$ and $\alpha > 0$. The Mittag-Leffler function with two parameters has the following form: $E_{\alpha,\beta}(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}$, where $z \in \mathbb{C}$, $\alpha > 0$ and $\beta > 0$. For $\beta = 1$, we have $E_\alpha(z) = E_{\alpha,1}(z)$. Also, $E_{1,1}(z) = e^z$. Moreover, the Laplace transform of the Mittag-Leffler function in two parameters is $\mathscr{L}\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha+\lambda}$, $(\mathrm{Re}\{s\} > |\lambda|^{\frac{1}{\alpha}})$, where t and s are respectively the variables in the time domain and Laplace domain, $\mathrm{Re}\{s\}$ denotes the real part of s, $\lambda \in \mathbb{R}$ and $\mathscr{L}\{\cdot\}$ stands for the Laplace transform [28].

C. The operator norm and $\lambda-$norm

The norm of an operator $\pounds$ is defined as: $\|\pounds\|_{\text{op}} = \sup\limits_{\|v\|=1} \|\pounds v\|$, where v is an unit vector and $\|\cdot\|$ can be an arbitrary vector norm. In this section, let $\|\cdot\| = \|\cdot\|_\infty$, it follows from the definition of operator norm that

$$
\|\pounds\|_{\text{op}} = \sup_{\|v\|=1} \|\pounds v\| = \sup_{w\neq 0} \frac{\|\pounds w\|_\infty}{\|w\|_\infty} \le 1 \Rightarrow \|\pounds w\|_\infty \le \|w\|_\infty,
$$

where w is an arbitrary non-zero vector [32], [33].

The $\lambda-$norm of a $n-$vector valued function $e(t)$ defined on $[0,T]$ is $\|e(t)\|_\lambda = \sup\limits_{0\le t\le T} \left\{ e^{-\lambda t} \max\limits_{1\le i\le n} |e_i(t)| \right\}$, where λ is a positive constant [1].

III. The Fractional-Order Nonlinear System

The fractional-order nonlinear system discussed in this paper is shown as

$$
y^{(\alpha)}(t) = f(t,y,u), \tag{3}
$$

where $\alpha \in (0,1)$, $y(0) \in \mathbb{R}^{n\times 1}, u \in \mathbb{R}^{m\times 1}$, $\cdot^{(\alpha)}$ denotes the α_{th}-order Riemann-Liouville or Caputo derivative with respect to t, and the piecewise continuous function f satisfies

$$
\left\|\frac{\partial f}{\partial y}\right\|_\infty \le c\|y\|_\infty, \quad \left\|\frac{\partial f}{\partial u}\right\|_\infty \le \gamma\|u\|_\infty, \tag{4}
$$

where $c, \gamma > 0$ and $\|\text{matrix}\|_\infty$ and $\|\text{vector}\|_\infty$ denote respectively the maximum norms of a matrix and a vector.

Remark 3.1: It can be seen that any fractional-order system with rational or commensurate orders can be included in (3). Moreover, it follows from (4) and the uniqueness and existence theorem of the fractional-order differential equations [28] that, for the fixed $y(0)$ and $u(t)$, there exists an unique solution of system (3).

Moreover, let the reference be $y_d(t)$, where $y_d(0) = y(0)$, and the fractional-order ILC updating law be

$$
u_{k+1}(t) = u_k(t) + \mathfrak{K}_1\{K_p(t)e(t)\} + \mathfrak{K}_2\{K_d(t)e_k^{(\alpha)}(t)\}, \tag{5}
$$

where $\mathfrak{K}_1$ and $\mathfrak{K}_2$ denote arbitrary linear operators, $\alpha \in (0,1)$, $K_p(t)$ and $K_d(t)$ are gain matrices, $k = 0,1,2,\cdots$, $t \in [0,T]$, $y_k(0) = y_d(0) = y(0)$,

$$
\begin{cases}
y_k^{(\alpha)}(t) = f(t,y_k,u_k), \\
y_d^{(\alpha)}(t) = f(t,y_d,u_d), \\
e_k(t) = y_d(t) - y_k(t),
\end{cases} \tag{6}
$$

and $u_d(t)$ and $y_d(t)$ denote the desired control effort and system output, respectively.

Based on the fractional-order nonlinear system (3) and the fractional-order ILC scheme (6), the following lemmas are introduced.

