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Relative Value of Measurements in a Discrete Decentralized LQG Framework

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Abstract— **A discrete linear multi-agent system is considered, and each agent in the system aims to minimize his own infinite-horizon quadratic cost. An attempt is made to quantify the relative value of information at each node in the network. This knowledge is relevant when the exchange of information between agents is costly or limited. First, a centralized approach to the estimation and control problem is taken. Then, a non-zero sum game is formulated, in which the strategies are the measurements selections. We impose that these be time periodic, which allows us to pose an equivalent finite-horizon game and to interpret the results as explicit data rates. The method is applied to the simple example of a string of cars, and provides an interesting validation to a heuristic assumption made in previous work on string stability.**

Index Terms— **Decentralized control, limited information, measurement selection**

I. Introduction

Due to potentially significant advantages and technological advances, decentralized control has generated a substantial interest in the last few decades ([1], [2]). However, a number of theoretical difficulties are introduced with decentralization. Among them, the question of the distribution of information at each of the nodes of the network at which the local controls are applied has a great impact on the performance of the system. Indeed, a non-classical information pattern, where the measurement sequence is not nested, was shown to lead to non-obvious consequences ([3]). Decentralized information is inherently non-classical, and requires the generalization of notions of stabilization under (local) feedback ([4], [5]) and controllability ([6]).

In large multiple-agent systems, the transmission of information between agents can be costly, or limited by bandwidth. Depending on the objective of the control problem, it seems reasonable that only a part of the total information about the system is relevant for an individual node. For example, in a formation of a large number of vehicles, information about distant vehicles might be ignored, or transmitted at a lower frequency than that of nearby vehicles.

The aim of this paper is to develop a methodology which quantifies the relative value of data from each measurement at all the nodes in the system and to provide efficient measurement selection under bandwidth constraint. The framework adopted is the following. The system is assumed linear, discrete, and perturbed by Gaussian white noise. The decision process is non-cooperative, so each agent attempts to minimize his own infinite-horizon quadratic cost. Each agent selects a subset of all the available measurements, and the selections are constrained to be periodic. From this local measurement information sequence, the agents construct their own filter of the entire state, on which the individual controllers are based.

Ideally, each agent would try to minimize his cost with respect to his choice of the state estimator, the control law, and the measurement sequence. However, a Nash solution with respect to all these free variables entering the optimizations is difficult, since the information pattern would become non-classical, and the separation principle would therefore no longer hold. To avoid this complexity, we choose to perform the optimizations sequentially. First, the filter is defined for all agents. Then, assuming all agents use this centralized state estimator, person-by-person optimal controllers are determined. Finally, the measurement sequences are sought, by formulating a periodic optimal control problem, where the controls are the explicit data rates of each measurement. The notion is to start from a solution assuming centrality, and discard information with limited relevance. This will yield a suboptimal solution to the general problem. A similar approach was taken for continuous time ([7]), but the discrete-time formulation allows for direct interpretation of the results as data exchange rates. Also, the optimal measurement sequences are for a given limit on the number of measurements per time-step, instead of adding a cost for communication in the performance objective ([7], [8]). Related work has been done in the single agent systems case: selecting a single measurement optimally with respect to an estimation error ([9]), determining optimal control when each measurement is costly ([10], [11], [12]), and simultaneous optimization of control and measurement under constrained total amount of measurement ([13]).

The paper is organized as follows. The problem is formulated in section II. Optimization of the estimation and control gains is done in section III. In section IV, the agents' measurement selection process is introduced, and a periodic optimal control problem is posed. Finally, the approach is tested on a simple example in section V.

II. Problem Formulation

A. Dynamics and Measurements

Consider a discrete linear dynamic system composed of M vehicles:

$$x[k+1] = Ax[k] + \sum_{j=1}^{M} B_j u_j[k] + \Gamma w[k] \tag{1}$$

$$y[k] = Cx[k] + v[k] \tag{2}$$

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where $x \in \mathbb{R}^n$ is the aggregate state vector of the M vehicle system, $u_j \in \mathbb{R}^{m_j}$, $j = 1 \cdots M$ is the control vector of the j^{th} vehicle, $w \in \mathbb{R}^p$ is the zero-mean Gaussian white process noise with power spectral density W, $y \in R^q$ is the vector of all available measurements, and $v \in \mathbb{R}^q$ is the zero-mean Gaussian white measurement noise with power spectral density V, assumed to be independent between measurements. Let $Y[k]$ be the measurement history

$$Y[k] \stackrel{def}{=} \{y[s] : 0 \leq s \leq k\} \tag{3}$$

At time k, each agent i will have access to a subset of the measurement history (3), in order to update its estimate $\hat{x}_i[k]$ of the entire state of the system. Each control $u_i[k]$ will be a function of $\hat{x}_i[k]$.

