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Lie Group Projection Operator Approach: Optimal Control on $T\mathrm{SO}(3)$

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***Abstract*— The Lie group projection operator approach is an iterative scheme for solving continuous-time optimal control problems on Lie groups. This work details the approach for optimal control problems on $T\mathrm{SO}(3)$, the tangent bundle of the special orthogonal group $\mathrm{SO}(3)$. The dynamics of a rigid satellite is used as illustrative example. Numerical simulations are presented and discussed.**

I. Introduction

In [1], [2], the authors have introduced an algorithm for solving continuous time optimal control problems for systems evolving on Lie groups (including, as a particular case, the flat space $\mathbb{R}^n$). The approach borrows from and expand the key results of the projection operator approach for optimization of trajectory functionals developed in [3].

In [2], the authors have discussed the implementation details of the approach for solving optimization problems on SO(3). Here the natural extension of that work is presented, showing how to construct an iterative scheme for solving optimal control problems on $T\mathrm{SO}(3)$, the tangent bundle of SO(3). The dynamics of a rigid satellite [4] is used as illustrative example.

The reader might be interested in comparing our approach with the *discrete-time* methods presented in [5], [6] for *mechanical systems* evolving on Lie groups.

The paper is organized as follows. In Section II, we introduce the notation and review the projection operator approach for Lie groups. In Section III, we formulate the optimization problem reviewing the systems dynamics of a rigid satellite and introducing a specific cost functional. In Section IV, we compute the second order approximation of the optimization problem that defines the core of the Lie group projection operator approach. Simulation results are presented in Section V. Conclusions and future research directions are presented in Section VI.

Due to space limitations, all results are stated without proofs. They can be readily obtained upon request by contacting the authors.

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II. Preliminaries

We assume that the reader is familiar with the theory of finite dimensional smooth manifolds, covariant differentiation [7], [8], [9], and matrix Lie groups [10], [11].

A. Notations and definitions

M, N	smooth manifolds
TM, T^*M	tangent and cotangent bundles
$f : M \to N$	(smooth) mapping from M to N
$\mathbf{D}f : TM \to TN.$	tangent map of f
∇	affine connection
$\nabla_X Y$	covariant derivative of the vector field Y in the direction X
D_t	covariant differentiation w.r.t. t
$\gamma(t)$, $t \in I$	curve ($I \subset \mathbb{R}$)
$\boldsymbol{P}_\gamma^{t_1 \leftarrow t_0} V_0$	parallel displacement of V_0 along γ
$\mathbb{D}^2 f(x) \cdot (\mathrm{v}, \mathrm{w})$	second covariant derivative of f [1]
G	Lie group
$\mathfrak{g}$	Lie algebra of G
e	group identity
$L_g x$, $R_g x$	left and right translations
gx, xg,	shorthand notation for $L_g x$ and $R_g x$
$g\mathrm{v}_x$, $\mathrm{v}_x g$	shorthand notation for $\mathbf{D}L_g(x) \cdot \mathrm{v}_x$ and $\mathbf{D}R_g(x) \cdot \mathrm{v}_x$
$[\cdot, \cdot]$	Lie bracket operation
Ad_g	adjoint representation of G on $\mathfrak{g}$
ad_ϱ	adjoint representation of $\mathfrak{g}$ onto itself ($\mathrm{ad}_\varrho \varsigma = [\varrho, \varsigma]$)
$\exp : \mathfrak{g} \to G$	exponential map
$\log : G \to \mathfrak{g}$	logarithm map (i.e., inverse of $\exp$ in a neighborhood of e)
$\mathrm{SO}(3)$	special orthogonal group
$\mathfrak{so}(3)$	Lie algebra of SO(3)
$\mathbb{R}^3_\times$	Lie algebra given by $\mathbb{R}^3$ with cross product as Lie bracket
$\wedge : \mathbb{R}^3_\times \mapsto \mathfrak{so}(3)$	Lie algebra isomorphism $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^\wedge \mapsto \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$
$\vee : \mathfrak{so}(3) \mapsto \mathbb{R}^3_\times$	inverse of $\wedge$

B. The Lie group projection operator approach

The projection operator approach for the optimization of trajectory functionals [3] is an iterative scheme to solve continuous-time nonlinear optimal control problems. In [1], the authors have shown how the approach can be generalized to work with a dynamical system defined on a Lie group G, that is, for a system in the form

$$\dot{g} = f(g,u,t) = g(t)\lambda(g(t),u(t),t)\,, \tag{1}$$

where $f : G \times \mathbb{R}^m \times \mathbb{R} \to TG$ is a control system on G and $\lambda : G \times \mathbb{R}^m \times \mathbb{R} \to \mathfrak{g}$, $\lambda(g,u,t) := g^{-1}f(g,u,t)$, is its *left-trivialization*. The approach, in its simplest formulation, can handle optimal control problems in the form

$$\min_{(g(\cdot),u(\cdot))} \int_0^T l(g(\tau),u(\tau),\tau)\,d\tau + m(g(T)) \tag{2}$$

subject to

$$\dot{g} = f(g,u,t), \tag{3}$$
$$g(0) = g_0, \tag{4}$$

with $l : G \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ the incremental cost, $m : G \to \mathbb{R}$ the terminal cost, and g_0 the initial condition. Modifications of the strategy for handling a terminal condition and mixed input-state constraints (through a barrier functional approach) are discussed, for control problems on $\mathbb{R}^n$, in [12] and [13].