Lemma 3.1: For the fractional-order nonlinear system (3), where f is a continuous differentiable function, it follows from (6) that $f_d - f_k = \left\{\frac{\partial f_i}{\partial y_j}\right\}_{y_j=\eta_{ij}(t)} e_k(t) + \left\{\frac{\partial f_i}{\partial u_l}\right\}_{u_l=\xi_{il}(t)} \delta u_k(t)$, where $k = 0,1,2,\cdots$, $f_d = f(t,y_d,u_d)$, $f_k = f(t,y_k,u_k)$, $\delta u_k = u_d - u_k$, and $\eta_{ij}(t)$ and $\xi_{il}(t)$ ($i,j \in \{1,2,\cdots,n\}$ and $l \in \{1,2,\cdots,m\}$) are defined in the following proof.

Proof: It follows from (6) that

$$
\begin{aligned}
&f_d - f_k = f(t,y_d,u_d) - f(t,y_k,u_k) \\
&= f(t,y_d,u_k) - f(t,y_k,u_k) + f(t,y_d,u_d) - f(t,y_d,u_k) \\
&= \left\{\frac{\partial f_i}{\partial y_j}\right\}_{y_j=\eta_{ij}(t)} e_k(t) + \left\{\frac{\partial f_i}{\partial u_l}\right\}_{u_l=\xi_{il}(t)} \delta u_k(t), \\
&= {}_kA(t)e_k(t) + {}_kB(t)\delta u_k(t),
\end{aligned} \tag{7}
$$

where ${}_kA \in \mathbb{R}^{n\times n}$, ${}_kB \in \mathbb{R}^{n\times m}$, and there exist functions $\xi_{ij}(t)$ and $\eta_{il}(t)$ satisfying $f_i(t,y_d,u_k) - f_i(t,y_k,u_k) = \sum_{j=1}^{m} \frac{\partial f_i}{\partial y_j}\Big|_{y_j=\eta_{ij}(t)} e_k(t)$, and $f_i(t,y_d,u_d) - f_i(t,y_d,u_k) = \sum_{l=1}^{n} \frac{\partial f_i}{\partial u_l}\Big|_{u_l=\xi_{il}(t)} \delta u_k(t)$. ■

Lemma 3.2: For the fractional-order nonlinear systems (6) and an arbitrary positive constant $q > 1/\alpha$, suppose $\left\|\frac{\partial f}{\partial u}\right\|_\infty \le \gamma\|u\|_\infty$, we have that there exists a large enough λ satisfying

$$
\|e_k\|_\lambda \le O(\lambda^{-1/q})\|\delta u_k\|_\lambda.
$$

Proof: Applying ${}_0^{RL}\mathscr{D}_t^{-\alpha}$ to both sides of equation (6), it follows from $y_k(0) = y_d(0)$, equations (1) and (2) and Lemma

3.1 that

$$\|e_k\|_\lambda = \sup_{0\le t\le T}\left\{e^{-\lambda t}\left\|\frac{t^{\alpha-1}}{\Gamma(\alpha)}*[f_d-f_k]\right\|_\infty\right\}$$
$$\le \sup_{0\le t\le T}\int_0^t \frac{e^{-\lambda(t-\tau)}(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\sup_{0\le\tau\le T} e^{-\lambda\tau}[c\|e_k(\tau)\|_\infty + \gamma\|\delta u_k(\tau)\|_\infty]\mathrm{d}\tau$$
$$\le [c\|e_k\|_\lambda+\gamma\|\delta u_k\|_\lambda]\cdot\sup_{0\le t\le T}\left[\int_0^t e^{-q\lambda(t-\tau)}\mathrm{d}\tau\right]^{1/q} \cdot\sup_{0\le t\le T}\left[\int_0^t \frac{(t-\tau)^{q(\alpha-1)/(q-1)}}{\Gamma^{q/(q-1)}(\alpha)}\mathrm{d}\tau\right]^{1-1/q}$$
$$=\frac{(1-e^{-q\lambda T})T^{\frac{q\alpha-1}{q}}\Gamma^{\frac{q-1}{q}}(\frac{q\alpha-1}{q-1})}{(q\lambda)^{1/q}\Gamma(\alpha)}[c\|e_k\|_\lambda+\gamma\|\delta u_k\|_\lambda],$$

where $q>1/\alpha$, the Hölder's inequality is applied to the above equation, and $\|\cdot\|_\infty$ denotes the maximum norm of $\cdot$. It follows that $\|e_k\|_\lambda \le O(\lambda^{-1/q})\|\delta u_k\|_\lambda$, where λ is large enough that $(q\lambda)^{1/q}\Gamma(\alpha)-c(1-e^{-q\lambda T})T^{\frac{q\alpha-1}{q}}\Gamma^{\frac{q-1}{q}}(\frac{q\alpha-1}{q-1})>0$ and $O(\lambda^{-1/q})=\frac{\gamma(1-e^{-q\lambda T})T^{\frac{q\alpha-1}{q}}\Gamma^{\frac{q-1}{q}}(\frac{q\alpha-1}{q-1})}{(q\lambda)^{1/q}\Gamma(\alpha)-c(1-e^{-q\lambda T})T^{\frac{q\alpha-1}{q}}\Gamma^{\frac{q-1}{q}}(\frac{q\alpha-1}{q-1})}$. ■

