2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

A Generalized Solution of H^∞ Control Problems for Preview and Delayed Systems

Akira Kojima

***Abstract*— For a broad class of H^∞ preview/delayed control problems, a solvability condition is newly established based on an analytic solution of operator Riccati equation. The condition is characterized by maximal eigenvalue of a compact operator and enables us to overcome the limitation of existing methods. An H^∞ control law is also clarified and some interpretation is provided based on the property of the analytic solution. Employing the advantage of proposed approach, a design method of H^2 control law is derived for the preview/delayed systems.**

I. INTRODUCTION

Design methods of H^∞ control law have been considered for broad class of infinite-dimensional systems and, for preview/delayed control problems, the solvabiliy condition and the analytic solution are characterized with finite-dimensional operations. The H^∞ control problems for delayed systems are solved via various approaches (e.g. [4], [9], [6], [7]) and, in the preview control problems, it is shown that the solvability condition is significantly simplified [10], [14], [5].

A state-space approach on appropriate function space is employed by [6], [7] and the H^∞ preview/delayed control law is derived by solving the operator Riccati equation. Although the state-space approach has an advantage of dealing with the preview/delayed strategies in a unified manner, the solvability condition for the general problem is still complicated because it requires to check whether all the eigenvalues of a compact operator are with nonnegative real values. Furthermore, the check method occasionally causes numerical instability as the existence region for the roots of correspondingly defined transcendental equation is unbounded.

In this paper, we focus on generalized H^∞ preview/ delayed control problems and newly establish a solvability condition with a control law. The solvability condition is characterized by maximal eigenvalue of a compact operator, which is given by a root of transcendental equation. The control problem covers multiple preview/delayed control systems in a unified manner and enables us to deal with rather complicated system such as unilateral delay systems [3]. Employing the advantage of the state-space approach, a design method of H^2 control law is derived for broad preview/delayed systems.

This paper is organized as follows. In Section II, a generalized H^∞ control problem is defined for preview and delayed systems, which inherits the structure investigated by [6]. The relation between the system description and the typical control problems is also discussed. In Section III, the operator Riccati equation which corresponds to the generalized H^∞ control problem is solved and, further, the solvability condition is newly established. In Section IV, the solution of H^2 preview/delayed control problem is clarified by employing the advantage of proposed approach. In Section V, the H^∞ performance of the preview and delayed control system is illustrated with numerical examples. The conclusion of this paper is presented in Section VI.

A.Kojima is with Faculty of System Design, Tokyo Metropolitan University, Asahigaoka 6-6, Hino-city, Tokyo 191-0065, Japan
akojima@sd.tmu.ac.jp

II. PROBLEM FORMULATION AND PRELIMINARIES

Define a full-information (FI) control problem for preview and delayed system:

$$\Sigma: \ \dot{x}(t) = Ax(t) + \sum_{i=0}^{d} B_1^i w(t-h_i) + \sum_{i=0}^{d} B_2^i u(t-h_i)$$
$$z(t) = \sum_{j=0}^{\ell} C_1^j x(t-\check{h}_j) + D_{12}u(t)$$
$$y(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & w^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{m_1}$, $u(t) \in \mathbb{R}^{m_2}$, $z(t) \in \mathbb{R}^p$, $y(t) \in \mathbb{R}^{n+m_2}$ are the state, the disturbance, the control input, the regulated output, and the measurement of the system, respectively. The system matrices are with appropriate dimensions and h_i $(i = 0, 1, \dots, d)$, $\check{h}_j$ $(j = 0, 1, \dots, \ell)$ are the time delays denoted in ascending order: $0 =: h_0 < h_1 < h_2 < \cdots < h_d := L$, $0 =: \check{h}_0 < \check{h}_1 < \check{h}_2 < \cdots < \check{h}_\ell := \check{L}$. The time delays in the disturbance w equivalently describe previewable reference signals and those in the regulated output enable to deal with general input/output delay systems (Remark 2). We prepare the auxiliary matrices:

$$A_c := A - B_2 D_{12}^{+} C_1,$$
$$B := \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \ B_1 := \sum_{i=0}^{d} B_1^i, \ B_2 := \sum_{i=0}^{d} B_2^i,$$
$$B^i := \begin{bmatrix} B_1^i & B_2^i \end{bmatrix} \ (i = 0, 1, \dots, d),$$
$$C_1 := \sum_{j=0}^{\ell} C_1^j, \ D_{12}^{+} := (D_{12}^{\mathrm{T}} D_{12})^{-1} D_{12}^{\mathrm{T}}$$
$$R_c := \begin{bmatrix} -\gamma^2 \cdot I_{m_1} & 0 \\ 0 & D_{12}^{\mathrm{T}} D_{12} \end{bmatrix}, \ N_c := I - D_{12} D_{12}^{+}. \quad (2)$$

and make following assumptions for the system Σ.

(H1) (A, B_2) is stabilizable,

(H2) D_{12} is full column rank,

(H3) $\operatorname{rank}\begin{bmatrix} A-j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n+m_2, \ \ \forall\omega\in\mathbb{R}$,

(H4) Σ satisfies the following conditions:

$$B^i R_c^{-1} B^{j\mathrm{T}} = 0, \ \ B_1^i N_c B_1^{j\mathrm{T}} = 0 \ \ (i\neq j), \quad (3)$$

$$B_2^i D_{12}^{+} C_1^j = 0 \ \ (i\neq 0 \text{ or } j\neq 0). \quad (4)$$

The assumption (H4) is broader than the orthogonal condition (e.g. [16]) and, from the practical viewpoint, typical problems are easily formulated along (H4).

The H^∞ control problem is to design a control law such that the resulting closed loop system satisfies the following conditions.

(C1) the closed-loop system is internally stable,

(C2) the transfer function of the closed loop system Σ_{zw} satisfies $\|\Sigma_{zw}\|_\infty < \gamma$ for a prescribed $\gamma > 0$.

Although the H^∞ control problem Σ is discussed by [6] with the orthogonal condition, the solvability condition is not fairly characterized as it requires direct calculation of minimal eigenvalue of a compact operator. More precisely, for the check of positive semi-definiteness of the compact operator, it needs to verify that all the eigenvalue keep nonnegative real values.

In the sequel, we focus on the system Σ under (H1)-(H4) and newly establish a solvability condition which overcomes the limitation of the existing result. The condition is characterized in terms of maximal eigenvalue of an auxiliary compact operator, which value is given by a root of transcendental equation. Employing the advantage of proposed approach, a solution of H^2 preview/delayed control problem is also clarified.

The relation to typical control problems is noted in the following remarks.