From an abstract point of view, the optimization scheme can be thought as a constrained Newton method in infinite dimension. The approach is based on (and derives its name from) the projection operator $\mathcal{P}$ [3], which is an operator that maps a generic curve $\xi(t) = (\alpha(t),\mu(t)) \in G \times \mathbb{R}^m$, $t > 0$, into a trajectory $\eta(t) = (g(t),u(t)) \in G \times \mathbb{R}^m$, $t > 0$, of the system (1). It is defined through the feedback system

$$\begin{aligned} \dot{g}(t) &= g(t)\lambda(g(t),u(t),t)\,, \quad g(0) = \alpha(0)\,, \\ u(t) &= \mu(t) + K(t)\big[\log(g(t)^{-1}\alpha(t))\big]\,, \end{aligned} \tag{5}$$

where $K(t) : \mathfrak{g} \to \mathbb{R}^m$ is a linear map, which can be thought as a standard linear feedback as soon as a basis is chosen for the Lie algebra $\mathfrak{g}$. It is straightforward to verify that $\mathcal{P}$ is indeed a projection, i.e., it satisfies $\mathcal{P}^2 := \mathcal{P} \circ \mathcal{P} = \mathcal{P}$. The projection operator $\mathcal{P}$ was used in [14] to show that the set of exponentially stabilizable trajectories $\mathcal{T}$ (of infinite extent) forms an infinite dimensional manifold, a fact that allows us to use vector space operations [15] to effectively explore it.

Given a *trajectory* $\xi(t) = (g(t),u(t))$ of the control system (1), its (left-trivialized) linearization is defined as the time-varying linear system

$$\dot{z}(t) = A(\xi(t),t)z(t) + B(\xi(t),t)v(t), \tag{6}$$

with $(z(t),v(t)) \in \mathfrak{g} \times \mathbb{R}^m$, $t \geq 0$ and where

$$A(\xi,t) := \mathbf{D}_1\lambda(g,u,t) \circ T_eL_g - \mathrm{ad}_{\lambda(g,u,t)}, \tag{7}$$
$$B(\xi,t) := \mathbf{D}_2\lambda(g,u,t). \tag{8}$$

From a geometric point of view, a trajectory $\zeta(t) = (z(t),v(t))$, $t \geq 0$, of the (left-trivialized) linearization about ξ should be regarded as a (left-trivialized) tangent vector to the trajectory manifold $\mathcal{T}$, a fact that is indicated as $\xi\zeta \in T_\xi\mathcal{T}$ [1].

The projection operator approach consists in applying the following iterative method

Algorithm (Projection operator Newton method)
given initial trajectory $\xi_0 \in \mathcal{T}$
for $i = 0, 1, 2, \ldots$

(search direction)
$$\zeta_i = \arg\min_{\xi_i\zeta \in T_{\xi_i}\mathcal{T}} \mathbf{D}h(\xi_i)\cdot\xi_i\zeta + \tfrac{1}{2}\,\mathbb{D}^2\tilde{h}(\xi_i)\cdot(\xi_i\zeta,\xi_i\zeta) \tag{9}$$
$$\gamma_i = \arg\min_{\gamma\in(0,1]} \tilde{h}(\xi_i\exp(\gamma\zeta_i)) \qquad \text{(step size)} \tag{10}$$
$$\xi_{i+1} = \mathcal{P}(\xi_i\exp(\gamma_i\zeta_i)) \qquad \text{(update)} \tag{11}$$

end

In (9), h is the cost functional appearing in (2) and $\tilde{h}$ is the functional obtained by composing h with the projection operator $\mathcal{P}$, i.e., $\tilde{h} := h\circ\mathcal{P}$. At each iterate, the search direction minimization (9) is performed on the tangent space $T_\xi\mathcal{T}$, that is, we search over the curves $\zeta(\cdot) = (z(\cdot),v(\cdot))$ that satisfies (6). Then, the step size subproblem (10) is considered. The classical *approximate* solution obtained using backtracking line search with Armijo condition [16, Chapter 3] can be used to compute the optimal step size γ_i. Finally, the update step (11) *projects* each iterate on to the trajectory manifold and the process restarts as long as termination conditions have not been met.

As explained in [1], the derivation of the above Newton algorithm on a generic Lie group has required the use of *covariant differentiation of mappings*, indicated with the symbol $\mathbb{D}$. (In fact, in [1], the covariant derivative is called "geometric derivative" as we were unaware at the time that the operator $\mathbb{D}$ should be interpreted as the covariant derivative of a two-point tensor [17]). This is related to the problem of constructing the Taylor-like expansion of a function between two smooth manifolds M_1 and M_2, each endowed with an affine connection.