B. Costs

Each agent i in the network attempts to minimize his cost J_i, with an infinite horizon. We assume the cost is quadratic in the state and the controls, having the form:

$$J_i = \lim_{N\to\infty} \frac{1}{N} E\left[\sum_{k=0}^{N-1} \left(x[k]^T Q_i x[k] + u_i[k]^T R_i u_i[k]\right)\right] \tag{4}$$

where $E[\cdot]$ is the expectation operator, and $Q_i = Q_i^T \geq 0$ and $R_i = R_i^T > 0$ for all i. The matrices Q_i should reflect the relative importance of certain linear combinations of the state, e.g. by weighing more the relative distance with a close agent versus the one with a farther agent in the network. This multi-cost formulation provides flexibility through the different possible choices of the matrices Q_i (perhaps more than a single-cost approach).

III. Centralized Estimator and Controller

A. Centralized Estimator

First, we assume that all agents have access to the entire available set of measurements $y[k]$ as well as all the agents' controls $u_j[k]$ at each time-step. With this assumption, we choose to use the standard discrete Kalman filter with infinite horizon, and each agent will have the same estimator gains. Denote $\hat{x}[k]$ the a posteriori estimate, and $\bar{x}[k]$ the a priori (or propagated) estimate at time k; then

$$\bar{x}[k+1] = A\hat{x}[k] + \sum_{j=1}^{M} B_j u_j[k] \tag{5}$$

$$\hat{x}[k+1] = \bar{x}[k+1] + K(y[k+1] - C\bar{x}[k+1]) \tag{6}$$

K is the steady-state Kalman gain, and is given by:

$$K = \bar{P}C^T(C\bar{P}C^T + V)^{-1} \tag{7}$$

where $\bar{P}$ is the *a priori* error variance, and is the solution to the algebraic Riccati equation:

$$0 = A^T\bar{P}A - \bar{P} + A\left[\bar{P}C^T(C\bar{P}C^T + V)^{-1}C\bar{P}\right]A^T + \Gamma W \Gamma^T \tag{8}$$

This can be rewritten in terms of the *a posteriori* error variance P:

$$K = PC^TV^{-1} \tag{9}$$

$$P = \left(\bar{P}^{-1} + C^TV^{-1}C\right)^{-1} \tag{10}$$

B. Nash Equilibrium Controllers

We assume that all agents use the same estimator, $\hat{x}[k]$, computed above. We now look to solve the multi-criteria optimization problem. We can compute the Nash equilibrium by solving the following problem:

$$\begin{array}{ll} \min_{u_1} & J_1(u_1, u_2^0 \cdots, u_M^0) \\ & \vdots \\ \min_{u_M} & J_M(u_1^0, \cdots u_{M-1}^0, u_M) \\ & \\ \text{s.t.} & (1) - (2), (5) - (8) \end{array} \tag{11}$$

Solutions to the finite-horizon version of this problem were derived both for continuous time ([14], [7]) and discrete time ([15]). It was also shown that when all the agents use the same measurements, the certainty equivalence principle holds ([15]). Therefore, in the remainder of the section, full-state feedback is assumed in determining the control gains. It is not immediate to conclude that the solution to the infinite-horizon problem is the limit of the solution to the finite-horizon case. However, we will now show that this is in fact the case when solutions exist and lead to a stable closed-loop system. Namely, if we define

$$A_{(-i)} = (A + \sum_{j\neq i} B_j G_j), \ \forall i = 1, \ldots, M \tag{12}$$

the optimal controls to the infinite-horizon problem are:

$$\forall i = 1, \ldots M, \ \ u_i = G_i x \tag{13}$$

where

$$\begin{aligned} G_i &= -(R_i + B_i^T S_i B_i)^{-1} B_i^T S_i A_{(-i)} &(14)\\ &= -R_i^{-1} B_i^T S_i (A + \sum_j B_j G_j) &(15)\\ S_i &= Q_i + A_{(-i)}^T S_i A_{(-i)} \\ &\quad - A_{(-i)}^T S_i B_i (R_i + B_i^T S_i B_i)^{-1} B_i^T S_i A_{(-i)} &(16)\\ &= Q_i + (A + \sum_j B_j G_j)^T S_i (A + \sum_{j\neq i} B_j G_j) &(17) \end{aligned}$$

To show this, first make a change of variable in the control:

$$u_i = G_i x + r_i \tag{18}$$

Next, note that if the closed-loop system is stable, $E(x[k])$ will be bounded, and a fortiori so will $E(x[k]^T S_i x[k])$. Therefore, using the following zero quantity:

$$H = \lim_{N\to\infty} \frac{1}{N} E\left[\sum_{k=0}^{N-1} (x[k+1]^T S_i x[k+1] - x[k]^T S_i x[k])\right], \tag{19}$$

the dynamics, and the equations for G_i and S_i, we have after some manipulation that:

$$J_i = J_i + H = \lim_{N\to\infty} \frac{1}{N} E\left[\sum_{k=0}^{N-1} r_i^T (R_i + B_i^T S_i B_i) r_i\right] + tr(\Gamma^T Q_i \Gamma W) \tag{20}$$

Since the quadratic form in (20) is positive definite, the minimum of J_i with respect to r_i is clearly attained for $r_i = 0$, which proves the claim.

