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Robust Control of Nonlinear Strict-feedback Systems with Measurement Errors

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Abstract— **This paper presents a new method for robust control of a class of uncertain nonlinear systems in strict-feedback form with state measurement errors. The measurement feedback control problem is solved by recursively designing input-to-state stability (ISS) induced nonlinear state estimators and virtual control laws. With the gain assignment technique, the closed-loop system can be transformed into an interconnection of ISS subsystems, the ISS and input-to-output stability (IOS) of which can be guaranteed by the cyclic-small-gain theorem. Moreover, the IOS gain from the measurement error of the system output (the first state) to the system output can be designed to be linear and arbitrarily close to the identity function.**

I. INTRODUCTION

Robustness with respect to measurement errors is essential for automatic control systems. Fundamentally different from linear systems, controllers for nonlinear systems often need to be carefully designed to achieve such robustness; see [1, Section 6.1] for examples. Input-to-state stability (ISS) invented by Sontag is a tool to describe how external inputs affect the internal stability of a nonlinear system, i.e., the robustness of a nonlinear system with respect to external inputs (see [2] for a tutorial). With the development of networked control, ISS with respect to measurement errors appears to be fundamental in several recent networked controller designs; see e.g., [3] on quantized control, [4] for time-delayed nonlinear systems, [5] on sampled-data control. It should be noted that measurement errors are usually non-smooth or even discontinuous signals. For example, a quantizer can be mathematically modeled as a discontinuous map.

Despite its importance to practical control problems, robust control of nonlinear systems with measurement errors has not yet been paid enough attention. Reference [6] studied the input-to-state stabilization of first-order nonlinear systems with measurement error. In [7], the conditions under which a system can be stabilized insensitively to small measurement errors were given. Reference [1] introduced a robust control design approach based on the well-known backstepping methodology (see the book [8] for an excellent introduction to backstepping) and flattened Lyapunov functions to the control of strict-feedback nonlinear systems with bounded measurement errors. In [9], a hybrid controller was developed for a class of nonlinear systems to achieve ISS with respect to measurement errors. Reference [10] studied nonlinear systems composed of two subsystems, one is ISS and the other one is input-to-state stabilizable with respect to the measurement error. In [10], the ISS of the closed-loop system is guaranteed by the gain assignment technique introduced in [11], [12], [13] and the nonlinear small-gain theorem proposed in [11], [14]. In our very recent work [15], we developed a fundamentally new approach by using set-valued maps to cope with the discontinuous measurement errors caused by sensor noise in decentralized control systems. However, most of the existing results of high-order systems consider bounded measurement errors. In addition, these results are not directly applicable to networked control systems design.

Reference [16] firstly announced an extension of the ISS small-gain theorem in [11], [14] for networks of discrete-time ISS systems. Shortly, [17] independently developed a matrix-small-gain criterion for networks with plus-type interconnections and mentioned the cyclic-small-gain condition. In [18], the dynamical network with max-type interconnections in the ISS gain formulation was systematically studied and more general cyclic-small-gain criteria were developed for networks of IOS systems. The ISS-Lyapunov based cyclic-small-gain theorem was developed in [19]. The cyclic-small-gain condition can be roughly described as follows: every loop-gain, i.e., the composition of the gain functions of the subsystems along every simple-loop in the network, is less than the identity function.

In practical industrial control applications, low-pass filters are usually employed to attenuate high-frequency measurement noise and to estimate the measured signals. Motivated by low-pass filters, in this paper, we develop new ISS-induced estimators to estimate the measured states, "polluted" by measurement errors. Based on the estimators, a dynamic state feedback control law will be designed to transform the closed-loop system into an interconnection of ISS subsystems. Then, the ISS property of the total interconnected system will be guaranteed by checking the loop-gains with the cyclic-small-gain theorem. The closed-loop system will also be designed to be input-to-output stable (IOS) from the measurement errors to the control errors. Moreover, it will be demonstrated that the IOS gain from the measurement error of the system output to the system output can be designed to be arbitrarily close to the identity

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function.

II. Problem Formulation

In this paper, we consider a class of nonlinear uncertain systems in the strict-feedback form:

$$\dot{x}_i = x_{i+1} + \Delta_i(\bar{x}_i), \quad i = 1, \ldots, n \tag{1}$$

$$x_{n+1} \stackrel{\text{def}}{=} u \tag{2}$$

$$x_i^m = x_i + d_i, \quad i = 1, \ldots, n \tag{3}$$

where $[x_1, \ldots, x_n]^T := x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $\bar{x}_i = [x_1, \ldots, x_i]^T$, $x_i^m \in \mathbb{R}$ is the disturbed measurement of x_i with measurement error $d_i \in \mathbb{R}$, and Δ_i's for $i = 1, \ldots, n$ are unknown locally Lipschitz continuous functions.

Assumption 1: For each Δ_i with $i = 1, \ldots, n$ in (1), there exists a known $\psi_{\Delta_i} \in \mathscr{K}_\infty$ such that for all $\bar{x}_i$,

$$|\Delta_i(\bar{x}_i)| \le \psi_{\Delta_i}(|\bar{x}_i|). \tag{4}$$

Remark 1: In our previous work, e.g., [15], the measurement errors d_i are assumed to be bounded by some unknown constants. We do not make such assumption in this paper.

The objective of this paper is to design a continuous dynamic state-measurement feedback controller of the form

$$\dot{\zeta} = \varphi(\zeta, x^m); \quad u = \lambda(\zeta) \tag{5}$$

such that the system (1)–(3) is made ISS with d_i's as the inputs. Moreover, the closed-loop system is input-to-output stable with d_i's as the inputs and x_1 as the output, and the IOS gain from d_1 to x_1 can be assigned arbitrarily close to the identity function.