Remark 1 (Preview control): A preview control problem:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_{1,0}w_0(t) + B_{1,1}w_1(t-L) + B_2u(t) \\ z(t) &= C_1x(t) + D_{12}u(t) \\ y(t) &= \begin{bmatrix} x^{\mathrm{T}}(t) & w^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}} \end{aligned} \quad (5)$$

is formulated by Σ with

$$w(t) := \begin{bmatrix} w_0(t) \\ w_1(t) \end{bmatrix}, B_1^0 = \begin{bmatrix} B_{1,0} & 0 \end{bmatrix}, B_1^1 = \begin{bmatrix} 0 & B_{1,1} \end{bmatrix}.$$

In the system (5), w_0, w_1 denote system uncertainty and previewable reference signal, respectively. Replacing the signal by $r(t) = w_1(t-L)$, it is observed that the future information of the reference $r(t+L)$ is included in the measurement $y(t)$. The assumption (H4) broadens the system description as structural condition between the regulated output and control input is relaxed. H^∞ control problems for unilateral delay systems [3] are also formulated with Σ by imposing transmission delays on both disturbance and control. ■

Remark 2 (Output feedback problem): Time delays in the regulated output enable to derive analytic solutions of control/filtering Riccati equations defined for multiple preview and input/output delay systems. The problem Σ provides fundamental result on the output feedback problems which include preview/delayed action. A preliminary result is discussed by [7] under the orthogonal condition. ■

In order to solve the H^∞ control problem Σ, we prepare a system description on appropriate function space. Introduce a Hilbert space $\mathcal{X} := \mathbb{R}^n \times L_2(-L,0;\mathbb{R}^n) \times L_2(-\check{L},0;\mathbb{R}^n)$ endowed with the inner product

$$\begin{aligned} \langle\psi,\phi\rangle &:= \psi^{0\mathrm{T}}\phi^0 \\ &\quad + \int_{-L}^0 \psi^{1\mathrm{T}}(\beta)\phi^1(\beta)\,d\beta + \int_{-\check{L}}^0 \psi^{2\mathrm{T}}(\beta)\phi^2(\beta)\,d\beta, \\ &\psi = (\psi^0,\psi^1,\psi^2)\in\mathcal{X}, \ \phi = (\phi^0,\phi^1,\phi^2)\in\mathcal{X}, \end{aligned} \quad (6)$$

the system Σ is described by the evolution equation [12]:

$$\begin{aligned} \hat{\Sigma}: \ \dot{\hat{x}}(t) &= \mathcal{A}\hat{x}(t) + \mathcal{B}_1w(t) + \mathcal{B}_2u(t) \\ z(t) &= \mathcal{C}_1\hat{x}(t) + D_{12}u(t) \\ \hat{y}(t) &= \begin{bmatrix} \hat{x}(t) \\ w(t) \end{bmatrix}. \end{aligned} \quad (7)$$

The operator $\mathcal{A}$ is an infinitesimal generator defined by

$$\begin{aligned} \mathcal{A}\phi &= \begin{bmatrix} A\phi^0 + \phi^1(-L) \\ \phi^{1\prime} \\ \phi^{2\prime} \end{bmatrix}, \\ \mathrm{D}(\mathcal{A}) &= \{\phi\in\mathcal{X} : \phi^1\in W^{1,2}(-L,0;\mathbb{R}^n), \\ &\quad \phi^2\in W^{1,2}(-\check{L},0;\mathbb{R}^n), \phi^1(0)=0, \phi^2(0)=\phi^0\} \end{aligned} \quad (8)$$

where $W^{1,2}(-L,0;\mathbb{R}^n)$ denotes the Sobolev space of $\mathbb{R}^n$-valued, absolutely continuous functions with square integrable derivatives on $[-L,0]$ (see e.g. [1], Chapter 2). For the subspaces:

$$\mathcal{V}^* := \{\psi\in\mathcal{X} : \psi^1\in W^{1,2}(-L,0;\mathbb{R}^n), \psi^1(-L)=\psi^0\}, \quad (9)$$

$$\mathcal{W} := \{\phi\in\mathcal{X} : \phi^2\in W^{1,2}(-\check{L},0;\mathbb{R}^n), \phi^2(0)=\phi^0)\}, \quad (10)$$

$\mathcal{W} = \mathrm{D}_\mathcal{V}(\mathcal{A})$, $\mathcal{V}^* = \mathrm{D}_{\mathcal{W}^*}(\mathcal{A}^*)$ hold and $\mathcal{W}$, $\mathcal{X}$, $\mathcal{V}$ are with continuous, dense injections satisfying $\mathcal{W}\subset\mathcal{X}\subset\mathcal{V}$ ([12], Remark 2.6). The input/output operators $\mathcal{B}_k\in\mathcal{L}(\mathbb{R}^{m_k},\mathcal{V})$ $(k=1,2)$, $\mathcal{C}_1\in\mathcal{L}(\mathcal{W},\mathbb{R}^p)$ are defined as follows:

$$\mathcal{B}_k^*\phi = \sum_{i=0}^{d} B_k^{i\mathrm{T}}\phi^1(-L+h_i), \ \phi\in\mathcal{V}^*, \quad (11)$$

$$\mathcal{C}_1\phi = \sum_{j=0}^{\ell} C_1^j\phi^2(-\check{h}_j), \ \phi\in\mathcal{W}. \quad (12)$$

The system $\hat{\Sigma}$ is in the class of Pritchard-Salamon systems [12], [13] and typical H^∞ (H^2) control problems are characterized with abstract Riccati equations. We introduce an operator Riccati equation

$$\begin{aligned} &\mathcal{S}\mathcal{A}_c\phi + \mathcal{A}_c^*\mathcal{S}\phi - \mathcal{S}\mathcal{B}R_c^{-1}\mathcal{B}^*\mathcal{S}\phi + \mathcal{C}_1^*N_c\mathcal{C}_1\phi = 0, \ \phi\in\mathcal{W} \\ &\mathcal{A}_c := \mathcal{A} - \mathcal{B}_2D_{12}^{+}\mathcal{C}_1, \ \ \mathcal{B} := \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \end{bmatrix} \end{aligned} \quad (13)$$

for $\hat{\Sigma}$ and clarify the solution of the H^∞ control problem.

For the system $\hat{\Sigma}$, the H^∞ control problem is solvable iff (13) has a stabilizing solution $\mathcal{S}\geq 0$ ($\mathcal{S}\in\mathcal{L}(\mathcal{V},\mathcal{V}^*)$)

such that $\mathcal{A}_c - \mathcal{B}R_c^{-1}\mathcal{B}^*\mathcal{S}$ generates an exponentially stable semigroup on $\mathcal{W}$, $\mathcal{V}$ [15]. Furthermore an H^∞ control law is given as follows:

$$u(t) = -(D_{12}^{\mathrm{T}}D_{12})^{-1}(\mathcal{B}_2^*\mathcal{S} + D_{12}^{\mathrm{T}}\mathcal{C}_1)\hat{x}(t). \tag{14}$$

We will establish analytic solution of (13) by introducing Hamiltonian operator representation (Section III). Some interpretations on the H^2 control problem are also provided by employing the solution of (13) (Section IV).

III. Main Result

Solve the H^∞ control problem Σ by providing analytic solution of (13). In highlight with [6], we newly characterize the solvability condition in terms of maximal eigenvalue of an auxiliary compact operator. Furthermore an integral kernel representation of $\mathcal{S} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ is provided which clarifies the related H^2 control problems (Section IV).

The following lemma is a foothold of our approach obtained along [5].

Lemma 3 ([5], Theorem 1): For a given $\gamma > 0$, the operator Riccati equation (13) has a stabilizing solution $\mathcal{S} \geq 0$ only if the Hamiltonian matrix

$$H = \begin{bmatrix} A_c & -BR_c^{-1}B^{\mathrm{T}} \\ -C_1^{\mathrm{T}}N_cC_1 & -A_c^{\mathrm{T}} \end{bmatrix} \tag{15}$$

does not have any eigenvalue on the imaginary axis. ■

Lemma 3 guarantees that there exists a full column rank matrix $V \in \mathbb{R}^{2n\times n}$:

$$\mathcal{X}_-(H) := \operatorname{Im} V : \; HV = V\Lambda_c, \; \Lambda_c : \text{ stable matrix}, \qquad V := \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \; V_1, V_2 \in \mathbb{R}^{n\times n} \tag{16}$$

if (13) has a stabilizing solution. We next introduce an auxiliary output delay form of Σ which yields a Hamiltonian operator representation.