IV. The Convergence Condition of the GFO-ILC Scheme

Allowing for the implementation and applications of the GFO-ILC methods, in this paper, the linear operator $\mathfrak{K}\{F(t)\}$ can be defined in the convolution form

$$\mathfrak{K}\{w(t)\}=\int_0^t h(t-\tau)w(\tau)\mathrm{d}\tau, \tag{8}$$

where the kernel function $h(t)$ is an integrable one on $[0,t]$, and $w(t)\in\mathbb{R}^n (n\in\mathbb{Z}^+)$ is a time varying vector with integrable components. Based on the above definitions, the following lemma is proved.

Lemma 4.1: For the linear operator $\mathfrak{K}$ in (8), it is a bounded linear operator if and only if there exists a positive constant c satisfying

$$\int_0^t |h(\tau)|\mathrm{d}\tau \le c, \quad (t\in[0,T]), \tag{9}$$

where $|\cdots|$ denotes the absolute value of $\cdots$. Moreover, for an arbitrary vector $w(t)\in\mathbb{R}^n$, we have $\|\mathfrak{K}\{w\}\|_\infty \le c\|w\|_\infty$.

Proof: The proof can be found in [32], [33]. ■

Now, let the generalized fractional-order ILC updating law be $\delta u_k(t)=u_d(t)-u_k(t)$ and

$$\Delta u_{k+1}(t) = u_{k+1}(t)-u_k(t) = \mathfrak{B}\{\tilde{K}(t)e(t)\}+h(t)*[K(t)e_k^{(\alpha)}(t)], \tag{10}$$

where $\mathfrak{B}$ is a bounded linear operator. It follows that

$$\delta u_{k+1}(t)=\delta u_k(t)-\mathfrak{B}\{\tilde{K}(t)e(t)\}-h(t)*[K(t)e_k^{(\alpha)}(t)]$$
$$=\mathscr{L}^{-1}\{1-h(s)\}*\delta u_k(t)+h(t)*\{[I-K(t)\ {}_kB(t)]\delta u_k(t)\} - \mathfrak{B}\{\tilde{K}(t)e(t)\}-h(t)*[K(t)\ {}_kA(t)e_k(t)],$$

where $h(s)$ is the Laplace transform of $h(t)$ and $\mathscr{L}^{-1}\{1-h(s)\}$ denotes the inverse Laplace transform of $1-h(s)$.

Applying the infinity norm to both sides of the above equation yields

$$\|\delta u_{k+1}(t)\|_\infty \le \|\mathscr{L}^{-1}\{1-h(s)\}*\delta u_k(t)\|_\infty + \|h(t)*\{[I-K(t)\ {}_kB(t)]\delta u_k(t)\}\|_\infty + \|\mathfrak{B}\{\tilde{K}(t)e(t)\}\|_\infty + \|h(t)*[K(t)\ {}_kA(t)e_k(t)]\|_\infty.$$

It follows from Lemma 4.1 that

$$\|\delta u_{k+1}(t)\|_\infty \le \int_0^t|\tilde{h}(\tau)|\mathrm{d}\tau\cdot\|\delta u_k(t)\|_\infty + \int_0^t|h(\tau)|\mathrm{d}\tau\cdot\|I-K(t)\ {}_kB(t)\|_\infty\cdot\|\delta u_k(t)\|_\infty + b\|\tilde{K}(t)\|_\infty\cdot\|e(t)\|_\infty+\int_0^t|h(\tau)|\mathrm{d}\tau\cdot\|K(t)\ {}_kA(t)\|_\infty\cdot\|e_k(t)\|_\infty$$

where $\tilde{h}(t)=\mathscr{L}^{-1}\{1-h(s)\}$. Then applying the $\lambda-$norm to the above equation yields

$$\|\delta u_{k+1}(t)\|_\lambda \le \int_0^T|\tilde{h}(\tau)|\mathrm{d}\tau\cdot\|\delta u_k(t)\|_\lambda + \int_0^T|h(\tau)|\mathrm{d}\tau\cdot\max_{t\in[0,T]}\|I-K(t)\ {}_kB(t)\|_\infty\cdot\|\delta u_k(t)\|_\lambda + b\max_{t\in[0,T]}\|\tilde{K}(t)\|_\infty\cdot\|e(t)\|_\lambda + \int_0^T|h(\tau)|\mathrm{d}\tau\cdot\max_{t\in[0,T]}\|K(t)\ {}_kA(t)\|_\lambda\cdot\|e_k(t)\|_\lambda.$$