The conditions which guarantee the existence of a solution to the coupled-Riccati equation satisfied by the S_i are unclear. Indeed, for each i the matrix $A_{(-i)}$ entering the equation for S_i depends on the solutions S_j, $j \neq i$. Likewise, if the existence of a solution is assumed, the closed-loop stability is not straightforward. What seems clear is that for all the costs J_i to be bounded, the states which are observable by any one Q_i must remain bounded. In particular, if $(A, [Q_1^T \dots Q_M^T]^T)$ is observable, then the closed-loop stability of the system is necessary for all the costs to be finite. All these considerations are the subject of current work.

IV. Measurement Optimization

A. Individual Estimators

We now assume that these gains G_j are known a priori by all agents, but that each agent does not know, by default, the actual controls $u_i[k]$ of the other agents (a priori knowledge of the control gains appears to yield a better performance than online knowledge of the exact control actions ([7])). Agent i's best estimate of agent j's control action $u_j[k] = G_j \hat{x}_j[k]$ is $G_j \hat{x}_i[k]$, where $\hat{x}_i[k]$ is agent i's a posteriori estimate of the state of the system at time k. Therefore, agent i's a priori estimator's update equation is:

$$\bar{x}_i[k+1] = A\hat{x}_i[k] + \sum_{j=1}^{M} B_j G_j \hat{x}_i[k] \tag{21}$$

Now, we want to express that at time step k, agent i only has access to a certain subset of measurements chosen among the available measurements $y[k]$. We do this by setting the innovation to zero when the measurement is not available, i.e., we define the "selector" matrix $L_i[k]$ of agent i at time k as a diagonal matrix whose diagonal terms are 1 (resp. 0) for the available (resp. unavailable) measurements. The a posteriori estimate's update equation for agent i is then:

$$\begin{aligned}\hat{x}_i[k+1] &= \bar{x}_i[k+1] + \dots \\ &\quad + KL_i[k+1](y[k+1] - C\bar{x}_i[k+1])\end{aligned} \tag{22}$$

B. Closed-loop System and Error Variance Differential Equation

Define the aggregate state:

$$\tilde{x}[k] = \left[x^T[k]\ \hat{x}_1^T[k]\ \dots\ \hat{x}_M^T[k]\right]^T \in \mathbb{R}^{(M+1)n} \tag{23}$$

We can write the closed-loop update equation (similar results can be found in [16]). For ease of notation, we drop the time index ($[k+1]$) for the L_i's. First, denote

$$\Omega_i = (I - KL_iC)(A + \sum_j B_j G_j) \tag{24}$$

Then, using (21) and (22):

$$\tilde{x}[k+1] = \tilde{A}_{CL}[k]\tilde{x}[k] + \tilde{\Gamma}[k]\begin{bmatrix} w[k] \\ v[k+1]\end{bmatrix} \tag{25}$$

where

$$\tilde{A}_{CL}[k] = \begin{bmatrix} I_n \\ KL_1C \\ \vdots \\ KL_MC \end{bmatrix}\begin{bmatrix} A & B_1G1 & \dots & B_MG_M\end{bmatrix} + \begin{bmatrix} 0_n & & & \\ & \Omega_1 & & \\ & & \ddots & \\ & & & \Omega_M \end{bmatrix} \tag{26}$$

and

$$\tilde{\Gamma}[k] = \begin{bmatrix} \Gamma & 0 \\ KL_1C\Gamma & KL_1 \\ \vdots & \vdots \\ KL_MC\Gamma & KL_M \end{bmatrix} \tag{27}$$

Let $\tilde{X}[k] = E[\tilde{x}[k]\tilde{x}^T[k]]$. The difference equation for $\tilde{X}$ is the Lyapunov equation:

$$\tilde{X}[k+1] = \tilde{A}_{CL}[k]\tilde{X}[k]\tilde{A}_{CL}^T[k] + \tilde{\Gamma}[k]\begin{bmatrix} W & 0 \\ 0 & V\end{bmatrix}\tilde{\Gamma}^T[k] \tag{28}$$

This deterministic matrix equation is a function of the matrices $L_i[k+1]$ of each agent i and at all time k. Note that $spectrum(A+\sum_j B_jG_j) \subset spectrum(\tilde{A}_{CL}[k])$ for all k, so that the stability of $A + \sum_j B_jG_j$ is necessary for the stability of $\tilde{A}_{CL}[k]$ (see appendix for proof).