Introduce an auxiliary state-space $\mathcal{X}^o := \mathbb{R}^n \times L_2(-L - \check{L}, 0; \mathbb{R}^n)$ and a state-transformation $\mathcal{G} \in \mathcal{L}(\mathcal{X}, \mathcal{X}^o)$:

$$\mathcal{G}\phi = \begin{bmatrix} (\mathcal{G}\phi)^0 \\ (\mathcal{G}\phi)^1 \end{bmatrix}, \quad \phi = (\phi^0, \phi^1, \phi^2) \in \mathcal{X} \tag{17}$$

$$(\mathcal{G}\phi)^0 = e^{A_cL}\phi^0 + \int_{-L}^0 e^{-A_c\beta}\phi^1(\beta)\, d\beta$$

$$(\mathcal{G}\phi)^1(\xi) = \begin{cases} e^{A_c(\xi+L)}\phi^0 + \displaystyle\int_{-L}^{\xi} e^{A_c(\xi-\beta)}\phi^1(\beta)\, d\beta, & -L \leq \xi \leq 0 \\ \phi^2(\xi + L), & -L - \check{L} \leq \xi \leq -L \end{cases}$$

which satisfies $\mathcal{G} \in \mathcal{L}(\mathcal{W}, \mathcal{W}^o)$, $\mathcal{G} \in \mathcal{L}(\mathcal{V}, \mathcal{X}^o)$. Then

$$\hat{x}^o(t) = \mathcal{G}\hat{x}(t), \quad \hat{x}(t) \in \mathcal{W} \tag{18}$$

holds and the system $\hat{\Sigma}$ is transformed to

$$\begin{aligned} \hat{\Sigma}^o : \; \dot{\hat{x}}^o(t) &= (\mathcal{A}_c^o + \mathcal{B}_2^o D_{12}^+ \mathcal{C}_1^o)\hat{x}^o(t) + \mathcal{B}_1^o w(t) + \mathcal{B}_2^o u(t) \\ z(t) &= \mathcal{C}_1^o\hat{x}^o(t) + D_{12}u(t) \\ y(t) &= \begin{bmatrix} \hat{x}^o(t) \\ w(t) \end{bmatrix} \end{aligned} \tag{19}$$

where $\mathcal{A}_c^o$ is an infinitesimal generator defined as follows:

$$\mathcal{A}_c^o\phi = \begin{bmatrix} A_c\phi^0 \\ \phi^{1'} \end{bmatrix}, \; \mathrm{D}(A_c) = \{\phi \in \mathcal{X}^o : \phi^1 \in W^{1,2}(-L - \check{L}, 0; \mathbb{R}^n), \phi^0 = \phi^1(0)\}. \tag{20}$$

For the space

$$\mathcal{W}^o := \{\phi \in \mathcal{X} : \phi^1 \in W^{1,2}(-L - \check{L}, 0; \mathbb{R}^n), \phi^1(0) = \phi^0\}, \tag{21}$$

$\mathcal{W}^o := \mathrm{D}(\mathcal{A}_c^o)$, $\mathrm{D}_{\mathcal{W}^o}(\mathcal{A}_c^{o*}) = \mathcal{X}^o$ hold and $\mathcal{W}^o$, $\mathcal{X}^o$ are with continuous, dense injections satisfying $\mathcal{W}^o \subset \mathcal{X}^o$ [12]. The input/output operators are given by

$$\mathcal{B}_1^o := \mathcal{G}\mathcal{B}_1, \; \mathcal{B}_2^o := \mathcal{G}\mathcal{B}_2, \tag{22}$$

$$\mathcal{C}_1^o \in \mathcal{L}(\mathcal{W}^o, \mathbb{R}^{p_1}) : \; \mathcal{C}_1^o\phi = \sum_{j=0}^{\ell} C_1^{\ell}\phi^1(-L - \check{h}_j). \tag{23}$$

Since $\mathcal{C}_1^o\mathcal{G} = \mathcal{C}_1$ follows from (12), (17), (23), the systems $\hat{\Sigma}$, $\hat{\Sigma}^o$ provide equivalent input-output map and the H^∞ control problem $\hat{\Sigma}$ is solvable iff $\hat{\Sigma}^o$ is solvable [6]. By (22), (23), it is also noted that the stabilizing solution $\mathcal{S} \geq 0$ of (13) is given by

$$\mathcal{S} = \mathcal{G}^*\mathcal{S}^o\mathcal{G} \tag{24}$$

where $\mathcal{S}^o \geq 0$ ($\mathcal{S}^o \in \mathcal{L}(\mathcal{X}^o)$) is a stabilizing solution of the auxiliary operator Riccati equation

$$\begin{aligned} &\mathcal{S}^o\mathcal{A}_c^o\phi + \mathcal{A}_c^{o*}\mathcal{S}^o\phi - \mathcal{S}^o\mathcal{B}^o R_c^{-1}\mathcal{B}^{o*}\mathcal{S}^o\phi + \mathcal{C}_1^{o*}N_c\mathcal{C}_1^o\phi = 0, \\ &\mathcal{B}^o := \begin{bmatrix} \mathcal{B}_1^o & \mathcal{B}_2^o \end{bmatrix}, \; \phi \in \mathcal{W}^o. \end{aligned} \tag{25}$$

For the system $\hat{\Sigma}^o$, a Hamiltonian operator representation is obtained from (20), (22), (23).

Lemma 4: Let $V \in \mathbb{R}^{2n\times n}$ be a full column rank matrix defined by (16). Then the Hamiltonian operator $\mathcal{H}^o \in \mathcal{L}(\mathcal{W}^o \times \mathcal{X}^o, \mathcal{X}^o \times \mathcal{W}^{o*})$:

$$\mathcal{H}^o := \begin{bmatrix} \mathcal{A}_c^o & -\mathcal{B}^o R_c^{-1}\mathcal{B}^{o*} \\ -\mathcal{C}_1^{o*}N_c\mathcal{C}_1^o & -\mathcal{A}_c^{o*} \end{bmatrix} \tag{26}$$

yields the equality

$$\mathcal{H}^o \begin{bmatrix} \mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2 \\ \mathcal{V}_2 \end{bmatrix} \phi = \begin{bmatrix} \mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2 \\ \mathcal{V}_2 \end{bmatrix} \mathcal{A}_{\Lambda_c}^o\phi, \quad \phi \in \mathrm{D}(\mathcal{A}_{\Lambda_c}^o) \tag{27}$$

$$\mathcal{V}_1 := \begin{bmatrix} V_1 & 0 \\ 0 & \mathcal{I} \end{bmatrix}, \; \mathcal{V}_2 := \begin{bmatrix} V_2 & 0 \\ 0 & \Theta \end{bmatrix}, \tag{28}$$

$$(\Theta\phi^1)(\xi) := \sum_{j=0}^{\ell} \chi_{[-L-\check{h}_j, 0]}(\xi) \cdot C_1^{j\mathrm{T}}N_cC_1^j\phi^1(\xi), \quad \phi^1 \in L_2(-L - \check{L}, 0; \mathbb{R}^n), \; -L - \check{L} \leq \xi \leq 0 \tag{29}$$

$$\Pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Pi_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{X},$$

$$(\Pi_1\phi^1)(\xi) = \sum_{i=0}^{d} \chi_{[-L+h_i, 0]}(\xi) \cdot B^iR_c^{-1}B^{i\mathrm{T}}\phi^1(\xi), \quad \phi^1 \in L_2(-L, 0; \mathbb{R}^n), \; -L \leq \xi \leq 0 \tag{30}$$

where χ is a characteristic function defined by $\chi_D(\beta) = \begin{cases} 1 & (\beta \in D) \\ 0 & (\beta \notin D) \end{cases}$. In (27), $\mathcal{A}^o_{\Lambda_c}$ is an infinitesimal generator:

$$\mathcal{A}^o_{\Lambda_c}\phi = \begin{bmatrix} \Lambda_c\phi^0 \\ \phi^{1'} \end{bmatrix}, \quad \mathrm{D}(\mathcal{A}^o_{\Lambda_c}) =$$
$$\{\phi \in \mathcal{X}^o : \phi^1 \in W^{1,2}(-L-\check{L}, 0; \mathbb{R}^{p_1}), V_1\phi^0 = \phi^1(0)\}$$
$$\Lambda_c : \text{ stable matrix defined by (16)} \tag{31}$$

which generates an exponentially stable semigroup on $\mathcal{X}^o$. ■

The equality (27) characterizes the condition such that (25) has a stabilizing solution or a positive semi-definite solution. In like manner of finite-dimensional systems (see e.g. [16]), the stabilizing solution of (25) is expressed as

$$\mathcal{S}^o = \mathcal{V}_2\,(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)^{-1} \tag{32}$$

and, further, (32) is positive semi-definite iff

$$\begin{aligned}\mathcal{Q} &:= (\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)^*\mathcal{S}^o(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)\\ &= (\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)^*\mathcal{V}_2 \geq 0\end{aligned} \tag{33}$$

holds. The condition (33) enables to avoid direct calculation of the eigenvalue problem for (32). Based on (32), (33), we will establish a solvability condition in terms of maximal eigenvalue of a compact operator.

Decompose the space $\mathcal{X}^o$ and the operators $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{G}$ as follows:

$$\begin{aligned}\mathcal{X}^o = \mathcal{X}^o_1 \times \mathcal{X}^o_2,\ \ &\mathcal{X}^o_1 := \mathbb{R}^n \times L_2(-L, 0; \mathbb{R}^n),\\ &\mathcal{X}^o_2 := L_2(-L-\check{L}, -L; \mathbb{R}^n),\end{aligned} \tag{34}$$

$$\mathcal{V}_1 = \begin{bmatrix}\mathcal{V}_{11} & 0\\ 0 & \mathcal{V}_{12}\end{bmatrix} \in \mathcal{L}(\mathcal{X}^o_1 \times \mathcal{X}^o_2), \tag{35}$$

$$\mathcal{V}_{11} = \begin{bmatrix}V_1 & 0\\ 0 & \mathcal{I}\end{bmatrix} \in \mathcal{L}(\mathcal{X}^o_1),\ \ \mathcal{V}_{12} = \mathcal{I} \in \mathcal{L}(\mathcal{X}^o_2)$$

$$\mathcal{V}_2 = \begin{bmatrix}\mathcal{V}_{21} & 0\\ 0 & \mathcal{V}_{22}\end{bmatrix} \in \mathcal{L}(\mathcal{X}^o_1 \times \mathcal{X}^o_2), \tag{36}$$

$$\mathcal{V}_{21} = \begin{bmatrix}V_1 & 0\\ 0 & \Theta_1\end{bmatrix} \in \mathcal{L}(\mathcal{X}^o_1),\ \ \mathcal{V}_{22} = \Theta_2 \in \mathcal{L}(\mathcal{X}^o_2)$$

$$(\Theta_1\phi^1)(\xi) = C_1^{\mathrm{T}}N_cC_1\phi^1(\xi),\ \ -L \leq \xi \leq 0,\quad \phi^1 \in L_2(-L, 0; \mathbb{R}^n)$$

$$(\Theta_2\phi^2)(\xi) = \sum_{j=0}^{\ell}\chi_{[-L-\check{h}_j, -L]}(\xi)C_1^{j\mathrm{T}}N_cC_1^j\phi^2(\xi),\quad -L \leq \xi \leq 0,\ \phi^2 \in L_2(-L, 0; \mathbb{R}^n)$$

$$\mathcal{G} = \begin{bmatrix}\mathcal{G}_1\\ \mathcal{G}_2\end{bmatrix} \in \mathcal{L}(\mathcal{X}, \mathcal{X}^o_1 \times \mathcal{X}^o_2),\quad \begin{matrix}\mathcal{G}_1 \in \mathcal{L}(\mathcal{X}, \mathcal{X}^o_1)\\ \mathcal{G}_2 \in \mathcal{L}(\mathcal{X}, \mathcal{X}^o_2)\end{matrix} \tag{37}$$

$$\mathcal{G}_1\phi = \begin{bmatrix}(\mathcal{G}_1\phi)^0\\ (\mathcal{G}_2\phi)^1\end{bmatrix},\quad \phi = (\phi^0, \phi^1, \phi^2) \in \mathcal{X}$$

$$(\mathcal{G}_1\phi)^0 = e^{A_cL}\phi^0 + \int_{-L}^{0} e^{-A_c\beta}\phi^1(\beta)\,d\beta$$

$$(\mathcal{G}_1\phi)^1(\xi) = e^{A_c(\xi+L)}\phi^0 + \int_{-L}^{\xi} e^{A_c(\xi-\beta)}\phi^1(\beta)\,d\beta,\quad -L \leq \xi \leq 0$$

$$(\mathcal{G}_2\phi)(\eta) = \phi^2(\eta + L),\ \ -L-\check{L} \leq \eta \leq -L.$$

The properties of $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ and $\mathcal{Q}$ are summarized by the following remarks.

Remark 5: On the space $\mathcal{X}^o_1 \times \mathcal{X}^o_2$, the operator $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ is expressed as

$$\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2 = \begin{bmatrix}\mathcal{V}_{11} + \mathcal{G}_1\Pi\mathcal{G}_1^*\mathcal{V}_{21} & 0\\ 0 & \mathcal{I}\end{bmatrix}, \tag{38a}$$

$$\mathcal{V}_{11} + \mathcal{G}_1\Pi\mathcal{G}_1^*\mathcal{V}_{21} = \mathcal{I} + \begin{bmatrix}V_1 - I & 0\\ 0 & 0\end{bmatrix} + \mathcal{G}_1\Pi\mathcal{G}_1^*\mathcal{V}_{21}. \tag{38b}$$

Since $\begin{bmatrix}V_1 - I & 0\\ 0 & 0\end{bmatrix}$ and $\mathcal{G}_1\Pi\mathcal{G}_1^*\mathcal{V}_{21}$ are compact, $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ has bounded inverse iff (38b) does not have any eigenvalue at origin. ■

Remark 6: The operator $\mathcal{Q}$ defined by (33) is given by

$$\begin{aligned}&\mathcal{Q} = \begin{bmatrix}\mathcal{Q}_1 & 0\\ 0 & \mathcal{Q}_2\end{bmatrix} \in \mathcal{L}(\mathcal{X}^o_1 \times \mathcal{X}^o_2),\\ &\mathcal{Q}_1 := \mathcal{V}_{11}^*\mathcal{V}_{21} + \mathcal{V}_{21}^*\mathcal{G}_1\Pi\mathcal{G}_1^*\mathcal{V}_{21},\quad \mathcal{Q}_2 := \Theta_2 \geq 0.\end{aligned} \tag{39}$$

Hence the condition (33) holds iff $\mathcal{Q}_1 \geq 0$. ■

Introducing a differential equation:

$$\begin{cases}\Phi_\lambda(0) = I\\ \frac{d}{dt}\Phi_\lambda(t) = H_j(\lambda)\Phi_\lambda(t),\ -L + h_j \leq t \leq -L + h_{j+1}\end{cases}$$

$$H_j(\lambda) := \begin{bmatrix} A_c & -\sum_{i=0}^{j} B^iR_c^{-1}B^{i\mathrm{T}}\\ -\frac{1}{\lambda}\cdot C_1^{\mathrm{T}}N_cC_1 & -A_c^{\mathrm{T}}\end{bmatrix},\quad (j = 0, 1, 2, \cdots, d-1), \tag{40}$$

the existence of stabilizing solution is clarified by the following theorem.