III. System dynamics and cost functional

The rotational dynamics of a rigid satellite with m gas jet actuators [4], $m \in \{0,1,3\}$, can be written as

$$\dot{R} = R\hat{\omega} \qquad R \in \mathrm{SO}(3) \tag{12}$$
$$\mathbb{I}\omega = (\mathbb{I}\omega)\times\omega + Cu, \qquad \omega \in \mathbb{R}^3 \tag{13}$$

where R is the rotational matrix expressing the attitude of the satellite relative to an inertial frame, ω the angular velocity in body frame, $\mathbb{I}$ the inertia matrix of the spacecraft in body coordinates, and $C \in \mathbb{R}^{3\times m}$ a matrix whose columns represent the axis about which the m control torques $u \in \mathbb{R}^m$ are applied by means of opposing pairs of gas jets.

Given $R \in \mathrm{SO}(3)$, let $\|I-R\|^2_{\bar{P}} := \mathrm{tr}((I-R)^T\bar{P}(I-R))$. Indicating with $g = (R,\omega) \in SO(3)\times\mathbb{R}^3$ the system state, we derive the expressions for the projection operator Newton method for the case where the incremental cost $l : (\mathrm{SO}(3)\times\mathbb{R}^3)\times\mathbb{R}^m\times\mathbb{R} \to \mathbb{R}$ and terminal cost $m : \mathrm{SO}(3)\times\mathbb{R}^3 \to \mathbb{R}$ are given by

$$\begin{aligned} l(g,u,\tau) := &\frac{1}{2}\|I - R_d^T(\tau)R\|^2_{\mathbf{Q}^R} \\ &+ \frac{1}{2}\|\omega_d(\tau)-\omega\|^2_{\mathbf{Q}^\omega} + \frac{1}{2}\|u_d(\tau)-u\|^2_{\mathbf{R}}, \end{aligned} \tag{14}$$

and

$$m(g) := \frac{1}{2}\|I - R_f^TR\|^2_{\mathbf{P}_f^R} + \frac{1}{2}\|\omega_f-\omega\|^2_{\mathbf{P}_f^\omega}. \tag{15}$$

where $((R_d, \omega_d), u_d)(t)$, $t \geq 0$ is a desired state-control curve, $(R_f, \omega_f) \in \mathrm{SO}(3) \times \mathbb{R}^3$ a reference terminal state, $\bar{\mathbf{Q}}^R$, $\mathbf{Q}^\omega$, $\bar{\mathbf{P}}_f^R$, and $\mathbf{P}_f^\omega$ positive semi-definite state weighting matrices and $\mathbf{R}$ a positive define control weighting matrix.

These weighing matrices allow to control, similarly to the standard LQ regulator, the trade off between the speed of convergence to the desired curve $(R_d(t), \omega_d(t))$, $t \in [0, T]$ and the amount of control u used.

Note the use of boldface to distinguish between the configuration matrix R and the control weight matrix $\mathbf{R}$.

IV. The Quadratic Approximation of the Optimal Control Problem

The rigid satellite state space $T\mathrm{SO}(3)$ can be made into a Lie group in different ways. In this section, we propose and discuss one particular choice of group structure and detail how to compute all the basic expressions that define the Lie group projection operator approach. In particular, we provide explicit expressions for the left-trivialized linearization and the quadratic approximation of $\tilde{h}$, the functional obtained from the composition of the cost functional h, given in (2), with the projection operator $\mathcal{P}$, given in (5).

A. The Lie group $\mathrm{SO}(3) \times \mathbb{R}^3$ and its Lie algebra

The dynamics (12)-(13) is defined on $T\mathrm{SO}(3)$, which can be identified with $\mathrm{SO}(3) \times \mathbb{R}^3$ via left translation. By choosing the operation

$$(R, \delta R) \cdot (S, \delta S) = (RS, RS(S^T \delta S + R^T \delta R)), \qquad (16)$$

with $(R, \delta R)$ and $(S, \delta S) \in T\mathrm{SO}(3)$, the manifold $T\mathrm{SO}(3)$ can be made into a Lie group. Defining $\omega = (R^T \delta R)^\vee$ and $\nu = (S^T \delta S)^\vee$, the above operation on $\mathrm{SO}(3) \times \mathbb{R}^3$ simply reads

$$(R, \omega) \cdot (S, \nu) = (RS, \nu + \omega). \qquad (17)$$

The unit element of $\mathrm{SO}(3) \times \mathbb{R}^3$ is $e = (I, 0)$ and for each $(R, \omega) \in \mathrm{SO}(3) \times \mathbb{R}^3$ its inverse is $(R^T, -\omega)$.