C. Periodic Optimal Control Formulation

With the notations introduced above, we can express the costs J_i as functions of $\tilde{X}$. Indeed, if

$$\tilde{X}_{00} = E[x[k]x^T[k]] \text{ and } \tilde{X}_{ij} = E[\hat{x}_i[k]\hat{x}_j^T[k]] \tag{29}$$

then

$$J_i = \lim_{N\to\infty} \frac{1}{N} E\left[\sum_{k=0}^{N-1}\Big(x[k]^TQ_ix[k] + u_i[k]^TR_iu_i[k]\Big)\right] \tag{30}$$

$$= \lim_{N\to\infty} \frac{1}{N} tr\left[\sum_{k=0}^{N-1}(Q_iE[x[k]x[k]^T] + \right.$$
$$\left. + G_i^TR_iG_iE[\hat{x}_i[k]\hat{x}_i[k]^T]\Big)\right] \tag{31}$$

$$= \lim_{N\to\infty} \frac{1}{N} tr\left[\sum_{k=0}^{N-1}(Q_i\tilde{X}_{00}[k] + G_i^TR_iG_i\tilde{X}_{ii}[k]\Big)\right] \tag{32}$$

We interpret the difference equation (28) as a controlled dynamical system with $\tilde{X}$ as the state, where the controls are the matrices $L_i[k]$. These matrices $L_i[k]$ are *assumed* periodic (data rate), but with an unknown period T. We optimize each J_i, and the optimization is performed on the controls as well as on the period. Since the L_i are assumed periodic with period T, so are $\tilde{A}_{CL}$ and $\tilde{\Gamma}$. Therefore, equation (28) has T-periodic coefficients.

We will now show that under certain conditions, this infinite-time optimal control problem is equivalent to a finite-time problem, which we will then be able to solve numerically. First, define the monodromy matrix Ψ for $\tilde{A}_{CL}$ as:

$$\Psi = \tilde{A}_{CL}[T-1] \times \ldots \times \tilde{A}_{CL}[0] \tag{33}$$

Claim: *If Ψ is stable, the limit in equation (32) can be rewritten as a finite sum:*

$$J_i = \lim_{N\to\infty} \frac{1}{N} tr\left[\sum_{k=0}^{N-1} \left(Q_i \tilde{X}_{00}[k] + G_i^T R_i G_i \tilde{X}_{ii}[k]\right)\right] \tag{34}$$

$$= \frac{1}{T} tr\left[\sum_{k=0}^{T-1} \left(Q_i \tilde{X}^*_{00}[k] + G_i^T R_i G_i \tilde{X}^*_{ii}[k]\right)\right] \tag{35}$$

where $\tilde{X}^*$ is the unique T-periodic solution of eq.(28).

Proof: It is a known result ([17]) that if monodromy matrix does not have any reciprocal eigenvalues, equation (28) has a unique T-periodic solution $\tilde{X}^*$. In particular, this is the case for a stable Ψ, since all eigenvalues will be inside the unit circle.

Next, define $\Delta\tilde{X}[k] = \tilde{X}[k] - \tilde{X}^*[k]$; then from eq.(28):

$$\Delta\tilde{X}[k+1] = \tilde{A}_{CL}[k]\, \Delta\tilde{X}[k]\, \tilde{A}^T_{CL}[k] \tag{36}$$

By immediate induction:

$$\Delta\tilde{X}[k+T] = \Psi\, \Delta\tilde{X}[k]\, \Psi \tag{37}$$

$$\Longrightarrow ||\Delta\tilde{X}[k+T]|| \leq ||\Psi||\, ||\Delta\tilde{X}[k]||\, ||\Psi|| \tag{38}$$

So if $||\Psi|| < 1$,

$$\Longrightarrow \lim_{k\to\infty} ||\tilde{X}[k] - \tilde{X}^*[k]|| = \lim_{k\to\infty} ||\Delta\tilde{X}[k]|| = 0 \tag{39}$$

This shows that although $\tilde{X}$ is not a periodic sequence, it converges, in the sense of eq.(39), to the periodic sequence $\tilde{X}^*$.

Now define the function $f_i : \mathbb{R}^{(M+1)n\times(M+1)n} \mapsto \mathbb{R}$ by

$$X \mapsto f_i(X) = tr(Q_i X_{00} + G_i^T R_i G_i X_{ii}) \tag{40}$$

where the subscripts on X represent, as before, the block selection, and the sequence

$$\tilde{Z}^*_i[k] = f_i(\tilde{X}^*[k]) \tag{41}$$

The following lemma (see appendix for proof), which is an extension of Cesaro's lemma to periodic sequences, will be of use:

Lemma 1 *Let $(u_n)_{n\in\mathbb{N}}$ be a T-periodic sequence, i.e., $\forall n \in \mathbb{N}, u_{n+T} = u_n$. Then:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} u_k = \frac{1}{T} \sum_{k=0}^{T-1} u_k = \mu \tag{42}$$

Since the sequence $\tilde{Z}^*_i$ is clearly T-periodic, we can conclude that:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \tilde{Z}^*_i[k] = \frac{1}{T} \sum_{k=0}^{T-1} \tilde{Z}^*_i[k] < +\infty \tag{43}$$

Note that the term on the right hand side of the equation above is exactly the term in eq.(35). To conclude the claim, we now present two lemmas, the proofs of which are in the appendix.