By using the Lemma 3.2, we arrive at $\|\delta u_{k+1}(t)\|_\lambda \le [\rho+O(\lambda^{-1/q})]\|\delta u_k(t)\|_\lambda$, where $\rho=\int_0^T|\tilde{h}(\tau)|\mathrm{d}\tau+\int_0^T|h(\tau)|\mathrm{d}\tau\cdot\max_{t\in[0,T]}\|I-K(t)\ {}_kB(t)\|_\infty$.

It can be seen from the above discussions that if $0\le\rho<1$, there must exist a large enough λ satisfying $\rho+O(\lambda^{-1/q})<1$, which implies that $\lim_{k\to\infty}\|\delta u_k(t)\|_\lambda\equiv 0$ for all $t\in[0,T]$. In order to derive the convergence condition of the generalized fractional-order iterative learning control scheme, the following theorem is introduced.

Theorem 4.1: For the fractional-order system (3) and the generalized fractional-order iterative learning control scheme (6) and (10), if there exists a constant $\rho\in[0,1)$ satisfying

$$\|I-K(t)\ {}_kB(t)\|_\infty \le \frac{\rho-\int_0^T|\tilde{h}(\tau)|\mathrm{d}\tau}{\int_0^T|h(\tau)|\mathrm{d}\tau}, \tag{11}$$

where $k=\{0,1,2,\cdots\}$, we have $\lim_{k\to\infty}y_k(t)=y_d(t)$ on $[0,T]$.

Proof: It can be easily seen from (11) that $\int_0^T|\tilde{h}(\tau)|\mathrm{d}\tau+\int_0^T|h(\tau)|\mathrm{d}\tau\cdot\|I-K(t)\ {}_kB(t)\|_\infty\le\rho$ holds for all $t\in[0,T]$ and $k=\{0,1,2,\cdots\}$. In other words, $\int_0^T|\tilde{h}(\tau)|\mathrm{d}\tau+\int_0^T|h(\tau)|\mathrm{d}\tau\cdot\max_{t\in[0,T]}\|I-K(t)\ {}_kB(t)\|_\infty\le\rho<1$. Therefore, $\|\delta u_{k+1}(t)\|_\lambda\le[\rho+O(\lambda^{-1/q})]\|\delta u_k(t)\|_\lambda$, and there exists a sufficient large λ satisfying $[\rho+O(\lambda^{-1/q})]<1$, which implies that, for all $t\in[0,T]$, $\lim_{k\to\infty}\|\delta u_k(t)\|=0$. Moreover, it follows from $y_d(0)=y_k(0)$ and the uniqueness and existence of the system equation that $\lim_{k\to\infty}y_k(t)=y_d(t), \quad (t\in[0,T])$. ■

Remark 4.2: In equation (10), if an extra term is added, i.e. $\Delta u_{k+1}(t)=h_1(t)*[K_1(t)e_k^{(\alpha)}(t)]+$

$h_2(t) * [K_2(t)e_k^{(\alpha)}(t)] + \mathfrak{B}\{\tilde{K}(t)e(t)\}$, it follows that $\|\delta u_{k+1}(t)\|_\lambda \le [\hat{\rho} + O(\lambda^{-1/q})]\|\delta u_k(t)\|_\lambda$, where

$$\hat{\rho} = \int_0^T |\hat{h}(\tau)|\mathrm{d}\tau + \int_0^T |h_1(\tau)|\mathrm{d}\tau \cdot \max_{t\in[0,T]} \|I - K_1(t)\ {}_kB(t)\|_\infty + \int_0^T |h_2(\tau)|\mathrm{d}\tau \cdot \max_{t\in[0,T]} \|I - K_2(t)\ {}_kB(t)\|_\infty,$$

and $\hat{h}(t) = \mathscr{L}^{-1}\{1 - h_1(s) - h_2(s)\}$. Therefore, the convergence condition becomes $\max_{i=1,2}\{\|I - K_i(t)\ {}_kB(t)\|_\infty\} \le \frac{\hat{\rho} - \int_0^T |\hat{h}(\tau)|\mathrm{d}\tau}{\int_0^T |h_1(\tau)|\mathrm{d}\tau + \int_0^T |h_2(\tau)|\mathrm{d}\tau}$, where $\hat{\rho} \in [0,1)$ and $k = \{0,1,2,\cdots\}$.