The left translation on $\mathrm{SO}(3) \times \mathbb{R}^3$ is

$$L_{(R,\omega)}(S, \nu) = (RS, \nu + \omega) \qquad (18)$$

with tangent map

$$T_{(S,\nu)} L_{(R,\omega)}(\delta S, \delta \nu) = (R\, \delta S, \delta \nu), \qquad (19)$$

which we also write in short form as

$$(R, \omega)(\delta S, \delta \nu) = (R\, \delta S, \delta \nu) \in T(SO(3) \times \mathbb{R}^3). \qquad (20)$$

Similar expressions hold for the right translation. A faithful representation of the matrix group $SO(3) \times \mathbb{R}^3$ is

$$\begin{bmatrix} R & 0 & 0 \\ 0 & I & \omega \\ 0 & 0 & 1 \end{bmatrix}. \qquad (21)$$

Clearly, the Lie algebra of $\mathrm{SO}(3) \times \mathbb{R}^3$ is $\mathfrak{so}(3) \times \mathbb{R}^3$ which we further identify with $\mathbb{R}^3_\times \times \mathbb{R}^3$ (see Section II for the definition of $\mathbb{R}^3_\times$). A generic Lie algebra element $(z^R, z^\omega) \in \mathbb{R}^3_\times \times \mathbb{R}^3$ has matrix representation given by

$$\begin{bmatrix} \widehat{z^R} & 0 & 0 \\ 0 & 0 & z^\omega \\ 0 & 0 & 0 \end{bmatrix} \qquad (22)$$

from which one can show that the exponential map $\widetilde{\exp} : \mathbb{R}^3_\times \times \mathbb{R}^3 \to \mathrm{SO}(3) \times \mathbb{R}^3$ is given by

$$\widetilde{\exp}(z^R, z^\omega) = (\exp(\widehat{z^R}), z^\omega) \in \mathrm{SO}(3) \times \mathbb{R}^3 \qquad (23)$$

with $\exp$ the standard exponential from $\mathfrak{so}(3) \to \mathrm{SO}(3)$.

Given $(R, \omega) \in \mathrm{SO}(3) \times \mathbb{R}^3$ and $(z^R, z^\omega) \in \mathbb{R}^3_\times \times \mathbb{R}^3$, the adjoint representation of the group into its Lie algebra is

$$\mathrm{Ad}_{(R,\omega)}(z^R, z^\omega) = (R z^R, z^\omega) \qquad (24)$$

The above expression is obtained differentiating the inner automorphism $I_g(h) = g h g^{-1}$ with respect to h at the identity. Given (z_1^R, z_1^ω) and $(z_2^R, z_2^\omega) \in \mathbb{R}^3_\times \times \mathbb{R}^3$, the adjoint representation of the Lie algebra $\mathbb{R}^3_\times \times \mathbb{R}^3$ onto itself is

$$\mathrm{ad}_{(z_1^R, z_1^\omega)}(z_2^R, z_2^\omega) = (\widehat{z_1^R} z_2^R, 0). \qquad (25)$$

The above expression can be obtained computing the commutator of the matrix representations of the two elements of the Lie algebra or by differentiating the Ad operator at the identity.

B. The left-trivialized dynamics and linearization

Recall that given the control system $\dot{g} = f(g, u, t)$, with $g \in G$, a Lie group, and $u \in \mathbb{R}^m$, the left-trivialization of f is defined as $\lambda(g, u, t) := g^{-1} f(g, u, t)$. From (12)-(13) and (19), we obtain

$$\begin{aligned} \lambda((R, \omega), u, t) &= (R^T, -\omega)(R\hat{\omega}, \mathbb{I}^{-1}(\mathbb{I}\omega \times \omega + Cu)) \\ &= (\omega, \mathbb{I}^{-1}\widehat{\mathbb{I}\omega}\,\omega + \mathbb{I}^{-1} C u) \in \mathbb{R}^3_\times \times \mathbb{R}^3. \end{aligned} \qquad (26)$$

Given $x \in \mathbb{R}^3$, define

$$H(x) := \mathbb{I}^{-1}(\widehat{\mathbb{I}x} - \widehat{x}\mathbb{I}). \qquad (27)$$

Note that H is a symmetric operator in the sense that $H(x)y = H(y)x$, $\forall x, y \in \mathbb{R}^3$. We can now state the following result

Proposition 4.1: The *left-trivialized linearization* of the control system (12)-(13) is given by

$$A(\xi, t) = \begin{bmatrix} -\widehat{\omega} & I \\ 0 & H(\omega) \end{bmatrix} \quad \text{and} \quad B(\xi, t) = \begin{bmatrix} 0 \\ \mathbb{I}^{-1} C \end{bmatrix} \qquad (28)$$

with $\xi = (g, u) = ((R, \omega), u) \in \mathrm{SO}(3) \times \mathbb{R}^m$ and $H(\omega)$ as in (27).