Lemma 2 *Let (u_n) and (v_n) be two sequences such that*

$$\lim_{n\to\infty} ||u_n - v_n|| = 0 \tag{44}$$

and let f be a given uniformly continuous *function. Then:*

$$\lim_{n\to\infty} ||f(u_n) - f(v_n)|| = 0 \tag{45}$$

Note: if f is continuous, but not uniformly, the result is no longer valid.

Lemma 3 *Let (u_n) and (v_n) be two sequences such that*

$$\lim_{n\to\infty} ||u_n - v_n|| = 0 \tag{46}$$

Then:

$$\lim_{N\to\infty} \frac{1}{N} \left[\sum_{n=0}^{N-1} ||u_n - v_n||\right] = 0 \tag{47}$$

From the previous two lemmas, we can conclude that if f is a *uniformly continuous* function and if

$$\lim_{n\to\infty} ||u_n - v_n|| = 0 \tag{48}$$

then:

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f(u_n) - \frac{1}{N} \sum_{n=0}^{N-1} f(v_n) \right\| = 0 \tag{49}$$

Indeed, we have that:

$$\left\| \frac{1}{N} \sum_{n=0}^{N-1} f(u_n) - \frac{1}{N} \sum_{n=0}^{N-1} f(v_n) \right\| \leq \frac{1}{N} \sum_{n=0}^{N-1} \left\| f(u_n) - f(v_n) \right\| \tag{50}$$

In particular, if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(v_n) < +\infty, \tag{51}$$

then we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(u_n) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(v_n) \tag{52}$$

We are now in a position to conclude, by noting that f_i is uniformly continuous, since it is linear, and that

$$J_i = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f_i(\tilde{X}[k]) \quad (53)$$

The stability condition on Ψ gives us eq.(39), which, combined with eq.(43), give us eq.(52):

$$J_i = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f_i(\tilde{X}[k]) \quad (54)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} Z_i^*[k] \quad (55)$$

$$= \frac{1}{T} \sum_{k=0}^{T-1} Z_i^*[k] \quad (56)$$

which is the claim. ■

Therefore, if we impose the monodromy matrix to be stable (which will guarantee a bounded closed-loop system), we can reformulate the infinite horizon optimal control problem as a finite horizon, periodic optimal control problem:

$$J_i^* = \min_{L_i} \frac{1}{T} tr \left[\sum_{k=0}^{T-1} (Q_i \tilde{X}_{00}^*[k] + G_i^T R_i G_i \tilde{X}_{ii}^*[k]) \right] \quad (57)$$

under the dynamics constraints:

$$\tilde{X^*}[k+1] = \tilde{A}_{CL}[k]\tilde{X^*}[k]\tilde{A}_{CL}^T[k] + \tilde{\Gamma}[k] \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \tilde{\Gamma}^T[k] \quad (58)$$

$$\tilde{X^*}[T] = \tilde{X^*}[0] \quad (59)$$

Note that the controls $L_i[k]$ enter the dynamics in the matrices $\tilde{A}_{CL}[k]$, and the minimization is over the set of controls such that Ψ is stable. First-order necessary conditions for optimality can be derived using notions of matrix maximum principle ([18]) and periodic optimal control ([19]), but are not presented here for clarity, since they do not lead to obvious closed-form solutions, and a global search for the Nash solution is performed numerically (see for example [20] for the computation of solutions to discrete periodic Lyapunov equations).

V. Example: String of Cars

We consider a string of M cars traveling is a straight line, as depicted in Fig. 1.

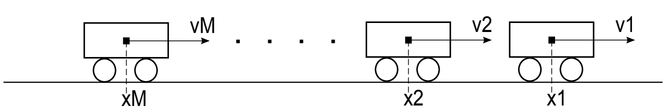


Fig. 1. *M car example*

We use a simplified dynamical model, in which the inertia is neglected, i.e., for each car's subsystem, the differential equation is:

$$\dot{x}_i(t) = u_i(t) \quad (60)$$

Assuming a nominal control $u_i^0(t)$ for each car, which keeps the separation between the cars to a desired value, we can linearize around these nominal trajectories, to obtain the differential equations for the deviation states:

$$\delta\dot{x}_i(t) = \delta u_i(t) \quad (61)$$

Finally, we consider the equivalent sampled-data system, with a sampling time T_s, to which process noise was added:

$$\delta x_i[k+1] = \delta x_i[k] + T_s \delta u_i[k] + \Gamma_i w[k] \quad (62)$$

The aggregate system is therefore

$$\delta x[k+1] = A\delta x[k] + \sum_i B_i \delta u_i[k] + \Gamma w[k] \quad (63)$$

$$\delta y[k] = C\delta x[k] + v[k] \quad (64)$$

where $A = I_M$, $B_i = T_s e_i$ (e_i being i-th vector of the standard canonical basis of $\mathbb{R}^N$), $C = I_M$, and $\Gamma = blkdiag(\Gamma_i, i = 1 \dots M)$.