In addition, if $\Delta u_{k+1}(t) = \sum_{i=1}^{N} h_i(t) * [K_i(t)e_k^{(\alpha)}(t)] + \mathfrak{B}\{\tilde{K}(t)e(t)\}$, the convergence condition is that there exists a constant $\check{\rho} \in [0,1)$ satisfying $\max_i\{\|I - K_i(t)\ {}_kB(t)\|_\infty\} \le \frac{\check{\rho} - \int_0^T |\check{h}(\tau)|\mathrm{d}\tau}{\sum_i \int_0^T |h_i(\tau)|\mathrm{d}\tau}$, where $t \in [0,T]$, $k = \{0,1,2,\cdots\}$ and $\check{h}(t) = \mathscr{L}^{-1}\{1 - \sum_i h_i(s)\}$.

Remark 4.3: In Theorem 4.1, let $h(t) = \delta(t)$ be the Dirac-Delta function, the ILC scheme is reduced to the classical D^α or PD^α ILC cases. Therefore, we have $\tilde{h}(t) = 0$ and $\int_0^T h(\tau)\mathrm{d}\tau = 1$ so that the convergence condition becomes $\|I - K(t)\ {}_kB(t)\|_\infty \le \rho < 1$. Especially when the system is

$$\begin{cases} x^{(\alpha)}(t) = A(t)x(t) + B(t)u(t), \\ y(t) = C(t)x(t), \end{cases}$$

the convergence condition becomes $\|I - K(t)C(t)B(t)\|_\infty \le \rho < 1$ [23], [24].

V. Revisit of the GFO-ILC scheme

Let's reconsider the generalized fractional-order ILC updating law in another form. Compare with (10), another generalized fractional-order ILC updating law to be discussed in the part is

$$\Delta u_{k+1}(t) = \mathfrak{B}\{\tilde{K}(t)e(t)\} + K(t)h(t) * e_k^{(\alpha)}(t). \tag{12}$$

Remark 5.1: It should be noted that, if $h(t) = \delta(t)$ is the Dirac-Delta function or $K(t) = K$ is a constant matrix, (10) and (12) are the same. Moreover, if $h(t) = \frac{t^q}{\Gamma(q+1)}, (q > -1)$, (12) becomes $\Delta u_{k+1}(t) = \mathfrak{B}\{\tilde{K}(t)e(t)\} + K(t)e_k^{(\beta)}(t)$, where $\beta = \alpha - q - 1 < \alpha$, which is corresponding to the PD^β ILC scheme [23].

Theorem 5.2: For the fractional-order system (3) and the generalized fractional-order iterative learning control scheme (6) and (12), if ${}_kB(t), k = 0,1,2,\cdots$ are full column rank matrices and there exists a constant $\rho_1 \in [0,1)$ satisfying

$$\|I - {}_{k+1}B(t)K(t)\|_\infty \le \frac{\rho_1 - \|\Theta(t)\|_\infty \int_0^T |\tilde{h}(\tau)|\mathrm{d}\tau - \|\Theta(t) - I\|_\infty \int_0^T |h(\tau)|\mathrm{d}\tau}{\int_0^T h(\tau)\mathrm{d}\tau},$$

where $k = \{0,1,2,\cdots\}$, we have $\lim_{k\to\infty} y_k(t) = y_d(t)$ on $[0,T]$.

Proof: In this case,

$$\begin{aligned}\delta u_{k+1}(t) =& \delta u_k(t) - K(t)h(t) * [\ {}_kB(t)\delta u_k(t)] \\ &- \mathfrak{B}\{\tilde{K}(t)e(t)\} - K(t)h(t) * [\ {}_kA(t)e_k(t)].\end{aligned}$$

If ${}_kB(t), k = 0,1,2,\cdots$, are full column rank matrices, multiplying ${}_{k+1}B(t)$ to both sides of the above equation yields

$$\begin{aligned}&{}_{k+1}B(t)\delta u_{k+1}(t) \\ =& \Theta(t)\ {}_kB(t)\delta u_k(t) - {}_{k+1}B(t)K(t)h(t) * [\ {}_kB(t)\delta u_k(t)] \\ &- {}_{k+1}B(t)\mathfrak{B}\{\tilde{K}(t)e(t)\} - {}_{k+1}B(t)K(t)h(t) * [\ {}_kA(t)e_k(t)],\end{aligned}$$

where $\Theta_{m\times m}(t) = {}_{k+1}B(t)[\ {}_kB^T(t)\ {}_kB(t)]^{-1}\ {}_kB^T(t)$. It can be proved that

$$\begin{aligned}&{}_{k+1}B(t)\delta u_{k+1}(t) \\ =& \Theta(t)\{\ {}_kB(t)\delta u_k(t) - h(t) * [\ {}_kB(t)\delta u_k(t)]\} \\ &+ [\Theta(t) - I]h(t) * [\ {}_kB(t)\delta u_k(t)] \\ &+ [I - {}_{k+1}B(t)K(t)]h(t) * [\ {}_kB(t)\delta u_k(t)] \\ &- {}_{k+1}B(t)\mathfrak{B}\{\tilde{K}(t)e(t)\} - {}_{k+1}B(t)K(t)h(t) * [\ {}_kA(t)e_k(t)].\end{aligned}$$