III. Modified Gain Assignment

The gain assignment technique has been proved to be extremely useful in small-gain based controller designs [11], [12], [13], [10]. In this section, we present a modified gain assignment lemma for the dynamic state-measurement feedback control design in this paper.

In this paper, we will transform the closed-loop system into an interconnection of first-order nonlinear systems in the following form:

$$\dot{\eta} = \phi(\eta, \omega_1, \ldots, \omega_m) + \bar{\kappa} \tag{6}$$

$$\eta^m = \eta + \omega_{m+1} \tag{7}$$

where $\eta \in \mathbb{R}$ is the state, $\bar{\kappa} \in \mathbb{R}$ is the control input, $\omega_1, \ldots, \omega_{m+1} \in \mathbb{R}$ represent external inputs, $\eta^m \in \mathbb{R}$ is the measurement of η, the nonlinear function $\phi(\eta, \omega_1, \ldots, \omega_m)$ is locally Lipschitz and satisfies

$$|\phi(\eta, \omega_1, \ldots, \omega_m)| \le \psi_\phi^\eta(|\eta|) + \sum_{k=1}^m \psi_\phi^{\omega_k}(|\omega_k|) \tag{8}$$

with $\psi_\phi^\eta, \psi_\phi^{\omega_1}, \ldots, \psi_\phi^{\omega_m} \in \mathscr{K}_\infty$. Define $\alpha_V(s) = \frac{1}{2}s^2$ for $s \in \mathbb{R}_+$.

Lemma 3.1 presents a modified gain assignment technique for system (6)–(7).

Lemma 3.1: Consider system (6)–(7). For any specified $0 < c < 1$, $\varepsilon > 0$, $\ell > 0$ and $\gamma_\eta^{\omega_1}, \ldots, \gamma_\eta^{\omega_m} \in \mathscr{K}_\infty$, we can design a measurement feedback control law

$$\bar{\kappa} = \kappa(\eta^m) = -\nu(|\eta^m|)\eta^m \tag{9}$$

where $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ is positive, nondecreasing and continuously differentiable on $(0, \infty)$, and satisfies

$$\begin{aligned}&(1-c)\nu((1-c)s)s \\ &\ge \psi_\phi^\eta(s) + \sum_{k=1}^m \psi_\phi^{\omega_k} \circ \left(\gamma_\eta^{\omega_k}\right)^{-1} \circ \alpha_V(s) + \frac{\ell}{2}s\end{aligned} \tag{10}$$

for all $s \ge \sqrt{2\varepsilon}$. Moreover, κ is continuously differentiable, odd, strictly decreasing and radially unbounded, and $V_\eta(\eta) = \alpha_V(|\eta|)$ satisfies the following implication:

$$\begin{aligned}&V_\eta(\eta) \ge \max_{k=1,\ldots,m+1} \left\{\gamma_\eta^{\omega_k}(|\omega_k|), \varepsilon\right\} \\ &\Rightarrow \nabla V_\eta(\eta)\dot{\eta} \le -\ell V_\eta(\eta)\end{aligned} \tag{11}$$

where $\gamma_\eta^{\omega_{m+1}}(s) = \alpha_V\left(\frac{s}{c}\right)$ for $s \in \mathbb{R}_+$.

Proof: Because $\psi_\phi^\eta(s) + \sum_{k=1}^m \psi_\phi^{\omega_k} \circ \left(\gamma_\eta^{\omega_k}\right)^{-1} \circ \alpha_V(s) + \frac{\ell}{2}s$ is a $\mathscr{K}_\infty$ function of s, from Lemma 1 in [13], for any $0 < c < 1$, $\varepsilon > 0$, we can find a $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ which is positive, nondecreasing and continuously differentiable on $(0, \infty)$, and satisfies (10) for all $s \ge \sqrt{2\varepsilon}$. It can be directly proved that the κ defined as $\kappa(r) = -\nu(|r|)r$ is odd, strictly decreasing, radially unbounded and continuously differentiable on $(-\infty, 0) \cup (0, \infty)$. With direct calculation, we have $\lim_{r\to 0^+} \frac{d\kappa(r)}{dr} = \lim_{r\to 0^-} \frac{d\kappa(r)}{dr}$ and κ is continuously differentiable.

Recall $V_\eta(\eta) = \alpha_V(|\eta|) = \frac{1}{2}\eta^2$. Consider the case of

$$V_\eta(\eta) \ge \max_{k=1,\ldots,m+1} \left\{\gamma_\eta^{\omega_k}(|\omega_k|), \varepsilon\right\}. \tag{12}$$

In this case, we have $|\omega_k| \le \left(\gamma_\eta^{\omega_k}\right)^{-1} \circ \alpha_V(|\eta|)$ for $k = 1, \ldots, m$, $\omega_{m+1} \le c\alpha_V^{-1}(V_\eta(\eta)) = c|\eta|$ and $|\eta| \ge \sqrt{2\varepsilon}$. Recall $\bar{\kappa} = \kappa(\eta^m) = \kappa(\eta + \omega_{m+1})$. With $0 < c < 1$ and $\omega_{m+1} \le c|\eta|$, when $\eta \ne 0$, we have $\text{sgn}(\eta^m) = \text{sgn}(\eta)$, $|\eta^m| = |\eta + \omega_{m+1}| \ge (1-c)|\eta|$ and $\nu(|\eta^m|)|\eta^m| \ge (1-c)\nu((1-c)|\eta|)|\eta|$.