In order to guarantee stability of the closed-loop system with full-information, we choose:

$$Q_1 = \epsilon e_i, \ \epsilon = 10^{-3} \quad (65)$$

$$Q_i = D_i^T D_i, \ i = 2 \dots M \quad (66)$$

where

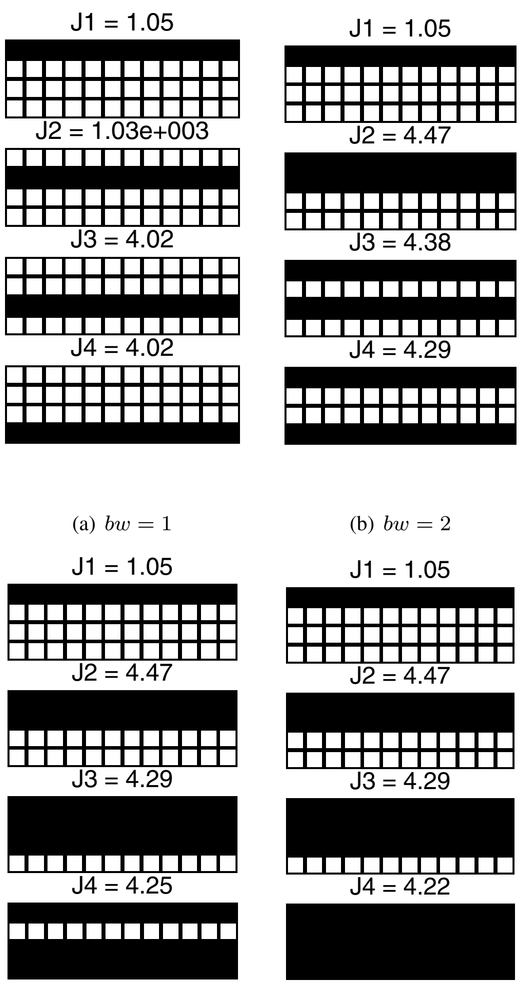
$$D_i = (e_{i-1} - e_i)^T$$

Finally, R_1 is picked equal to 10^2, and $R_i = 1, \ i = 2 \dots M$. The lead vehicle's weighing matrix Q_1 reflects the fact that the lead car only cares about stabilizing its own state. The other vehicles aim to minimize the separation error with the vehicle in front of them. The ratio ϵ/R_1 will have an influence on how aggressively the lead car will try to keep its state close to zero, at the expense of a high control. A large ratio will weight the norm of x_1 more heavily, so the lead car will track the zero more closely. The knowledge of this by the other agents will allow them to disregard the state of the lead car completely. By "trickle" effect down the string, each car will only need to focus on keeping its own state stable, which amounts to the decoupling of the systems, and does not require any exchange of information. We are more interested in the case where coupling is introduced via the costs. Therefore, we prefer to choose a small ratio, which will give the lead car more freedom to divert from the origin, and require exchange of information within the system.

With a slight abuse of terminology, we call "bandwidth" the maximum number of measurements allowed per time-step for an individual vehicle. Figs. 2 and 3 show the Nash measurement sequences over 12 time-steps, with time being represented along the horizontal axis. In each diagram representing the measurement sequence of an individual car, each line represents one coordinate of the total available measurement vector. In our example, we chose $C = I_M$, so there are M lines per diagram. A black cell signifies that the measurement corresponding to that line is available at the time-step corresponding to that particular column, while a white cell means this measurement is not available.

We first look at the case of time-invariant measurement schemes, i.e., we set $T = 1$, and we vary the upper bound on the bandwidth at each time-step. The results for a string of 4 cars are displayed in Fig. 2. We only show the schemes which use the least possible bandwidth, although making possibly more measurements does not deteriorate the performance. Fig. 2(a) shows that if only one measurement is allowed for all time, each car will measure its own position. However, in this case, the performance of the second car is very poor, since it cannot observe the state of the lead car, which directly affects its cost. Its only recourse is to rely on the knowledge that the lead car will keep its state near the origin, and bring its own state to the origin as well.