Theorem 7: Let $V \in \mathbb{R}^{2n\times n}$ be a full column rank matrix defined by (16). The operator Riccati equation (25) has a stabilizing solution $\mathcal{S}^o \in \mathcal{L}(\mathcal{X}^o)$ iff the matrix

$$V_s := \begin{bmatrix}I & 0\end{bmatrix}\Phi_1(-L)V \tag{41}$$

is nonsingular where $\Phi_1(\cdot)$ is a solution of (40) with $\lambda = 1$. Furthermore the stabilizing solution is given by (32). ■

Proof: We first show that there exists a stabilizing solution $\mathcal{S}^o$ in (25) iff the operator (38a) does not have any eigenvalue at origin (Remark 5).

($\Rightarrow$) Suppose there exists a stabilizing solution of (25). Applying $[-\mathcal{S}^o,\ \mathcal{I}]$ to (27), then using (25), we have

$$\mathcal{A}^{o*}_S\mathcal{F}_S\phi + \mathcal{F}_S\mathcal{A}^o_{\Lambda_c}\phi = 0,\quad \phi \in \mathrm{D}(\mathcal{A}^o_{\Lambda_c}), \tag{42a}$$

$$\mathcal{F}_S := \mathcal{S}^o\mathcal{V}_1 - \mathcal{V}_2 + \mathcal{S}^o\mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2, \tag{42b}$$

$$\mathcal{A}^o_S := \mathcal{A}_o - \mathcal{B}^oR_c^{-1}\mathcal{B}^{o*}\mathcal{S}^o. \tag{42c}$$

Since $\mathcal{A}^o_S$ and $\mathcal{A}^o_{\Lambda_c}$ generate exponentially stable semigroups on $\mathcal{X}^o$, the Lyapunov equation (42a) requires $\mathcal{F}_S = 0$ ([2],[15] Lemma 2.32). We verify by contradiction that the operator (38a) is invertible. Suppose (38a) has an eigenvalue 0 and

$$(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)v = 0 \tag{43}$$

holds for $v = (v^0, v^1) \neq 0$. Then the equalities

$$\mathcal{F}_S v = -\mathcal{V}_2 v = 0, \quad \mathcal{V}_1 v = 0 \tag{44}$$

follow from (42b), (43) and, further, (44) yields contradiction

$$\exists v^0 \neq 0 : V v^0 = 0, \;\; v = (v^0, v^1) \in \mathcal{X}^o, \tag{45}$$

since V is full column rank by (16). Hence (38a) is invertible.

($\Leftarrow$) Since (27) holds and (38a) is assumed to be invertible, (32) is a solution of (25). Applying $[\mathcal{I},\ 0]$ to (27), we have

$$\mathcal{A}_S^o(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)\phi = (\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)\mathcal{A}_{\Lambda_c}^o\phi, \; \phi \in \mathrm{D}(\mathcal{A}_{\Lambda_c}^o) \tag{46}$$

where $\mathcal{A}_{\Lambda_c}^o$ generates exponentially stable semigroups on $\mathcal{X}_o$. Thus $\mathcal{A}_{So}$ also generates exponentially stable semigroups and (32) provides the stabilizing solution.

Secondly by contraposition, we show that (38a) is invertible iff (41) is nonsingular. By the equality $(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)v = 0$, $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ has an eigenvalue 0 iff there exists $v \neq 0$ such that

$$\mathcal{V}_1 v = -\mathcal{G}\Pi u, \qquad u = \mathcal{G}^*\mathcal{V}_2 v \tag{47}$$

hold. In the following, we verify that the matrix V_s is singular iff $v \neq 0$ exist. Introducing auxiliary variables

$$p(\xi) := \int_{-L}^{\xi} e^{A_c(\xi-\beta)}\{-(\Pi_1 u^1)(\beta)\}\, d\beta \tag{48a}$$

$$q(\beta) := e^{-A_c^{\mathrm{T}}\beta}V_2 v^0 + \int_{\beta}^{0} e^{A_c^{\mathrm{T}}(\xi-\beta)}(\Theta v^1)(\xi)\, d\xi \tag{48b}$$

to the left and right equalities in (47), respectively, we have

$$\begin{bmatrix} p'(\xi) \\ q'(\xi) \end{bmatrix} = H_i(1)\begin{bmatrix} p(\xi) \\ q(\xi) \end{bmatrix}, \quad \begin{matrix} -L + h_i \leq \xi \leq -L + h_{i+1} \\ (i = 0, 1, \cdots, d-1) \end{matrix} \tag{49}$$

with the following conditions:

$$\begin{bmatrix} p(-L) \\ q(-L) \end{bmatrix} = \Phi_1(-L)\begin{bmatrix} p(0) \\ q(0) \end{bmatrix}, \tag{50a}$$

$$\begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = V v^0, \quad p(-L) = 0. \tag{50b}$$

It is also noted that the left equality in (47) includes

$$v^1(\xi) = \begin{cases} p(\xi), & -L \leq \xi \leq 0 \\ 0, & -L - \check{L} \leq \xi \leq 0 \end{cases}. \tag{51}$$

By (50a), (50b), we finally obtain the equality:

$$\begin{bmatrix} I & 0 \end{bmatrix}\Phi_1(-L)V v^0 = V_s v^0 = 0. \tag{52}$$

If $v^0 = 0$, (49), (50b) yield $(p, q) = 0$ and, further with (51), we have $v^1 = 0$. Hence $v = 0$ if $v^0 = 0$. Conversely $v = 0$ includes $v^0 = 0$. Thus, $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ is invertible iff the matrix V_s is nonsingular.

The representation (32) is obtained in the proof of necessity. ■

Finally we clarify an alternative condition of $\mathcal{S}^o \geq 0$ by transforming (33) to maximal eigenvalue condition of a compact operator.

Theorem 8: Let $\lambda_{\max}$ be maximal root of the transcendental equation

$$\det V_p(\lambda) = 0,$$
$$V_p(\lambda) := \begin{bmatrix} I & 0 \end{bmatrix}\Phi_\lambda(-L)\left\{(\lambda - 1)\cdot\begin{bmatrix} I \\ 0 \end{bmatrix} + V V_2^{\mathrm{T}}\right\} \tag{53}$$

where $\Phi_\lambda(\cdot)$ is a solution of (40). The stabilizing solution (32) is positive semi-definite ($\mathcal{S}^o \geq 0$) iff maximal root of (53) satisfies $\lambda_{\max} \leq 1$. ■

For the proof of Theorem 8, we prepare a preliminary result which characterizes the condition (33) in terms of maximal eigenvalue of a compact operator.

Lemma 9: The condition (33) is satisfied iff

$$\Gamma\Delta\Gamma^* \leq \mathcal{I} \tag{54}$$

holds for $\Delta \in \mathcal{L}(\mathcal{X}_o^1)$, $\Gamma \in \mathcal{L}(\mathcal{X}_o^1, \mathbb{R}^n \times L_2(-L, 0; \mathbb{R}^{p_1}))$:

$$\Delta := \begin{bmatrix} I - V_1^{\mathrm{T}}V_2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} V_2^{\mathrm{T}} & 0 \\ 0 & \mathcal{I} \end{bmatrix}\mathcal{G}_1\Pi\mathcal{G}_1^*\begin{bmatrix} V_2 & 0 \\ 0 & \mathcal{I} \end{bmatrix}, \tag{55}$$

$$\Gamma := \begin{bmatrix} I & 0 \\ 0 & N_c C_1 \cdot \mathcal{I} \end{bmatrix}. \tag{56}$$

■

Proof: For the operator $\mathcal{Q}_1$ defined by (39), the equality

$$\mathcal{Q}_1 = \Gamma^*(\mathcal{I} - \Gamma\Delta\Gamma^*)\Gamma \tag{57}$$

follows from (55), (56). We will show that the conditions (54) and $\mathcal{Q}_1 = \Gamma^*(\mathcal{I} - \Gamma\Delta\Gamma^*)\Gamma \geq 0$ are equivalent. The condition (54) derives $\mathcal{Q}_1 \geq 0$ directly. By contradiction, we verify (57) derives (54). Note that the operator

$$\Gamma^+ := \begin{bmatrix} I & 0 \\ 0 & (N_c C_1)^+ \cdot \mathcal{I} \end{bmatrix} \tag{58}$$

preserves equalities $\Gamma\Gamma^+\Gamma = \Gamma$ and $(\Gamma\Gamma^+)^* = \Gamma\Gamma^+$.