Using (8), (10) and the discussion above, for any $\bar{\kappa}$ satisfying (9), in the case of (12), we have

$$\begin{aligned}&\nabla V_\eta(\eta)(\phi(\eta, \omega_1, \ldots, \omega_m) + \bar{\kappa}) \\ &= \eta(\phi(\eta, \omega_1, \ldots, \omega_m) - \nu(|\eta^m|)\eta^m) \\ &\le |\eta||\phi(\eta, \omega_1, \ldots, \omega_m)| - |\eta|\nu(|\eta^m|)|\eta^m| \\ &\le |\eta|\Big(\psi_\phi^\eta(|\eta|) + \sum_{k=1}^m \psi_\phi^{\omega_k}(|\omega_k|) \\ &\qquad - (1-c)\nu((1-c)|\eta|)|\eta|\Big) \\ &\le |\eta|\Big(\psi_\phi^\eta(|\eta|) + \sum_{k=1}^m \psi_\phi^{\omega_k} \circ \left(\gamma_\eta^{\omega_k}\right)^{-1} \circ \alpha_V(|\eta|) \\ &\qquad - (1-c)\nu((1-c)|\eta|)|\eta|\Big) \\ &\le -\frac{\ell}{2}|\eta|^2 = -\ell V_\eta(\eta).\end{aligned} \tag{13}$$

This ends the proof. ■

IV. Dynamic State-measurement Feedback Control Design

In this section, we recursively construct the dynamic state-measurement feedback controller to transform the closed-loop system into an interconnection of ISS subsystems. Specifically, given the e_i-subsystem with x_{i+1} as the virtual control input, we design an estimator with state $\hat{e}_i$ to estimate e_i such that the estimation error subsystem with state $\tilde{e}_i = \hat{e}_i - e_i$ is ISS. Then, we design a virtual control law $\kappa_{12}(\hat{e}_i)$ to render the $\hat{e}_i$-subsystem to be ISS. A new state e_{i+1} is defined as $e_{i+1} = x_{i+1} - \kappa_{12}(\hat{e}_i)$. A new system with state $[\hat{e}_1, \tilde{e}_1, \ldots, \hat{e}_n, \tilde{e}_n]^T$ is constructed from the x-system. With the employment of the estimators, the problems caused by the discontinuity of the measurement errors are solved.

For each $\tilde{e}_i$-subsystem and each $\hat{e}_i$-subsystem, we define the following ISS-Lyapunov function candidates, respectively:

$$V_{\tilde{e}_i}(\tilde{e}_i) = \alpha_V(|\tilde{e}_i|), \quad i = 1, \ldots, n, \tag{14}$$

$$V_{\hat{e}_i}(\hat{e}_i) = \alpha_V(|\hat{e}_i|), \quad i = 1, \ldots, n, \tag{15}$$

where $\alpha_V(s) = \frac{1}{2}s^2$ for $s \in \mathbb{R}_+$. In the following discussions, we simply use $V_{\tilde{e}_i}$ and $V_{\hat{e}_i}$ instead of $V_{\tilde{e}_i}(\tilde{e}_i)$ and $V_{\hat{e}_i}(\hat{e}_i)$, respectively.

For convenience of notations, define $\hat{e}_0 = e_0 = 0$ and $d_0 = 0$.

For $i = 1, \ldots, n$, the e_i's are recursively defined as

$$e_1 = x_1 \tag{16}$$

$$e_i = x_i - \kappa_{(i-1)2}(\hat{e}_{i-1}), \quad i = 2, \ldots, n \tag{17}$$

where $\kappa_{(i-1)2} : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable, odd, strictly decreasing and radially unbounded function, and $\hat{e}_{i-1}$ is an estimate of e_{i-1}.

Since x_i (and thus e_i) is not available for feedback, we define

$$e_1^m = x_1^m \tag{18}$$

$$e_i^m = x_i^m - \kappa_{(i-1)2}(\hat{e}_{i-1}), \quad i = 2, \ldots, n \tag{19}$$

and construct the following estimator for e_i:

$$\dot{\hat{e}}_i = \kappa_{i1}(\hat{e}_i - e_i^m) + x_{i+1}^m \tag{20}$$

where $\hat{e}_i \in \mathbb{R}$ is an estimate of e_i and $\kappa_{i1} : \mathbb{R} \to \mathbb{R}$ is an odd and strictly decreasing function to be determined later.

The structure of the $\hat{e}_i$-subsystem is shown in Fig. 1.

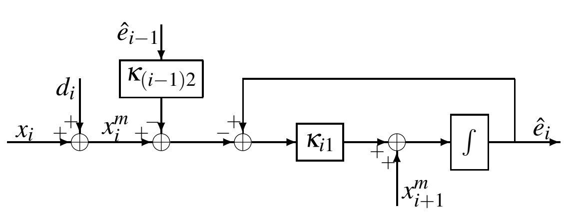

Fig. 1. The estimator for e_i: the $\hat{e}_i$-subsystem.

Define

$$\tilde{e}_i = \hat{e}_i - e_i \tag{21}$$

as the estimation error for e_i. Taking the derivative of $\tilde{e}_i$ and using $x_i^m = x_i + d_i$ and $x_{i+1}^m = x_{i+1} + d_{i+1}$, direct calculation yields:

$$\begin{aligned} \dot{\tilde{e}}_i &= \dot{\hat{e}}_i - \dot{e}_i \\ &= \dot{\hat{e}}_i - \dot{x}_i + \frac{\partial \kappa_{(i-1)2}(\hat{e}_{i-1})}{\partial \hat{e}_{i-1}} \dot{\hat{e}}_{i-1} \\ &= \kappa_{i1}(\tilde{e}_i - d_i) + d_{i+1} + x_{i+1} - \Delta_i(\bar{x}_i) - x_{i+1} \\ &\quad + \frac{\partial \kappa_{(i-1)2}(\hat{e}_{i-1})}{\partial \hat{e}_{i-1}} \dot{\hat{e}}_{i-1}, \end{aligned} \tag{22}$$

or equivalently,

$$\dot{\tilde{e}}_i = \Delta_{i1}^*(\hat{e}_1, \tilde{e}_1, \ldots, \hat{e}_i, \tilde{e}_i, d_{i-1}, d_i, d_{i+1}) + \kappa_{i1}(\tilde{e}_i - d_i) \tag{23}$$

where

$$\begin{aligned} &\Delta_{i1}^*(\hat{e}_1, \tilde{e}_1, \ldots, \hat{e}_i, \tilde{e}_i, d_{i-1}, d_i, d_{i+1}) \\ &= d_{i+1} - \Delta_i(\bar{x}_i) + \frac{\partial \kappa_{(i-1)2}(\hat{e}_{i-1})}{\partial \hat{e}_{i-1}} \dot{\hat{e}}_{i-1}. \end{aligned} \tag{24}$$