Fig. 2(b) shows that if two measurements are allowed at each time-step, the lead car will not improve by measuring any other state, while the other two cars will measure their own position, as well as the position of the lead car. This tells us that the position of the lead car is more important than, say, the position of the car right in front, even though the latter appears directly in the cost, while the former does not.



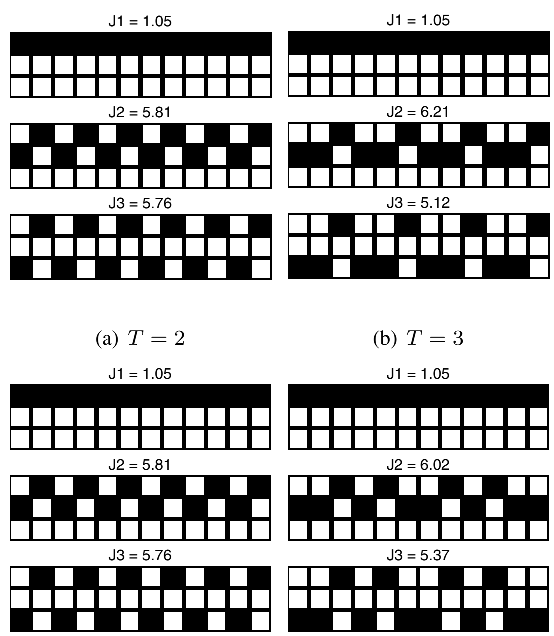
(a) $bw = 1$ (b) $bw = 2$

(c) $bw = 3$ (d) $bw = 4$

Fig. 2. *4 Cars, varying bandwidth, time-invariant*

This is a somewhat non-intuitive result. It was shown that this information pattern could yield string stability ([21]) under the appropriate control law; our methodology additionally provides a sense of optimality to this topology. It must be noted that the special role of the lead vehicle here is due to its matrix Q_1 rather that its position within the string. If yet another measurement is allowed per time-step, the position of the car right in front is selected next, as shown in Fig. 2(c). It appears from Fig. 2(d) that if more measurements are available, the cars will measure the states of the cars in front of them further and further up the string, until saturation of the bandwidth. It is also worth noting that the improvement in performance for each agent is marginal for a bandwidth greater than 2.

Next, we consider a string of 3 cars and vary the period, while limiting the bandwidth to 1. The results are shown in Fig. 3. Note that for every T-periodic solution, there are $T-1$ solutions, which are circular permutations of it, which yield the same costs; we only plot one of these equivalent solutions for clarity. We can observe that if allowed to switch measurements for different time-steps, the cars will replicate the invariant measurement sequence with a bandwidth of 2 over a period. Indeed, aside from the lead car, which only measures its own position, the other cars measure both their position and the position of the lead car. As shown in Figs. 3(a) and 3(c), when the time period is even, the two measurements are used alternatively, and equally frequently. It appears from Figs. 3(b) and 3(d) that, for each car, the measurement of its own position is slightly more important than that of the lead car. Indeed, when the time period is odd, each car measures its own position more often than the



(a) $T = 2$ (b) $T = 3$

(c) $T = 4$ (d) $T = 5$

Fig. 3. *3 Cars, varying period, bandwidth of 1*

position of the lead car. It may seem surprising that each car does not use all the available measurements, even when possible to do so over a period. This is probably due to the fact that measuring additional parts of the state only improves the performance slightly, if at all, as seen in the previous example.

VI. Conclusion

The methodology presented in this paper gives insight into the relative value of the different available measurements for each agent of a discrete time multi-agent linear system. The decision making is person-by-person optimal, which means each agents aims to minimize his own cost function, over an infinite time period. The problem was separated in two distinct steps: first, the Nash solution was solved centrally, assuming full information. The measurement selection strategies were then obtained as periodic optimal controls of a finite-horizon problem, which was shown to be equivalent to the original problem. The method assesses the relative importance of each measurement by gradually decreasing the available transmission bandwidth. Introducing time-periodicity for measurement selections helps to overcome very restrictive bandwidth limits, as well as interpret the solutions as data exchange rates in a straightforward way. The example presented illustrates the potential of the method to exhibit optimality properties of a given information pattern, such as the one assumed in [21]. Determining necessary and sufficient conditions for the existence of Nash solutions for the games formulated in both steps is the subject of current effort. Also, the measurement selections are solved numerically, and the dimension of the problem increases rapidly with the size of the measurement vector and the time-period, so implementing a search algorithm (such as [22]) to find the Nash solutions for larger problems would be useful.