Suppose (57) holds and there exists $y \in \mathcal{X}_1^o$ such that

$$\langle y, (\mathcal{I} - \Gamma\Delta\Gamma^*)y\rangle < 0 \tag{59}$$

holds. Then a contradiction

$$\begin{aligned} \langle \tilde{y}, \Gamma^*(\mathcal{I} - \Gamma\Delta\Gamma^*)\Gamma\tilde{y}\rangle &= \langle y, \Gamma^{+*}\Gamma^*(\mathcal{I} - \Gamma\Delta\Gamma^*)\Gamma\Gamma^+ y\rangle \\ &= \langle y, (\Gamma\Gamma^+ - \Gamma\Delta\Gamma^*)y\rangle \\ &\leq \langle y, (\mathcal{I} - \Gamma\Delta\Gamma^*)y\rangle < 0 \end{aligned} \tag{60}$$

is derived for $\tilde{y} := \Gamma^+ y$. Thus (54) is derived from (57). ■

Proof of Theorem 8: Since $\Gamma\Delta\Gamma^*$ is compact (see Remark 5), we clarify the condition such that $\lambda_{\max}(\Gamma\Delta\Gamma^*) \leq 1$ holds.

The equality $\lambda v = \Gamma\Delta\Gamma^* v$ is equivalently expressed as

$$\lambda v = \begin{bmatrix} I - V_1^{\mathrm{T}}V_2 & 0 \\ 0 & 0 \end{bmatrix} v - \begin{bmatrix} V_2^{\mathrm{T}} & 0 \\ 0 & \Gamma_1 \end{bmatrix}\mathcal{G}_1\Pi f, \tag{61a}$$

$$f = \mathcal{G}_1^*\begin{bmatrix} V_2 & 0 \\ 0 & \Gamma_1^* \end{bmatrix} v. \tag{61b}$$

For $\lambda \neq 0, 1$, we will show that there exists $v \neq 0$ which satisfies (61a), (61b) iff the matrix (53) is singular.

Introducing auxiliary variables

$$p(\xi) := \int_{-L}^{\xi} e^{A_c(\xi-\beta)}\{-(\Pi_1 f^1)(\beta)\}\, d\beta \tag{62a}$$

$$q(\beta) := e^{-A_c^{\mathrm{T}}\beta} V_2 v^0 + \int_{\beta}^{0} e^{A_c^{\mathrm{T}}(\xi-\beta)} C_1^{\mathrm{T}} N_c v^1(\xi)\, d\xi \tag{62b}$$

to (61a) and (61b), respectively, we have the following relations:

$$p'(\xi) = A_c p(\xi) - \sum_{i=0}^{d} \chi_{[-L+h_i,0]}(\xi) \cdot B^i R_c^{-1} B^{i\mathrm{T}} f^1(\xi), \quad -L \le \xi \le 0 \tag{63a}$$

$$p(-L) = 0 \tag{63b}$$

$$\{(\lambda-1)\cdot I + V_1^{\mathrm{T}} V_2\} v^0 = V_2^{\mathrm{T}} p(0) \tag{63c}$$

$$\lambda v^1(\xi) = N_c C_1 p(\xi) \tag{63d}$$

$$q'(\beta) = -A_c^{\mathrm{T}} q(\beta) - C_1^{\mathrm{T}} N_c v^1(\beta), \quad -L \le \beta \le 0 \tag{63e}$$

$$q(0) = V_2 v^0 \tag{63f}$$

$$f^0 = q(-L) \tag{63g}$$

$$f^1(\beta) = q(\beta), \quad -L \le \beta \le 0. \tag{63h}$$

The differential equations:

$$\begin{bmatrix} p'(\xi) \\ q'(\xi) \end{bmatrix} = H_i(\lambda) \begin{bmatrix} p(\xi) \\ q(\xi) \end{bmatrix}, \quad \begin{matrix} -L+h_i \le \xi \le -L+h_{i+1} \\ (i=0,1,\cdots,d-1) \end{matrix} \tag{64}$$

follow from (63a), (63d), (63e), (63h), and, further, the boundary conditions:

$$\begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = \Phi_\lambda^{-1}(-L) \begin{bmatrix} p(-L) \\ q(-L) \end{bmatrix} \tag{65a}$$

$$\begin{bmatrix} p(-L) \\ q(-L) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} f^0 \tag{65b}$$

$$\begin{bmatrix} V_2 V_2^{\mathrm{T}} & -(\lambda-1)\cdot I - V_2 V_1^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = 0 \tag{65c}$$

follow from (64) and (63b), (63c), (63f), (63g). By (65a), (65b), (65c), we finally obtain a condition:

$$\tilde{V}_p(\lambda) f^0 = 0, \tag{66a}$$

$$\tilde{V}_p(\lambda) := \begin{bmatrix} V_2 V_2^{\mathrm{T}} & -(\lambda-1)\cdot I - V_2 V_1^{\mathrm{T}} \end{bmatrix} \Phi_\lambda^{-1}(-L) \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{66b}$$

If $f^0 = 0$, (64), (65b) yield $(p,q)=0$ and, further with (63c), (63d), (63f), we have $v=(v^0,v^1)=0$ for $\lambda \ne 0,1$. Hence $v=0$ if $f^0=0$. Conversely, if $v=0$, (61b) derives $f=(f^0,f^1)=0$. Thus $\lambda \ne 0,1$ is the eigenvalue of $\mathcal{Q}_1$ iff the matrix $\tilde{V}_p(\lambda)$ is nonsingular.

Since $V_p(\lambda) = -\tilde{V}_p^{\mathrm{T}}(\lambda)$ is derived by substituting

$$\Phi_\lambda^{-1}(-L) = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \Phi_\lambda^{\mathrm{T}}(-L) \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \tag{67}$$

to (66b), the eigenvalues $\lambda \ne 0,1$ are given by (53). ■

By Theorems 7, 8 and (24), the analytic solution of (13) is given with the base of stable eigenspace (16). The following theorem summarizes the solvability condition and provides an integral kernel representation of

$$\mathcal{S} = \mathcal{G}^* \mathcal{V}_2 (\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)^{-1} \mathcal{G}. \tag{68}$$

Theorem 10: For a given $\gamma > 0$, the operator Riccati equation (13) has a stabilizing solution $\mathcal{S} \ge 0$ iff (a1), (a2) are satisfied.

(a1) The Hamiltonian matrix (15) does not have eigenvalues on the imaginary axis.

(a2) The matrix (41) is nonsingular and maximal root of (53) satisfies $\lambda_{\max} \le 1$.