With Assumption 1 satisfied, we can find $\psi_{\Delta_{i1}^*}^{\hat{e}_k}, \psi_{\Delta_{i1}^*}^{\tilde{e}_k} \in \mathcal{K}_\infty$ with $k = 1, \ldots, i$ and $\psi_{\Delta_{i1}^*}^{d_{i-1}}, \psi_{\Delta_{i1}^*}^{d_i}, \psi_{\Delta_{i1}^*}^{d_{i+1}} \in \mathcal{K}_\infty$ such that

$$\begin{aligned} &|\Delta_{i1}^*(\hat{e}_1, \tilde{e}_1, \ldots, \hat{e}_i, \tilde{e}_i, d_{i-1}, d_i, d_{i+1})| \\ &\le \sum_{k=1}^{i} \left(\psi_{\Delta_{i1}^*}^{\hat{e}_k}(|\hat{e}_k|) + \psi_{\Delta_{i1}^*}^{\tilde{e}_k}(|\tilde{e}_k|) \right) \\ &\quad + \psi_{\Delta_{i1}^*}^{d_{i-1}}(|d_{i-1}|) + \psi_{\Delta_{i1}^*}^{d_i}(|d_i|) + \psi_{\Delta_{i1}^*}^{d_{i+1}}(|d_{i+1}|). \end{aligned} \tag{25}$$

By using $x_i^m = x_i + d_i$ and $x_{i+1}^m = x_{i+1} + d_{i+1}$, we can rewrite the $\hat{e}_i$-subsystem as

$$\dot{\hat{e}}_i = \kappa_{i1}(\tilde{e}_i - d_i) + d_{i+1} + x_{i+1}. \tag{26}$$

Consider x_{i+1} as the virtual control input of the $\hat{e}_i$-subsystem. We design a virtual control law $\kappa_{i2}(\hat{e}_i)$, with $\kappa_{i2} : \mathbb{R} \to \mathbb{R}$ continuously differentiable, odd, strictly decreasing and radially unbounded, to be determined later. Define

$$e_{i+1} = x_{i+1} - \kappa_{i2}(\hat{e}_i). \tag{27}$$

Note that x_{i+1} (and thus e_{i+1}) is not available for feedback. We use $\hat{e}_{i+1}$ to denote the estimate of e_{i+1} and define $\tilde{e}_{i+1} = \hat{e}_{i+1} - e_{i+1}$. Then, the $\hat{e}_i$-subsystem (26) can be rewritten as

$$\dot{\hat{e}}_i = \Delta_{i2}^*(\tilde{e}_i, \tilde{e}_{i+1}, \hat{e}_{i+1}, d_i, d_{i+1}) + \kappa_{i2}(\hat{e}_i) \tag{28}$$

where

$$\begin{aligned} &\Delta_{i2}^*(\tilde{e}_i, \tilde{e}_{i+1}, \hat{e}_{i+1}, d_i, d_{i+1}) \\ &= \kappa_{i1}(\tilde{e}_i - d_i) + d_{i+1} + \hat{e}_{i+1} - \tilde{e}_{i+1}. \end{aligned} \tag{29}$$

Because κ_{i1} is odd and strictly decreasing, we can find $\psi_{\Delta_{i2}}^{\tilde{e}_i}, \psi_{\Delta_{i2}}^{\tilde{e}_{i+1}}, \psi_{\Delta_{i2}}^{\hat{e}_{i+1}}, \psi_{\Delta_{i2}}^{d_i}, \psi_{\Delta_{i2}}^{d_{i+1}} \in \mathcal{K}_\infty$ such that

$$\begin{aligned} &|\Delta_{i2}^*(\tilde{e}_i, \tilde{e}_{i+1}, \hat{e}_{i+1}, d_i, d_{i+1})| \\ &= \psi_{\Delta_{i2}}^{\tilde{e}_i}(|\tilde{e}_i|) + \psi_{\Delta_{i2}}^{\tilde{e}_{i+1}}(|\tilde{e}_{i+1}|) + \psi_{\Delta_{i2}}^{\hat{e}_{i+1}}(|\hat{e}_{i+1}|) \\ &\quad + \psi_{\Delta_{i2}}^{d_i}(|d_i|) + \psi_{\Delta_{i2}}^{d_{i+1}}(|d_{i+1}|). \end{aligned} \tag{30}$$

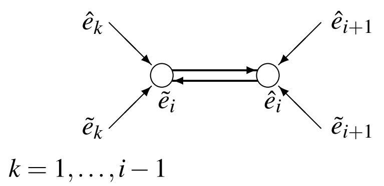

Fig. 2. The interconnection in each $\check{e}_i$-system ($i = 1, \dots, n$).

Denote $\check{e}_i = [\hat{e}_i, \tilde{e}_i]^T$ for $i = 1, \dots, n$. The interconnection in the $\check{e}_i$-subsystem is shown in Fig. 2. Denote $\bar{\check{e}}_i = [\check{e}_1^T, \dots, \check{e}_i^T]^T$ for $i = 1, \dots, n$.