Applying the $\lambda-$norm to the above equation, it follows from Lemma 3.2 and ${}_kB(t), k = 0,1,2,\cdots$ are full column rank matrices that $\|{}_{k+1}B(t)\delta u_{k+1}(t)\|_\lambda \le [\rho_1 + O(\lambda^{-1/q})]\|{}_kB(t)\delta u_k(t)\|_\lambda$, where $\rho_1 = \|\Theta(t)\|_\infty \int_0^t |\tilde{h}(\tau)|\mathrm{d}\tau + \|\Theta(t) - I\|_\infty \int_0^t |h(\tau)|\mathrm{d}\tau + \|I - {}_{k+1}B(t)K(t)\|_\infty \int_0^t h(\tau)\mathrm{d}\tau$. Therefore, if $\rho_1 < 1$, there must exist a sufficient large λ satisfying $\lim_{k\to\infty} \|{}_kB(t)\delta u_k(t)\|_\lambda = 0$. It then follows from ${}_kB(t), k = 0,1,2,\cdots$ are full column rank matrices and the uniqueness and existence of the system equation that $\lim_{k\to\infty} y_k(t) = y_d(t), \quad (t \in [0,T])$. ∎

Corollary 5.3: Let $h(t) = \delta(t)$ be the Dirac-Delta function, it can be seen from Theorems (4.1) and (5.2) that the convergence condition is either $\|I_m - K(t)\ {}_kB(t)\|_\infty < 1$, where $k = \{0,1,2,\cdots\}$ or $\|I_n - {}_{k+1}B(t)K(t)\|_\infty < 1$, where $k = \{0,1,2,\cdots\}$, ${}_kB(t)$ and ${}_{k+1}B(t)$ are full column rank matrices, and the ILC updating law is

$$\Delta u_{k+1}(t) = \mathfrak{B}\{\tilde{K}(t)e(t)\} + K(t)e_k^{(\alpha)}(t). \tag{13}$$

In other words, (13) guarantees the convergence of the ILC scheme if one of the following conditions is satisfied:

- There exists a $K(t)$ satisfying $\|I_m - K(t)\ {}_kB(t)\|_\infty < 1$, where $k = \{0,1,2,\cdots\}$.
- There exists a $K(t)$ satisfying $\|I_n - {}_{k+1}B(t)K(t)\|_\infty < 1$, where $k = \{0,1,2,\cdots\}$, $\mathrm{rank}(\ {}_{k+1}B(t)) = m$ and ${}_{k+1}B(t) \in \mathbb{R}^{n\times m}$.

Proof: The proof comes from Theorems (4.1) and (5.2). ∎

Remark 5.4: For the state-space system

$$\begin{cases} x^{(\alpha)}(t) = A(t)x(t) + B(t)u(t), \\ y(t) = C(t)x(t), \end{cases}$$

if it is controllable, i.e. $C(t)B(t)$ is invertible a.e. on $t \in [0,T]$, let the ILC updating law be $\Delta u_{k+1}(t) = \tilde{K}(t)e(t) + K(t)e_k^{(\alpha)}(t)$, we have the convergence condition of the ILC scheme is either $\|I - K(t)C(t)B(t)\|_\infty < 1$ or $\|I - C(t)B(t)K(t)\|_\infty < 1$. Moreover, when $C(t)B(t)$ is not a full column rank matrix, $\|I - K(t)C(t)B(t)\|_\infty < 1$ can still guarantee the convergence. However, $\|I - C(t)B(t)K(t)\|_\infty < 1$ does not work in some circumstances.

VI. The Adaptiveness and Implementation of the GFO-ILC

A. The adaptiveness of GFO-ILC

In most of the previous references regarding the design of a D^α or PD^α type iterative learning control scheme, the constant α must be known in prior which is strictly equal to the system order α. However, in practice, α is always determined using experimental data or empirical values so that it's very difficult to obtain the accurate value of it. Moreover, suppose the order of the system is α, if $\beta > \alpha$, the convergence of D^β or PD^β type iterative learning control scheme may not be guaranteed. Therefore, in this section, a generalized GFO-ILC scheme is discussed which satisfies the convergence condition in spite of the knowledge of the system order.