C. Second order approximation of the cost functional

The search direction subproblem (9) consists in minimizing the functional $\mathbf{D}h(\xi) \cdot \xi\zeta + \frac{1}{2}\mathbb{D}^2\tilde{h}(\xi) \cdot (\xi\zeta, \xi\zeta)$ over the set of trajectories of the left-trivialized linearization

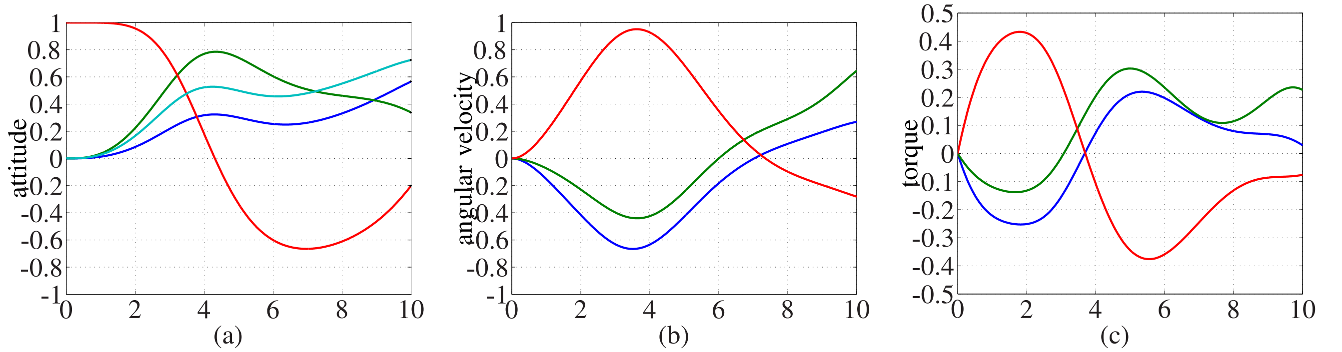

Fig. 1. Desired trajectory $\xi_d(t) = ((R_d(t), \omega_d(t)), u_d(t)) \in (\mathrm{SO}(3) \times \mathbb{R}^3) \times \mathbb{R}^3$, $t \in [0, 10]$. Part (a) shows the unit quaternion representation of the desired attitude $R_d(\cdot)$; Part (b) shows the desired body angular velocity $\omega_d(\cdot)$; Part(c) shows the desired input $u_d(\cdot)$.

of (12)-(13) about ξ. As shown in [2], this optimal control problem can be written in matrix form as

$$\min_{(z,v)(\cdot)} \int_0^T a(\tau)^T z(\tau) + b(\tau)^T v(\tau) + \frac{1}{2}\begin{bmatrix} z(\tau) \\ v(\tau) \end{bmatrix}^T W(\tau) \begin{bmatrix} z(\tau) \\ v(\tau) \end{bmatrix} dt$$
$$+ a_1^T z(T) + \frac{1}{2} z(T)^T P_1 z(T), \tag{29}$$

subject to the dynamic constraint

$$\dot{z}(t) = A(\xi(t), t) z(t) + B(\xi(t), t) v(t) \tag{30}$$
$$z(0) = 0. \tag{31}$$

where (30) is the left-trivialized linearization given in (6)–(7), $a^T(t)$, $b^T(t)$, a_1, P_1, satisfy, respectively,

$$\begin{aligned} a^T(t) z &= \mathbf{D}_1 l(g(t), u(t), t) \cdot g(t) z, \\ b^T(t) v &= \mathbf{D}_2 l(g(t), u(t), t) \cdot v, \\ a_1^T z &= \mathbf{D} m(g(T)) \cdot g(t) z, \\ z^T P_1 z &= \mathbb{D}^2 m(g(T)) \cdot (g(T) z, g(T) z). \end{aligned}$$

The matrix $W(t)$ is symmetric and its explicit expression will be given, for the specific dynamics of our interest, in the sequel.

Proposition 4.2: Given the incremental cost (14) and terminal cost (15), $a(t)$, $b(t)$, a_1 and P_1 are

$$\begin{aligned} a^T(t) &= [2 q_v^T(t) \mathbf{Q}^R (q_s(t) I + \hat{q}_v(t)), (\omega(t) - \omega_d(t))^T \mathbf{Q}^\omega] \\ b^T(t) &= (u(t) - u_d(t))^T \mathbf{R} \\ a_1^T &= [2 \kappa_v^T \mathbf{P}_f^R (\kappa_s I + \hat{\kappa}_v), (\omega(T) - \omega_1)^T \mathbf{P}_f^\omega] \\ P_1 &= \begin{bmatrix} (\kappa_s I + \hat{\kappa}_v)^T \mathbf{P}_f^R (\kappa_s I + \hat{\kappa}_v) - (\kappa_v^T \mathbf{P}_f^R \kappa_v) I & 0 \\ 0 & \mathbf{P}_f^\omega \end{bmatrix} \end{aligned}$$

where $(q_s, q_v) \in \mathbb{R} \times \mathbb{R}^3$ denotes one of the two unit quaternions associated to the rotational matrix $R_d(t)^T R(t)$, $(\kappa_s, \kappa_v) \in \mathbb{R} \times \mathbb{R}^3$ is one of the two unit quaternions associated to the rotational matrix $R_1^T R(T)$, $\mathbf{Q}^R := \mathrm{tr}(\bar{\mathbf{Q}}^R) I - \bar{\mathbf{Q}}^R$, and $\mathbf{P}_f^R := \mathrm{tr}(\bar{\mathbf{P}}_f^R) I - \bar{\mathbf{P}}_f^R$

Before stating the main result concerning the explicit expression of the matrix W, we state two technical lemmas.