Given the $\bar{\check{e}}_i$-subsystem, we choose $\gamma_{\tilde{e}_i}^{\hat{e}_k}, \gamma_{\tilde{e}_i}^{\tilde{e}_k} \in \mathscr{K}_\infty$ with $k = 1, \dots, i$ such that the compositions of the gain functions along all the simple loops through the $\tilde{e}_i$-subsystem in the $[\bar{\check{e}}_{i-1}^T, \tilde{e}_i]^T$-subsystem are less than the identity function.

Consider the $\tilde{e}_i$-subsystem defined in (23). Using Lemma 3.1, for any specified constants $0 < c_{i1} < 1$, $\varepsilon_{i1} > 0$ and $\ell_{i1} > 0$, any specified $\gamma_{\tilde{e}_i}^{\hat{e}_i}, \gamma_{\tilde{e}_i}^{d_{i-1}}, \gamma_{\tilde{e}_i}^{d_i}, \gamma_{\tilde{e}_i}^{d_{i+1}} \in \mathscr{K}_\infty$, and the $\gamma_{\tilde{e}_i}^{\hat{e}_k}, \gamma_{\tilde{e}_i}^{\tilde{e}_k} \in \mathscr{K}_\infty$ for $k = 1, \dots, i$ chosen above, we design κ_{i1} in the form of (9) such that $V_{\tilde{e}_i}$ satisfies

$$V_{\tilde{e}_i} \ge \max_{k=1,\dots,i-1} \left\{ \begin{array}{l} \gamma_{\tilde{e}_i}^{\tilde{e}_k}(V_{\tilde{e}_k}), \gamma_{\tilde{e}_i}^{\hat{e}_k}(V_{\hat{e}_k}), \gamma_{\tilde{e}_i}^{\hat{e}_i}(V_{\hat{e}_i}), \\ \gamma_{\tilde{e}_i}^{d_{i-1}}(|d_{i-1}|), \gamma_{\tilde{e}_i}^{d_i}(|d_i|), \\ \gamma_{\tilde{e}_i}^{d_{i+1}}(|d_{i+1}|), \varepsilon_{i1} \end{array} \right\}$$
$$\Rightarrow \nabla V_{\tilde{e}_i} \dot{\tilde{e}}_i \le -\ell_{i1} V_{\tilde{e}_i} \tag{31}$$

where

$$\gamma_{\tilde{e}_i}^{d_i} = \alpha_V \left(\frac{s}{c_{i1}} \right) \tag{32}$$

for $s \in \mathbb{R}_+$.

Given the $[\bar{\check{e}}_{i-1}, \tilde{e}_i]^T$-subsystem, we choose $\gamma_{\hat{e}_i}^{\tilde{e}_i} \in \mathscr{K}_\infty$ such that the compositions of the gain functions along all the simple loops through the $\hat{e}_i$-subsystem in the $\bar{\check{e}}_i$-subsystem are less than the identity function.

Consider the $\hat{e}_i$-subsystem defined in (28). Using Lemma 3.1, for any specified constants $\varepsilon_{i2} > 0$ and $\ell_{i2} > 0$, any specified $\gamma_{\hat{e}_i}^{\tilde{e}_{i+1}}, \gamma_{\hat{e}_i}^{\hat{e}_{i+1}} \in \mathscr{K}_\infty$ and the $\gamma_{\hat{e}_i}^{\tilde{e}_i}$ chosen above, we design κ_{i2} in the form of (9) such that $V_{\hat{e}_i}$ satisfies

$$V_{\hat{e}_i} \ge \max \left\{ \begin{array}{l} \gamma_{\hat{e}_i}^{\tilde{e}_i}(V_{\tilde{e}_i}), \gamma_{\hat{e}_i}^{\hat{e}_{i+1}}(V_{\hat{e}_{i+1}}), \gamma_{\hat{e}_i}^{\tilde{e}_{i+1}}(V_{\tilde{e}_{i+1}}), \\ \gamma_{\hat{e}_i}^{d_i}(|d_i|), \gamma_{\hat{e}_i}^{d_{i+1}}(|d_{i+1}|), \varepsilon_{i2} \end{array} \right\} \tag{33}$$
$$\Rightarrow \nabla V_{\hat{e}_i} \dot{\hat{e}}_i \le -\ell_{i2} V_{\hat{e}_i}. \tag{34}$$

In the case of $i = n$, $x_{i+1} = x_{i+1}^m = u$, $\hat{e}_{i+1} = e_{i+1} = 0$ and $d_{i+1} = 0$. The dynamic state measurement feedback control law is designed as

$$u = \kappa_{n2}(\hat{e}_n). \tag{35}$$

Remark 2: In standard backstepping, we usually design a virtual control law in the form of $\kappa_i(e_i)$ for the e_i-subsystem. Due to the measurement error d_i, e_i is not available for feedback, and an intuitive modification is $\kappa_i(e_i + d_i)$. However, the discontinuity of d_i makes it impossible to take the derivative of $\kappa_i(e_i + d_i)$, and the differentiation-based standard backstepping cannot proceed.

Remark 3: In our very recent paper [15], we introduced a set-valued map based static feedback control design to overcome the problem caused by the discontinuity of the measurement errors. The dynamics of the closed-loop system are represented with differential inclusions. That method is effective for only bounded measurement errors and can only achieve local ISS. In this paper, by introducing the estimators, we do not make any assumption on the bounds of the measurement errors. Moreover, with the help of the estimators, the closed-loop system can be directly represented with differential equations.

V. STABILITY ANALYSIS AND MAIN RESULTS

The system digraph of the $[\check{e}_1, \dots, \check{e}_n]^T$-system is shown in Fig. 3.