If (a1), (a2) hold, the stabilizing solution $\mathcal{S} \ge 0$ of (68) is expressed as

$$(\mathcal{S}v)^0 = G(-L,-L)v^0 + \int_{-L}^{0} G(-L,\beta) v^1(\beta)\, d\beta$$

$$(\mathcal{S}v)^1(\xi) = G(\xi,-L)v^0 + \int_{-L}^{0} G(\xi,\beta) v^1(\beta)\, d\beta$$

$$(\mathcal{S}v)^2(\beta) = \sum_{j=0}^{\ell} \chi_{[-\check{h}_j,0]}(\beta) \cdot C_1^{j\mathrm{T}} N_c C_1^j v^2(\beta)$$

$$-L \le \xi \le 0, -\check{L} \le \beta \le 0, v=(v^0,v^1,v^2) \tag{69}$$

$$G(\xi,\beta) = \begin{cases} \begin{bmatrix} 0 & I \end{bmatrix} \Phi_1(\xi) V_s^R \Phi_1^{-1}(\beta) \begin{bmatrix} I \\ 0 \end{bmatrix}, & \xi \le \beta \\ -\begin{bmatrix} 0 & I \end{bmatrix} \Phi_1(\xi) V_s^N \Phi_1^{-1}(\beta) \begin{bmatrix} I \\ 0 \end{bmatrix}, & \xi \ge \beta \end{cases} \tag{70}$$

$$V_s^R := V V_s^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \Phi_1(-L), \quad V_s^N := I - V_s^R. \tag{71}$$

Furthermore the H^∞ control law (14) is given as follows:

$$u(t) = -D_{12}^{+} \sum_{j=0}^{\ell} C_1^j x(t-\check{h}_j) - (D_{12}^{\mathrm{T}} D_{12})^{-1} \sum_{i=0}^{d} B_2^{i\mathrm{T}} \times \left\{ G(-L+h_i,-L)x(t) + \int_{-L}^{0} G(-L+h_i,\beta) v_t(\beta)\, d\beta \right\}$$

$$v_t(\beta) := \sum_{k=0}^{d} \chi_{[-L,-L+h_k]}(\beta) \cdot B^k \begin{bmatrix} w(t+\beta+L-h_k) \\ u(t+\beta+L-h_k) \end{bmatrix}. \tag{72}$$

■

Proof: The conditions (a1), (a2) are derived by Lemmas 3-4 and Theorems 7-8. We will derive the representation of (69) from (68). By (68), the equality $f = \mathcal{S}v$ is expressed as

$$\mathcal{V}_1 w = \mathcal{G}(v - \Pi f), \quad f = \mathcal{G}^* \mathcal{V}_2 w. \tag{73}$$

Introducing auxiliary variables

$$p(\xi) := e^{A_c(\xi+L)} v^0 + \int_{-L}^{\xi} e^{A_c(\xi-\beta)}\{v^1(\beta) - (\Pi_1 f^1)(\beta)\}\, d\beta \tag{74a}$$

$$q(\beta) := e^{-A_c^{\mathrm{T}}\beta} V_2 v^0 + \int_{\beta}^{0} e^{A_c^{\mathrm{T}}(\xi-\beta)} (\Theta v^1)(\xi)\, d\xi \tag{74b}$$

to the left and right equalities in (73), respectively, we have

$$f^0 = q(-L) \tag{75a}$$

$$f^1(\xi) = q(\xi), \quad -L \le \xi \le 0 \tag{75b}$$

$$f^2(\beta) = \sum_{i=0}^{\ell} \chi_{[-\check{h}_i,0]}(\beta) \cdot C_1^{i\mathrm{T}} N_c C_1^i v^2(\beta), \quad -\check{L} \le \beta \le 0 \tag{75c}$$

and the following equalities

$$\begin{bmatrix} p(\xi) \\ q(\xi) \end{bmatrix} = \Phi_1(\xi) \begin{bmatrix} p(0) \\ q(0) \end{bmatrix} - \int_\xi^0 \Phi_1(\xi)\Phi_1^{-1}(\beta) \begin{bmatrix} I \\ 0 \end{bmatrix} v^1(\beta)\, d\beta \tag{76}$$

$$\begin{bmatrix} p(-L) \\ q(-L) \end{bmatrix} = \begin{bmatrix} v^0 \\ f^0 \end{bmatrix}, \quad \begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} w^0 \tag{77}$$

where $\Phi_1(\cdot)$ is defined by (40). Substituting (77) to

$$\begin{bmatrix} p(-L) \\ q(-L) \end{bmatrix} = \Phi_1(\xi) \begin{bmatrix} p(0) \\ q(0) \end{bmatrix} - \int_\xi^0 \Phi_1(\xi)\Phi_1^{-1}(\beta) \begin{bmatrix} I \\ 0 \end{bmatrix} v^1(\beta)\, d\beta, \tag{78}$$

then premultiplying $[I \ \ 0]$ to both sides of (78), we obtain

$$w^0 = V_s^{-1} v^0 + V_s^{-1} \begin{bmatrix} I & 0 \end{bmatrix} \Phi_1(-L)\Phi_1^{-1}(\beta) \begin{bmatrix} I \\ 0 \end{bmatrix} v^1(\beta)\, d\beta \tag{79}$$

where V_s is nonsingular by Theorem 7. By (76), (77), (79), the solution

$$q(\xi) = G(\xi, -L)v^0 + \int_{-L}^0 G(\xi,\beta) v^1(\beta)\, d\beta \tag{80}$$

is obtained. Thus (69) is derived by (75), (80).

Since the internal state $\hat{x}(t) = (\hat{x}_t^0, \hat{x}_t^1, \hat{x}_t^2) \in \mathcal{W}$ corresponds to

$$\hat{x}_t^0 := x(t), \quad \hat{x}_t^1 := v_t,$$

$$v_t(\beta) = \sum_{j=0}^{d} \chi_{[-L,-L+h_j]}(\beta) \cdot B^j \begin{bmatrix} w(t+\beta+L-h_j) \\ w(t+\beta+L-h_j) \end{bmatrix}, \quad -L \le \beta \le 0 \tag{81}$$

the control law (72) is obtained from (14), (11), (69). ■

IV. H^2 Control Problem

The state-space approach discussed in Section III has advantage of providing solutions of H^2 control problems in a unified manner. In this section, we derive an H^2 control law for the preview/delayed system Σ. The result obtained here is a generalization of [11], [8] which deal with multiple preview actions.

Focus on the H^2 full-information (FI) control problem defined by Σ with (H1)-(H3) and ($\tilde{\mathrm{H}}$4)

($\tilde{\mathrm{H}}$4) The system Σ satisfies the following condition.

$$B_2^i (D_{12}^{\mathrm{T}} D_{12})^{-1} B_2^{j\mathrm{T}} = 0 \ \ (i \ne j) \tag{82}$$

$$B_2^i D_{12}^{+} C_1^j = 0 \ \ (i \ne 0 \text{ or } j \ne 0) \tag{83}$$

The H^2 optimal control law and the performance are characterized by the following theorem.