Lemma 4.3: The second covariant derivative of the incremental cost l, given in (14), about $\xi = (g, u) = ((R, \omega), u) \in$

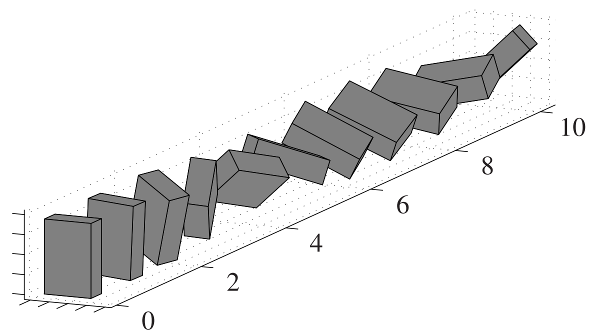

Fig. 2. Snapshots of the desired attitude trajectory $R_d(t)$, $t \in [0, 10]$.

$(\mathrm{SO}(3) \times \mathbb{R}^3) \times \mathbb{R}^m$, is

$$\mathbb{D}^2 l_\tau(\xi) \cdot (\xi\zeta_1, \xi\zeta_2) = \begin{bmatrix} z_1^R \\ z_1^\omega \\ v_1 \end{bmatrix}^T \begin{bmatrix} M(\tau) & 0 & 0 \\ 0 & \mathbf{Q}^\omega & 0 \\ 0 & 0 & \mathbf{R} \end{bmatrix} \begin{bmatrix} z_2^R \\ z_2^\omega \\ v_2 \end{bmatrix} \tag{32}$$

where $l_\tau(\xi) := l(\xi, \tau)$,

$$\xi\zeta_i = ((R, \omega), u)((z_i^R, z_i^\omega), v_i), \qquad i = \{1, 2\},$$

with $((z_i^R, z_i^\omega), v_i) \in (\mathbb{R}^3_\times \times \mathbb{R}^3) \times \mathbb{R}^m$, $i \in \{1, 2\}$, and

$$M(\tau) := (q_s I + \widehat{q}_v)^T \mathbf{Q}^R (q_s I + \widehat{q}_v) - (q_v^T \mathbf{Q}^R q_v) I, \tag{33}$$

with $q = (q_s, q_v) \in \mathbb{R} \times \mathbb{R}^3$ one of the two unit quaternions corresponding to the rotation matrix $R_d^T(\tau) R(\tau)$ and $\mathbf{Q}^R := tr(\bar{\mathbf{Q}}^R) I - \bar{\mathbf{Q}}^R$.

Lemma 4.4: The second covariant derivative of the left trivialized vector field λ, given in (26), is given by

$$\mathbb{D}^2 \lambda_\tau(\xi) \cdot (\xi\zeta_1, \xi\zeta_2) = \Big(0, H(z_2^\omega) z_1^\omega\Big), \tag{34}$$

where $\xi = (g, u) = ((R, \omega), u) \in (\mathrm{SO}(3) \times \mathbb{R}^3) \times \mathbb{R}^m$, $\lambda_\tau(g, u) := \lambda(g, u, \tau)$, $\zeta_i = (z_i, v_i) = ((z_i^R, z_i^\omega), v_i) \in (\mathbb{R}^3_\times \times \mathbb{R}^3) \times \mathbb{R}^m$, $i = \{1, 2\}$, and H as in (27).

We are ready to state the main result of this section.

Proposition 4.5: The matrix $W(\tau)$, appearing in (29), can

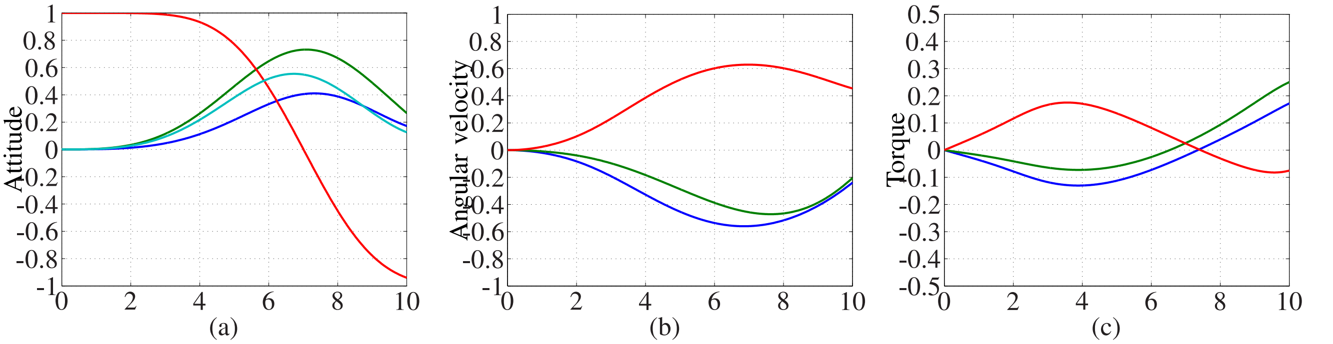

Fig. 3. Initial trajectory $\xi_0(t) = ((R_0(t), \omega_0(t)), u_0(t)) \in (\mathrm{SO}(3) \times \mathbb{R}^3) \times \mathbb{R}^3$, $t \in [0, 10]$. Part (a) shows the unit quaternion representation of the desired attitude $R_d(\cdot)$; Part (b) shows the desired body angular velocity $\omega_d(\cdot)$; Part (c) shows the desired input $u_d(\cdot)$.