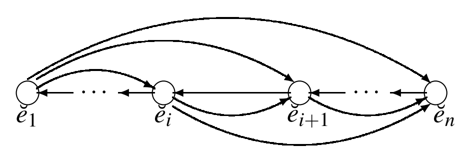

Fig. 3. The system digraph of the $[\check{e}_1, \dots, \check{e}_n]^T$-system.

According to the recursive design, given the $\bar{\check{e}}_{i-1}$-system, by designing κ_{i1} for the $\tilde{e}_i$-subsystem, we can assign the ISS gains $\gamma_{\tilde{e}_i}^{\hat{e}_k}, \gamma_{\tilde{e}_i}^{\tilde{e}_k}$'s for $k = 1, \dots, i-1$ such that all the simple loops in the $[\bar{\check{e}}_{i-1}^T, \tilde{e}_i]^T$-system through the $\tilde{e}_i$-subsystem satisfy the cyclic-small-gain condition. By designing κ_{i2} for the $\hat{e}_i$-subsystem, we can assign the ISS gain $\gamma_{\hat{e}_i}^{\tilde{e}_i}$ such that the simple loop in the $\bar{\check{e}}_i$-system through the $\tilde{e}_i$-subsystem satisfies the cyclic-small-gain condition. Through the recursive control design procedure, the e-system satisfies the cyclic-small-gain condition and is ISS [18], [19].

For the e-system, we construct an ISS-Lyapunov function in the following form:

$$V_e(e) = \max_{i=1,\dots,n} \{ \sigma_{i1}(V_{\tilde{e}_i}(\tilde{e}_i)), \sigma_{i2}(V_{\hat{e}_i}(\hat{e}_i)) \} \tag{36}$$

where $\sigma_{12} = \mathrm{Id}$, and σ_{i1} with $i = 1, \dots, n$ and σ_{i2} with $i = 2, \dots, n$ are compositions of $\hat{\gamma}_{(\cdot)}^{(\cdot)}$'s which are smooth on $(0, \infty)$ and slightly larger than the corresponding $\gamma_{(\cdot)}^{(\cdot)}$'s, and still satisfy the cyclic-small-gain condition. Here it is not necessary to give an explicit representation of the σ_{i1} and σ_{i2} to analyze the effect of the measurement errors.

Correspondingly, the influence from d_i and ε_i for $i = 1, \dots, n$ can be represented as:

$$\theta = \max_{i=1,\dots,n} \left\{ \begin{array}{l} \sigma_{i1} \circ \gamma_{\tilde{e}_i}^{d_{i-1}}(|d_{i-1}|), \sigma_{i1} \circ \gamma_{\tilde{e}_i}^{d_i}(|d_i|), \\ \sigma_{i1} \circ \gamma_{\tilde{e}_i}^{d_{i+1}}(|d_{i+1}|), \sigma_{i1}(\varepsilon_{i1}), \\ \sigma_{i2} \circ \gamma_{\hat{e}_i}^{d_i}(|d_i|), \sigma_{i2} \circ \gamma_{\hat{e}_i}^{d_{i+1}}(|d_{i+1}|), \sigma_{i2}(\varepsilon_{i2}) \end{array} \right\} \tag{37}$$

From the Lyapunov-based cyclic-small-gain theorem, it holds that

$$V_e(e) \ge \theta \Rightarrow \nabla V_e(e) \dot{e} \le -\alpha_e(V_e(e)) \tag{38}$$

wherever ∇V_e exists, with α_e positive definite.

By default, $\gamma_{\tilde{e}_{n+1}}^{d_n} := 0$, $\gamma_{\tilde{e}_0}^{d_1} := 0$, $\gamma_{\hat{e}_0}^{d_1} := 0$, $\sigma_{(n+1)1} := 0$ and $\sigma_{01} := 0$.

Define

$$\gamma_e^{d_i}(s) = \max \left\{ \begin{array}{l} \sigma_{(i+1)1} \circ \gamma_{\tilde{e}_{i+1}}^{d_i}(s), \sigma_{i1} \circ \gamma_{\tilde{e}_i}^{d_i}(s), \\ \sigma_{(i-1)1} \circ \gamma_{\tilde{e}_{i-1}}^{d_i}(s), \sigma_{i2} \circ \gamma_{\hat{e}_i}^{d_i}(s), \\ \sigma_{(i-1)2} \circ \gamma_{\hat{e}_{i-1}}^{d_i}(s) \end{array} \right\} \tag{39}$$

$$\varepsilon = \max_{i=1,\dots,n} \left\{ \sigma_{i1}(\varepsilon_{i1}), \sigma_{i2}(\varepsilon_{i2}) \right\}. \tag{40}$$

Then, the θ defined in (37) can be equivalently represented as

$$\theta = \max_{i=1,\dots,n} \left\{ \gamma_e^{d_i}(|d_i|), \varepsilon \right\}. \tag{41}$$

By choosing the $\gamma_{(\cdot)}^{(\cdot)}$'s small enough, we can make σ_{i1} for $i = 1,\dots,n$ and σ_{i2} for $i = 2,\dots,n$ small enough such that

$$\begin{aligned} &\sigma_{i1} \circ \gamma_{\tilde{e}_i}^{d_i}(s) \\ &\geq \max \left\{ \begin{array}{l} \sigma_{(i+1)1} \circ \gamma_{\tilde{e}_{i+1}}^{d_i}(s), \sigma_{(i-1)1} \circ \gamma_{\tilde{e}_{i-1}}^{d_i}(s), \\ \sigma_{i2} \circ \gamma_{\hat{e}_i}^{d_i}(s), \sigma_{(i-1)2} \circ \gamma_{\hat{e}_{i-1}}^{d_i}(s) \end{array} \right\}. \end{aligned} \tag{42}$$