Theorem 11: Define the Hamiltonian matrix (15) and the differential equation (40) replacing by

$$R_c^{-1} = \begin{bmatrix} 0_{m_1 \times m_1} & 0 \\ 0 & (D_{12}^{\mathrm{T}} D_{12})^{-1} \end{bmatrix} \ \ (\gamma \to \infty). \tag{84}$$

The H^2 optimal control law is given by (72). Furthermore, the optimal H^2 performance $\gamma_{\mathrm{opt}} \ge 0$ is expressed as

$$\gamma_{\mathrm{opt}}^2 = \mathrm{trace}(X),$$

$$X = \sum_{i=0}^{d}\sum_{j=0}^{d} B_1^{i\mathrm{T}} G(-L+h_i, -L+h_j) B_1^j. \tag{85}$$

where $G(\cdot,\cdot)$ is the integral kernel defined by (70). ■

Proof: It is noted that the system response against the disturbance

$$w(t) = \delta(t) \cdot \bar{w}, \quad \bar{w} \in \mathbb{R}^{m_1} \tag{86}$$

is equivalently described by $\hat{\Sigma}$ with the initial state

$$\hat{x}(0) = \mathcal{B}_1 \bar{w} \in \mathcal{V}. \tag{87}$$

Hence, by [12], [13], an optimal control which minimizes

$$J(\bar{w}) = \int_0^\infty \|z(t)\|^2\, dt \tag{88}$$

is given by (72). Employing Theorem 10, the optimal value of (88) is expressed as

$$J_{\mathrm{opt}}(\bar{w}) = \langle \mathcal{B}\bar{w}, \mathcal{S}\mathcal{B}\bar{w} \rangle_{\mathcal{V},\mathcal{V}^*} = \bar{w}^{\mathrm{T}} \mathcal{B}^* \mathcal{S}\mathcal{B}\bar{w} = \bar{w}^{\mathrm{T}} X \bar{w}. \tag{89}$$

Since the optimal H^2 performance is given by

$$\gamma_{\mathrm{opt}}^2 = \sum_{i=0}^{m_1} J_{\mathrm{opt}}(e_i) = \mathrm{trace}(X), \tag{90}$$

the control law (72) is also optimal for (90) and the equality (85) is obtained. ■

V. Example

Define an H^∞ preview and delayed control problem:

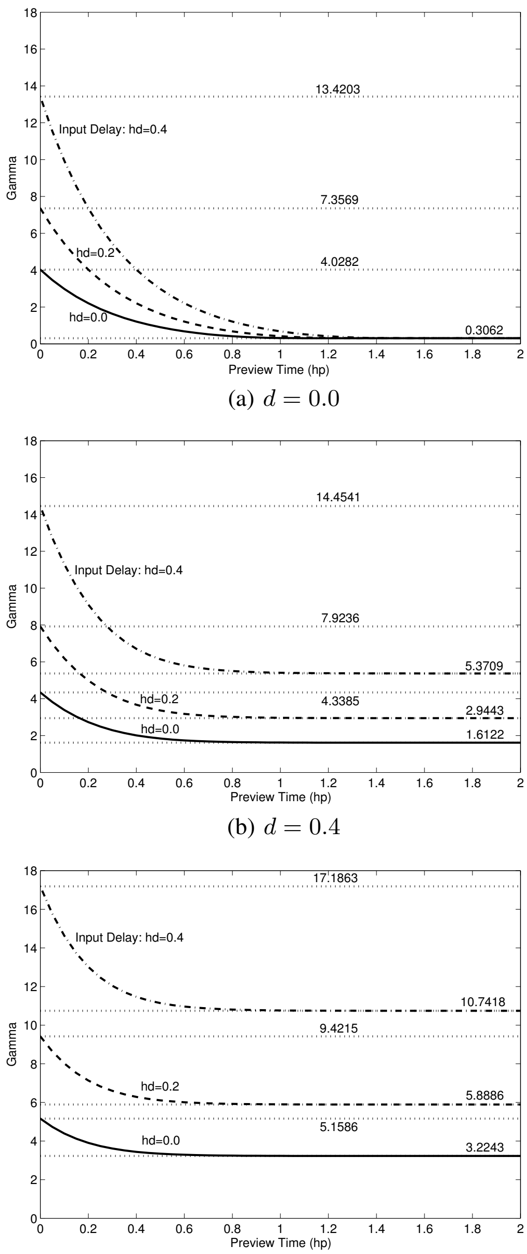
$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ d \end{bmatrix} w_0(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_1(t-h_p) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t-h_d)$$

$$z(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \ d = 0.0, 0.4, 0.8 \tag{91}$$

where w_1 is h_p unit-time previewable reference and w_0 is the uncertainty which disturbs the previewable signal w_1. Furthermore h_d unit-time delay is imposed on the control u. We will investigate the optimal performance γ_{opt} in terms of (h_p, h_d).

Based on Theorem 10, the achievable H^∞ performance for (91) is obtained by Fig.1 (a)-(c). Fig.1 (a) summarizes the performance for the case $d = 0.0$ and it is observed that the curves coincide by sliding aside. This feature arises from the fact that the common input delays $\min(h_p, h_d)$ can be pushed out to the regulated output channel.

While in the cases $d = 0.4, 0.8$ (Fig.1 (b), (c)), the relation between the preview and delay times are rather complicated and the H^∞ performance is not sufficiently recovered even if long preview time is employed.



(a) $d = 0.0$

(b) $d = 0.4$

(c) $d = 0.8$

Fig. 1. H^∞ performance vs. preview/delay times.

VI. Conclusion

A generalized H^∞ preview/delayed control problem is solved and a solvability condition is newly established. The solvability condition overcomes the limitation of [6], which causes numerical instability, and enables to deal with preview/delayed control problems in a unified manner. An H^2 control law is also clarified by employing the integral kernel representation for the solution of the operator Riccati equation. The solution of H^∞ output feedback control problem is derived by exploring the duality of control/filtering operator Riccati equations.

References

[1] G.Buttazzo and M.Giaquinta, One-dimensional variational problems, Oxford University Press (1998)

[2] R.Datko: Extending a theorem of A.M.Liapunov to Hilbert space, J. Mathematical Analysis and Applications, 32, pp.610-616 (1970)

[3] T.Fujinaka and M.Araki: Discrete-time optimal control of systems with unilateral time-delays, Automatica, 23-6, pp. 763-765 (1987)

[4] K.Kashima and Y.Yamamoto: On standard H^∞ control problems for systems with infinitely many unstable poles, Systems and Control Letters, 57-4, pp.309-314 (2008)

[5] A.Kojima and S.Ishijima: H^∞ performance of preview control systems, Automatica, 39-4, pp.693-701 (2003)

[6] A.Kojima and S.Ishijima: Formulas on preview and delayed H^∞ control, IEEE Trans. Automat. Control, AC-51-12, pp.1920-1937 (2006)

[7] A.Kojima and S.Ishijima: Generalized preview and delayed H^∞ control: output feedback case, IEEE Conference on Decision and Control, pp.5770-5775 (2005)

[8] A.Kojima and K.Yokoyama: H^2 performance on multiple preview compensation and initial state setting, 19th International Symposium on Mathematical Theory of Networks and Systems (MTNS 2010)

[9] G.Meinsma and L.Mirkin: H^∞ control of systems with multiple I/O delays via decomposition to adobe problems, IEEE Trans. Automat. Control, AC-50-2, pp.199-211 (2005)

[10] L.Mirkin: On the H^∞ fixed-lag smoothing: how to exploit the information preview, Automatica 39, pp.1495-1504 (2003)

[11] A.A.Moelja and G.Meinsma: H2 control of previewsystems, Automatica, Vol.42, No.4, pp.945-952 (2006)

[12] A.J.Pritchard and D.Salamon: The linear-quadratic control problem for retarded systems with delays in control and observation, IMA J. Math. Contr. Inform. 2, pp.335-361 (1985)

[13] A.J.Pritchard and D.Salamon: The linear quadratic control problem for infinite dimensional systems with unbounded input and output operators, SIAM J. Contr. Optim., 25-1, pp.121-144 (1987)

[14] G.Tadmor and L.Mirkin: H^∞ control and estimation with preview—Part I: matrix ARE solutions in continuous time, IEEE Trans. Automat. Control, AC-50-1, pp.19-28 (2005)

[15] B. van Keulen: H^∞-control for distributed parameter systems: A state-space approach, Birkhäuser Boston (1993)

[16] K.Zhou with J.C.Doyle and K.Glover: Robust and optimal control, Prentice Hall (1996)