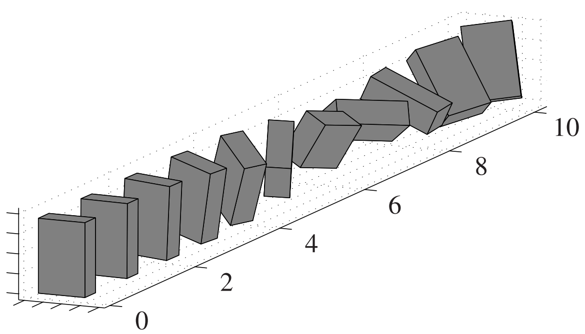

Fig. 4. Snapshots of the initial attitude trajectory $R_0(t)$, $t \in [0, 10]$.

be written as

$$\begin{bmatrix} M(\tau) & -1/2\widehat{p^R} & 0 \\ 1/2\widehat{p^R} & \mathbf{Q}^\omega + E(p^\omega) & 0 \\ 0 & 0 & R \end{bmatrix} \tag{35}$$

where $(q_s, q_v) \in \mathbb{R} \times \mathbb{R}^3$ is one of the two unit quaternions associated to the rotational matrix $R_d(\tau)^T R(\tau)$, $p = (p^R, p^\omega) \in \mathbb{R}^6$ (more correctly, $p \in (\mathbb{R}^3_\times \times \mathbb{R}^3)^*$) is the solution at time τ of the *stabilized adjoint* equation

$$-\dot{p}(t) = A_{cl}(t)^T p(t) + a(t) - K(t)^T b(t), \tag{36}$$

$$p(T) = a_1, \tag{37}$$

where $A_{cl}(t) := A(\xi(t), t) - B(\xi(t), t)\, K(t)$, with $A(\xi(t), t)$ and $B(\xi(t), t)$ the left-trivialized linearization (28), $K(t)$ the feedback that defines the projection operator, $a(t)$, $b(t)$ and a_1 as in Proposition 4.2.

V. Numerical results

This section demonstrates the effectiveness of the Lie group projection operator approach on $T\mathrm{SO}(3)$ for solving the optimal control problem defined in Section III for the fully actuated case. We choose as desired curve $\xi_d(t) = ((R_d(t), u_d(t)), u_d(t)) \in (\mathrm{SO}(3) \times \mathbb{R}^3) \times \mathbb{R}^3$, $t \geq [0, T]$, a nontrivial trajectory of the system. This choice has the advantage that we can check easily if the optimization scheme is converging to the solution of the optimization problem which is clearly the trajectory ξ_d itself.

In (12)–(13), we set $\mathbb{I} = \mathrm{diag}(0.9, 1.0, 1.1)$, $C = I$, and the initial condition $R(0) = \mathrm{diag}(-1, 1, -1)$ and $\omega(0) = (0, 0, 0)^T$. The optimization horizon is $T = 10$. The desired curve $\xi_d(t) = ((R_d(t), \omega_d(t)), u_d(t)) \in (\mathrm{SO}(3) \times \mathbb{R}^3) \times \mathbb{R}^3$, is the trajectory of (12)–(13) obtained by projecting through (5) the curve $(\alpha(t), \mu(t)) = ((S(t), 0), 0) \in (\mathrm{SO}(3) \times \mathbb{R}^3) \times \mathbb{R}^3$, $t \in [0, 10]$, where the rotational matrix $S(t)$ corresponds to the unit quaternion $q(t)/\|q(t)\|$ with $q(t) = (\sin(0.1t), \sin(0.3t), \cos(0.5t), \sin(0.2t))$ and the feedback $K(t)$ is identically equal the constant matrix

$$K_0 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

The trajectory $\xi_d(t)$, $t \in [0, 10]$, is shown in Figure 1. Figure 2 provides a visual representation of the desired attitude matrix R_d using a rectangular box. The width and height of the box (corresponding to the y and z body axes, respectively) are two and three times the depth (the x body axis). The box is centered at the point $(t, 0, 0)$ and 11 snapshots in the interval $t \in [0, 10]$ (one for each second) are shown. The optimization algorithm is started with the initial trajectory ξ_0 which is obtained similarly to ξ_d but with a (constant) feedback set to $0.1K_0$. The resulting trajectory $\xi_0 = ((R_0(t), \omega_0(t)), u_0(t))$ is shown in Figure 3. Note that ξ_0 is quite different from ξ_d (not at $t = 0$, where they are clearly equal). A visual representation of ξ_0 is given in Figure 4. The weighting matrices in the incremental cost (14) are $\bar{\mathbf{Q}}^R = 1/2\mathrm{tr}(\mathbf{Q}^R) - \mathbf{Q}^R$, with $\mathbf{Q}^R = \mathrm{diag}(10.0, 10.0, 10.0)$, $\mathbf{Q}^\omega = \mathrm{diag}(10.0, 10.0, 10.0)$, and $\mathbf{R} = \mathrm{diag}(1.0, 1.0, 1.0)$. The weighing matrices in the terminal cost (15) are given by $\bar{\mathbf{P}}^R = 1/2\mathrm{tr}(\mathbf{P}^R) - \mathbf{P}^R$, with $\mathbf{P}^R = \mathrm{diag}(13.0, 13.0, 13.0)$, and $\mathbf{P}^\omega = \mathrm{diag}(4.0, 4.0, 4.0)$ and the state penalty is "centered" at $R_f = R_d(T)$ and $\omega_f = \omega_d(T)$.