In this way, we achieve

$$\theta = \max_{i=1,\dots,n} \left\{ \sigma_{i1} \circ \gamma_{\tilde{e}_i}^{d_i}(|d_i|), \varepsilon \right\}. \tag{43}$$

Property (38) implies that there exists a $\beta_e \in \mathcal{KL}$ such that

$$V_e(e(t)) \leq \max \left\{ \beta_e(V_e(e(t_0)), t - t_0), \sup_{t_0 \leq \tau \leq t} (\theta(\tau)) \right\} \tag{44}$$

where

$$\theta(\tau) = \max_{i=1,\dots,n} \left\{ \sigma_{i1} \circ \gamma_{\tilde{e}_i}^{d_i}(|d_i(t)|), \varepsilon \right\}. \tag{45}$$

From the definition of V_e in (38), using $\sigma_{12} = \mathrm{Id}$, we have

$$\begin{aligned} |x_1| &= |e_1| = |\hat{e}_1 - \tilde{e}_1| \leq |\hat{e}_1| + |\tilde{e}_1| \\ &= \alpha_V^{-1}(V_{\hat{e}_1}(\hat{e}_1)) + \alpha_V^{-1}(V_{\tilde{e}_1}(\tilde{e}_1)) \\ &\leq \alpha_V^{-1} \circ \sigma_{12}^{-1}(V_e(e)) + \alpha_V^{-1} \circ \sigma_{11}^{-1}(V_e(e)) \\ &= (\alpha_V^{-1} + \alpha_V^{-1} \circ \sigma_{11}^{-1})(V_e(e)). \end{aligned} \tag{46}$$

Define

$$\bar{\gamma}_{x_1}^{d_i} = (\alpha_V^{-1} + \alpha_V^{-1} \circ \sigma_{11}^{-1}) \circ \sigma_{i1} \circ \gamma_{\tilde{e}_i}^{d_i}, \quad i = 1,\dots,n \tag{47}$$

$$\bar{\beta}_{x_1} = (\alpha_V^{-1} + \alpha_V^{-1} \circ \sigma_{11}^{-1}) \circ \beta_e \tag{48}$$

$$\bar{\varepsilon}_{x_1} = (\alpha_V^{-1} + \alpha_V^{-1} \circ \sigma_{11}^{-1})(\varepsilon). \tag{49}$$

Then, from (44) and (45), we obtain

$$\begin{aligned} |x_1(t)| \leq &(\alpha_V^{-1} + \alpha_V^{-1} \circ \sigma_{11}^{-1})(V_e(e(t))) \\ \leq &\max \Big\{ \bar{\beta}_{x_1}(V_e(e(t_0)), t - t_0), \\ &\sup_{t_0 \leq \tau \leq t} \big(\max_{i=1,\dots,n} \bar{\gamma}_{x_1}^{d_i}(|d_i(\tau)|) \big), \varepsilon \Big\}. \end{aligned} \tag{50}$$

Thus, the closed-loop system is IOS with x_1 as the output and the IOS gain from d_i to x_1 is $\bar{\gamma}_{x_1}^{d_i}$ [11].

Recall $\gamma_{\tilde{e}_1}^{d_1}(s) = \alpha_V(s/c_{11})$ for $s \in \mathbb{R}_+$. From the definition of $\bar{\gamma}_{x_1}^{d_1}$ in (47) with $i = 1$, we have

$$\begin{aligned} \bar{\gamma}_{x_1}^{d_1} &= (\alpha_V^{-1} + \alpha_V^{-1} \circ \sigma_{11}^{-1}) \circ \sigma_{11} \circ \gamma_{\tilde{e}_1}^{d_1} \\ &= (\mathrm{Id} + \alpha_V^{-1} \circ \sigma_{11} \circ \alpha_V) \left(\frac{s}{c_{11}} \right). \end{aligned} \tag{51}$$

Note that for $i = 1,\dots,n$, each σ_{i1} is a composition of $\hat{\gamma}_{(\cdot)}^{(\cdot)}$'s, which can be chosen arbitrarily small. Thus, the IOS gain $\bar{\gamma}_{x_1}^{d_i}$ for $i = 2,\dots,n$ can be designed arbitrarily small. If we also choose c_{11} arbitrarily close to one, and σ_{11} arbitrarily small, then $\bar{\gamma}_{x_1}^{d_1}$ is arbitrarily close to the identity function.

The main result of this paper is summarized in the following theorem.

Theorem 1: With Assumption 1 satisfied, the system (1)–(3) can be input-to-state stabilized with the dynamic state measurement feedback control law defined in (17), (19), (20) and (35). Moreover, the close-loop system is IOS with the measurement errors $d_1,\dots,d_n$ as inputs and x_1 as output, the IOS gain from d_1 to x_1 can be designed to be arbitrarily close to the identity function, and the IOS gain from $d_2,\dots,d_n$ to x_1 can be designed to be arbitrarily small.

VI. An Example

To verify the main result of this paper, consider the following second-order nonlinear system:

$$\dot{x}_1 = x_2 \tag{52}$$

$$\dot{x}_2 = 0.2x_2^2 + u \tag{53}$$

$$x_1^m = x_1 + d_1; \quad x_2^m = x_2 + d_2. \tag{54}$$

For the sake of simplicity, assume $d_2 = 0$.

Define $e_1 = x_1$. Following the design procedure in Section IV, we have

$$\dot{\hat{e}}_1 = \kappa_{11}(\tilde{e}_1 - d_1) + \hat{e}_2 - \tilde{e}_2 + \kappa_{12}(\hat{e}_1) \tag{55}$$

$$\dot{\tilde{e}}_1 = \kappa_{11}(\tilde{e}_1 - d_1) \tag{56}$$

where $\hat{e}_1$ is the estimate of e_1, $\tilde{e}_1 = \hat{e}_1 - e_1$, and $\hat{e}_2$ and $\tilde{e}_2$ will be defined later.