Lemma 6.1: For an arbitrary positive constant λ and a constant $\beta \in (0,1]$, it can be proved that $\int_0^T t^{\beta-1}E_{\beta,\beta}(-\lambda t^\beta)\mathrm{d}t$ is an increasing function to T and $\int_0^\infty t^{\beta-1}E_{\beta,\beta}(-\lambda t^\beta)\mathrm{d}t = \frac{1}{\lambda}$.

Proof: It can be proved by using the properties of Mittag-Leffler function that $T^\beta E_{\beta,\beta+1}(-\lambda T^\beta) = \int_0^T t^{\beta-1}E_{\beta,\beta}(-\lambda t^\beta)\mathrm{d}t = \frac{1}{\lambda}\left[1-E_\beta(-\lambda T^\beta)\right]$, where $E_\beta(0) = 1$, $E_\beta(-\lambda T^\beta)$ is a monotonic decreasing function to T, and $\lim_{T\to\infty} E_\beta(-\lambda T^\beta) = 0$. ■

Theorem 6.1: For the fractional-order nonlinear system (3), where $0 < a \le \alpha \le b \le 1$, and the GFO-ILC scheme (6) and $u_{k+1}(t) = u_k(t) + K(t)\hat{h}(t) * e(t)$, let $\hat{h}(t) = \lambda t^{\gamma-1}E_{\beta,\gamma}(-\lambda t^\beta)$, where $\lambda > 0$, $\beta \in (0,1]$ and $\gamma \in (-a, 1-b]$, it follows from Theorem 4.1 that the convergence condition becomes

$$\|I - {}_kB(t)K(t)\|_\infty \le \frac{\rho - \int_0^T |\tilde{h}(\tau)|\mathrm{d}\tau}{\int_0^T |h(\tau)|\mathrm{d}\tau},$$

where $\rho \in (0,1)$, $h(t) = \lambda t^{\beta-1}E_{\beta,\beta}(-\lambda t^\beta)$, and $\tilde{h}(\tau) = \frac{\mathrm{d}E_\beta(-\lambda t^\beta)}{\mathrm{d}t} < 0$. Moreover, for an arbitrary positive constants T and $\rho \in (0,1)$, there must exist a λ satisfying the convergence condition.

Proof: Let $h(t) = \lambda t^{\beta-1}E_{\beta,\beta}(-\lambda t^\beta)$ and $\gamma = \beta - \alpha$, where α is the system order, it follows from $\alpha \in [a,b]$ and $\beta \in (0,1]$ that $\gamma \in (-a, 1-b]$ and $\hat{h}(t) = h^{(\alpha)}(t)$. Moreover, it follows from Theorem 5.2 that $\int_0^T |\tilde{h}(\tau)|\mathrm{d}\tau = \int_0^T |h(\tau)|\mathrm{d}\tau = 1 - E_\beta(-\lambda T^\alpha)$. It follows that $\frac{\rho - \int_0^T |\tilde{h}(\tau)|\mathrm{d}\tau}{\int_0^T |h(\tau)|\mathrm{d}\tau} = \frac{\rho}{\int_0^T |h(\tau)|\mathrm{d}\tau} - 1$. Therefore, for an arbitrary $T > 0$ and $\rho \in (0,1)$, suppose $\|I - K(t)_kB(t)\|_\infty$ is bounded, there must exists a sufficient small λ satisfying $\int_0^T |h(\tau)|\mathrm{d}\tau = 1 - E_\beta(-\lambda T^\alpha) \le \frac{\rho}{\|I - {}_kB(t)K(t)\|_\infty + 1}$, which implies the conclusion. ■

Remark 6.2: It can be seen from the above theorem that for an unknown $\alpha \in (0,1]$, we can always let $\gamma = 0$, i.e. $\hat{h}(t) = \lambda \frac{\mathrm{d}E_\beta(-\lambda t^\beta)}{\mathrm{d}t}$, so that the convergence condition can be guaranteed.

B. The implementation of $h(t)$

The implementation of the kernel function $h(t)$ will bring us a lot of conveniences in the realization of fractional-order iterative learning control schemes.

For example, it was shown in Subsection VI-A that the application of $h(t) = \lambda t^{\beta-1}E_{\beta,\beta}(-\lambda t^\beta)$ in the GFO-ILC scheme guarantees the convergence and adaptiveness simultaneously by choosing the proper value of λ. Now, the implementation of it can be realized using the idea of fractional-order element networks, which is shown below.