Appendix

Proof: (**Lemma 1**) First, note that by clear induction:

$$\forall n \in \mathbb{N}, \frac{1}{T}\sum_{k=n}^{n+T-1} u_k = \mu \tag{67}$$

Now define the two following sequences:

$$\sigma_n = \frac{1}{n}\sum_{k=0}^{n-1} u_k, \;\; v_n = n\sigma_n - n\mu \tag{68}$$

We show that v_n is T-periodic:

$$\begin{aligned} v_n &= n\sigma_n - n\mu && (69)\\ &= \sum_{k=0}^{n-1} u_k - n\mu && (70)\\ v_{n+T} &= (n+T)\sigma_{n+T} - (n+T)\mu && (71)\\ &= \sum_{k=0}^{n+T-1} u_k - n\mu - T\mu && (72)\\ &= \sum_{k=0}^{n-1} u_k + \sum_{k=n}^{n+T-1} u_k - n\mu - T\mu && (73)\\ &= n\sigma_n + T\mu - n\mu - T\mu && (74)\\ &= n\sigma_n - n\mu = v_n && (75)\end{aligned}$$

Therefore, we have that:

$$\forall n \in \mathbb{N}\; m \leq v_n \leq M \tag{76}$$

where

$$m = \min(v_0, \dots, v_{T-1}),\;\; M = \max(v_0, \dots, v_{T-1}) \tag{77}$$

So:

$$\begin{aligned} m \leq v_n &\leq M && (78)\\ m \leq n\sigma_n - n\mu &\leq M && (79)\\ \frac{m}{n} \leq \sigma_n - \mu &\leq \frac{M}{n} && (80)\\ & && (81)\end{aligned}$$

Since both $\frac{m}{n}$ and $\frac{M}{n}$ converge to 0 as $n \to +\infty$, we get that

$$\sigma_n - \mu \to 0 \tag{82}$$

or equivalently:

$$\sigma_n \to \mu \tag{83}$$

■

Proof: (**Lemma 2**) By definition, f uniformly continuous means:

$$\forall \epsilon > 0, \exists \delta, \forall x, y, ||x-y|| \leq \delta \implies ||f(x)-f(y)|| \leq \epsilon \tag{84}$$

(As opposed to regular continuity, where the δ can be dependent on x, uniform continuity requires the existence of a single δ which works for all x and y.)
Since $\lim_{n\to\infty} ||u_n - v_n|| = 0$:

$$\exists N_0, \forall n \geq N_0, ||u_n - v_n|| \leq \delta \tag{85}$$

Therefore,

$$\forall \epsilon > 0, \exists N_0, \forall n \geq N_0 \; ||f(u_n) - f(v_n)|| \leq \epsilon \tag{86}$$

which is exactly the desired result. ■

Proof: (**Lemma 3**) Define the sequences

$$\begin{aligned} \alpha_n &= ||u_n - v_n|| && (87)\\ \beta_n &= 1 && (88)\end{aligned}$$

Then

$$\forall n \in \mathbb{N},\, 0 \leq \alpha_n = o(1) = o(\beta_n) \tag{89}$$

Call the sequences of finite sums:

$$s_N = \sum_{n=0}^{N-1} \alpha_n \tag{90}$$

$$\sigma_N = \sum_{n=0}^{N-1} \beta_n \tag{91}$$

Then it can be shown that:

- if $(s_N)_{N\in\mathbb{N}}$ converges, it is bounded, so that

$$\lim_{N\to\infty} \frac{1}{N} s_N = 0 \tag{92}$$

- if $(s_N)_{N\in\mathbb{N}}$ diverges, so does $(\sigma_N)_{N\in\mathbb{N}}$, and we have:

$$s_N = o(\sigma_N) = o(N) \tag{93}$$

$$\implies \lim_{N\to\infty} \frac{1}{N} s_N = 0 \tag{94}$$

■

Proof: **Fact:** $\sigma(A + \sum_j B_j G_j) \subset \sigma(\tilde{A}_{CL}[k])$ where $\sigma(\Lambda)$ denotes the set of eigenvalues of the matrix Λ. Define

$$T = \begin{bmatrix} I & 0 & \dots & 0 \\ I & \ddots & & 0 \\ \vdots & & I & \\ I & 0 & & \ddots \end{bmatrix} \implies T^{-1} = \begin{bmatrix} I & 0 & \dots & 0 \\ -I & \ddots & & 0 \\ \vdots & & I & \\ -I & 0 & & \ddots \end{bmatrix}$$

Then it is straightforward to see that:

$$T^{-1}\tilde{A}_{CL}T = \begin{bmatrix} A + \sum_j B_j G_j & B_1 G_1 & \dots & B_M G_M \\ 0 & \ddots & & \\ \vdots & & A_{err} & \\ 0 & & & \ddots \end{bmatrix} \tag{95}$$

where

$$A_{err} = \begin{bmatrix} KL_1 C \\ \vdots \\ KL_M C \end{bmatrix} \begin{bmatrix} B_1 G1 & \dots & B_M G_M \end{bmatrix} + \text{blkdiag}((I - KL_i C)(A + \sum_j B_j G_j)) \tag{96}$$

Eq.(95) proves the claim. ■

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