At each iteration of the optimization algorithm (9)–(11), the feedback K defining the projection operator $\mathcal{P}$ is redesigned solving a standard LQR problem with linear dynamics given by the transverse linearization about the current trajectory ξ and constant weighting matrices for the state and control identically equal to the identity. The Newton algorithm (9)–(11) converges to the the optimal solution

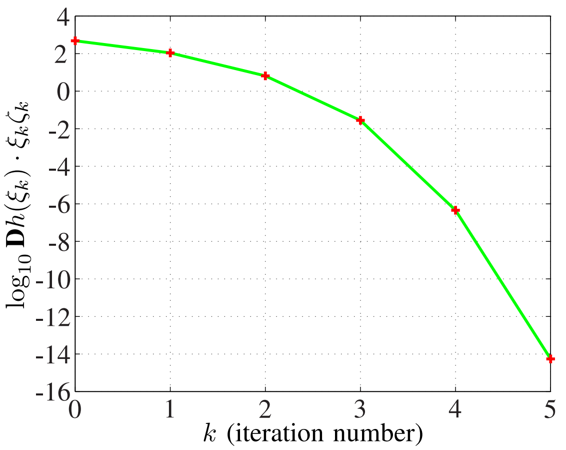

Fig. 5. Descent. The plot shows $\log_{10} -\mathbf{D}h(\xi_k)\cdot\xi_k\zeta_k \approx \log_{10}(h(\xi_k) - h(\xi_{k+1}))$ as a function of the number of iterations.

$\xi^* = \xi_d$ with quadratic convergence rate as one can see in Figure 5. To integrate the differential equations required at each iteration of the algorithm, we used the function `ode45` of Mathworks Matlab, storing all the trajectories with a sampling period of 0.01s. The absolute and relative tolerances of the ODE solver is set to 10^{-12} and 10^{-12}, respectively. The termination condition is $-\mathbf{D}h(\xi_k)\cdot\xi_k\zeta_k \approx h(\xi_k) - h(\xi_{k+1}) \leq 10^{-10}$. The algorithm takes about 2.3 seconds to run on a laptop equipped with a Intel Core 2 Duo CPU P8600 2.40 GHz. (Reducing the error tolerance for the ODE solver to standard values and setting the termination condition to 10^{-7} one can easily reach an execution time way below one second.) The algorithm is coded as a main m-function which calls a series of S-functions written in `C`.

VI. Conclusions

In this paper we have detailed how to construct the quadratic approximation of an optimal control problem on $T\mathrm{SO}(3)$. Numerical simulations have been presented to show the effectiveness of the method and the quadratic convergence rate.

The second covariant derivative of the left trivialized dynamics λ of the rigid satellite has a relatively simple expression due to the fact that λ does not depend on the configuration R. Our long term goal is to address constrained dynamical path planning problems for aerial and marine vehicles, for which the general expression is required. The derivation we have presented is sufficiently detailed that should allow the interested reader to derive the expression for the second covariant derivative in the general case.

This work has shown that the choice of the Lie group structure given to the tangent bundle $TSO(3)$ is important. Initially, we had chosen the operation

$$\begin{aligned}(R,\delta R)\cdot(S,\delta S) &= (RS, R\delta S + \delta R S)\\ &= (RS, RS(S^T\delta S + \mathrm{Ad}_{S^{-1}} R^T\delta R))\end{aligned}$$

obtained by differentiation of the standard operation on $\mathrm{SO}(3)$. This choice make $T\mathrm{SO}(3)$ into what is commonly called the *tangent group* of $SO(3)$. Initially, we thought that this was the best group structure to choose but direct computations have shown that, in terms of complexity of the expressions obtained for the Lie group projection operator approach, the one which has been presented in the paper is much more simpler. This leaves open the question of how to choose the "best" group structure given an optimal control problem defined on the tangent space of a Lie group.

Rigid satellite dynamics has attracted the attention of the control community in particular for the problem of controllability and stabilizability in the underactuated case [4], [18], [19]. As the linearization of the dynamics of the underactuated rigid satellite about a constant trajectory is not controllable, some care is to be taken in trying to solve numerically, e.g., an optimal stabilization problem. We have done some simulation in this context with the Projection Operator approach, but further investigation is needed to fully understand the obtained results.

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