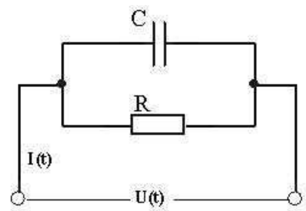

Fig. 1. A fractional-order element network composed of a fractional-order capacitor and a resistor.

Figure 1 shows a typical fractional-order RC circuit, which is arranged by a fractional-order capacitor and a resistor in parallel. The constitutive equation of the fractional-order capacitor is $i(t) = \frac{U^{(\gamma)}(t)}{C}$, where $i(t)$ and $U(t)$ denote the current and voltage, respectively. Therefore, it can be proved that the relation of $I(t)$ and $U(t)$ for the whole circuit is $U(t) = [t^{\gamma-1}E_{\gamma,\gamma}(-R^{-1}t^\gamma)] * I(t)$, where $*$ denotes the convolution. Let $\gamma = \beta$ and $\lambda = R^{-1}$, the function $t^{\gamma-1}E_{\gamma,\gamma}(-R^{-1}t^\gamma)$ in the above equation becomes $h(t)/\lambda$.

It can be seen that most of the linear GFO-ILC updating laws can be constructed by the fractional-order element networks. Moreover, in the linear robust ILC problems, the various fractional-order filters can also be realized by using the method of fractional-order element networks.

VII. Numerical Simulations

Let the fractional-order nonlinear system be

$$y^{(3/4)}(t) = y^{3/5}(t) + u, \tag{14}$$

where $y(0) = 0$. Given the reference $y_d(t) = 12t^2(1-t)$ and $t \in [0,1]$, use the GFO-ILC updating law

$$u_{k+1}(t) = u_k(t) + \hat{h}(t) * e(t), \tag{15}$$

where $\hat{h}(t) = t^{\gamma-1}E_{\beta,\gamma}(-t^\beta)$, $\gamma = 0.15$, $\beta = 0.9$ and $u_0(t) = 0$, then we arrive at the following two figures (Figure 2 and Figure 3).

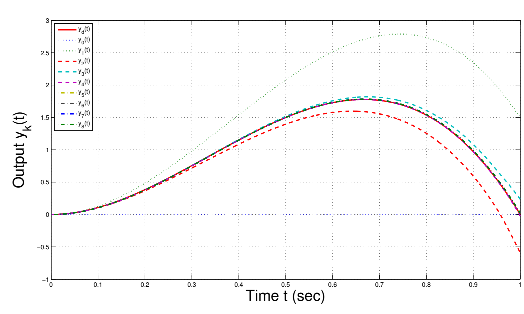

Fig. 2. The system output $y_k(t)$ for different iteration $k \in \{0,1,2,\cdots,8\}$.

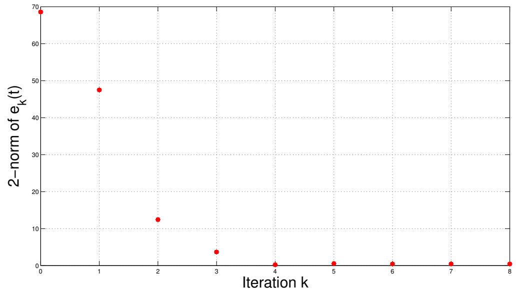

Fig. 3. The two norms of $e_k(t) = y_d(t) - y_k(t)$, $k \in \{0, 1, 2, \cdots, 8\}$.

It can be seen that the GFO-ILC updating law (15) guarantees the convergence for fractional-order system (14). The tracking error decreases to very small values ($\|y_d(t) - y_4(t)\|_2 = 0.2374$) when $k \geq 4$.

Remark 7.1: In the above example, $\gamma = 0.15$ and the GFO-ILC algorithm is convergent. It follows from Theorem 6.1 that we can arbitrary choose $\gamma \in [0, 0.15]$ so that the convergence can be guaranteed as well.

VIII. Conclusion and Future Works

In this paper, we discussed the convergence conditions of the linear GFO-ILC schemes for fractional-order nonlinear systems. Most of the previous works on fractional-order ILC fell into the scheme of this paper. For the fractional-order linear state-space systems, when $C(t)B(t)$ was invertible, the equivalence of $\|I - K(t)C(t)B(t)\|$ and $\|I - C(t)B(t)K(t)\|$ was proved. Moreover, an adaptive GFO-ILC updating law was proposed, which guaranteed the convergence conditions in spite of the knowledge of system order. The method of fractional-order element networks was used to the implementation of GFO-ILC schemes, which extended the applications of GFO-ILC in practical ways. Lastly, a numerical simulation example was provided to validate the concepts.

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