Consider the $\tilde{e}_1$-subsystem. Clearly, $\Delta_{11}^* = 0$. We choose $c_{11} = 0.8$ and $\ell_{11} = 0.02$. Then, the κ_{11} is designed in the form of $\kappa_{11}(r) = -\nu_{11}(|r|)r$ with ν_{11} satisfying

$$(1 - c_{11})\nu_{11}((1 - c_{11})s)s \geq 0.01s. \tag{57}$$

Then, we choose $\nu_{11}(s) = 0.05$ for $s \in \mathbb{R}_+$ and $\kappa_{11}(r) = -0.05r$ for $r \in \mathbb{R}$.

With κ_{11} designed, we have $\Delta_{12}^*(\tilde{e}_1, \tilde{e}_2, \hat{e}_2, d_1) = -0.05\tilde{e}_1 + 0.05d_1 + \hat{e}_2 - \tilde{e}_2$. Thus, $\psi_{\Delta_{12}^*}^{\tilde{e}_1}(s) = 0.05s$, $\psi_{\Delta_{12}^*}^{d_1}(s) = 0.05s$, $\psi_{\Delta_{12}^*}^{\hat{e}_2}(s) = s$ and $\psi_{\Delta_{12}^*}^{\tilde{e}_2}(s) = s$. Choose $\ell_{12} = 0.02$, $\gamma_{\hat{e}_1}^{\tilde{e}_1}(s) = s$, $\gamma_{\hat{e}_1}^{\tilde{e}_2}(s) = 0.99s$, $\gamma_{\hat{e}_1}^{d_1}(s) = 0.5s^2$ and $\gamma_{\hat{e}_1}^{\hat{e}_2}(s) = s$. Then, the κ_{12} is designed in the form of $\kappa_{12}(r) = -\nu_{12}(|r|)r$ with $\nu_{12}(s) = 2.11$ for $s \in \mathbb{R}_+$.

Define $e_2 = x_2 - \kappa_{12}(\hat{e}_1)$. The estimator for e_2 is designed in the following form:

$$\dot{\hat{e}}_2 = \kappa_{21}(\tilde{e}_2) + u \tag{58}$$

where $\hat{e}_2$ is the estimate of e_2. Define $\tilde{e}_2 = \hat{e}_2 - e_2$. By directly taking the derivative of $\tilde{e}_2$, we have

$$\dot{\tilde{e}}_2 = \Delta^*_{21}(\tilde{e}_1, \hat{e}_1, \tilde{e}_2, \hat{e}_2, d_1) + \kappa_{21}(\tilde{e}_2) \tag{59}$$

where $|\Delta^*_{21}(\tilde{e}_1, \hat{e}_1, \tilde{e}_2, \hat{e}_2, d_1)|$ satisfies $|\Delta^*_{21}(\tilde{e}_1, \hat{e}_1, \tilde{e}_2, \hat{e}_2, d_1)| \le 0.1055|\tilde{e}_1| + 1.7344|\hat{e}_1|^2 + 4.4521|\hat{e}_1| + 0.822|\tilde{e}_2|^2 + 2.11|\tilde{e}_2| + 0.822|\hat{e}_2|^2 + 2.11|\hat{e}_2| + 0.1055|d_1|$. Thus, we have $\psi^{\hat{e}_2}_{\Delta^*_{21}}(s) = 0.822s^2 + 2.11s$, $\psi^{\tilde{e}_2}_{\Delta^*_{21}}(s) = 0.822s^2 + 2.11s$, $\psi^{\hat{e}_1}_{\Delta^*_{21}}(s) = 1.7344s^2 + 4.4521s$, $\psi^{\tilde{e}_1}_{\Delta^*_{21}}(s) = 0.1055s$ and $\psi^{d_1}_{\Delta^*_{21}}(s) = 0.1055s$. We choose $\ell_{21} = 0.02$, $\gamma^{\hat{e}_1}_{\tilde{e}_2}(s) = 0.99s$, $\gamma^{\hat{e}_2}_{\tilde{e}_2}(s) = 0.99s$, $\gamma^{\tilde{e}_1}_{\tilde{e}_2}(s) = 0.99s$ and $\gamma^{d_1}_{\tilde{e}_2}(s) = 0.5s^2$. Then, the γ_{21} is designed in the form of $\kappa_{21}(r) = -\nu_{21}(|r|)r$ with $\nu_{21}(s) = 3.3784s + 8.7876$ for $s \in \mathbb{R}_+$.

With κ_{21} designed, we have $\Delta^*_{22}(\tilde{e}_2) = -3.3784|\tilde{e}_2|\tilde{e}_2 - 8.7876\tilde{e}_2$. Thus, $\psi^{\tilde{e}_2}_{\Delta^*_{22}}(s) = 3.3784s^2 + 8.7876s$. We choose $\ell_{22} = 0.02$ and $\gamma^{\tilde{e}_2}_{\tilde{e}_2}(s) = 0.99s$. Then, the κ_{22} is designed in the form of $\kappa_{22}(r) = -\nu_{22}(|r|)r$ with $\nu_{22}(s) = 3.3784s + 8.7976$ for $s \in \mathbb{R}_+$.

In the construction of the ISS-Lyapunov function for the closed-loop system, we can choose all $\sigma_{(\cdot)} = \mathrm{Id}$. With direct calculation following the procedure in Section V, we have $\bar{\gamma}^{d_1}_{x_1}(s) = 2.5s$ for $s \in \mathbb{R}_+$.

Simulation results with $d_1 = 0.3\mathrm{sgn}(\sin 0.2t) + 1.7$ shown in Figs. 4–5 are in accordance with the theoretical design.

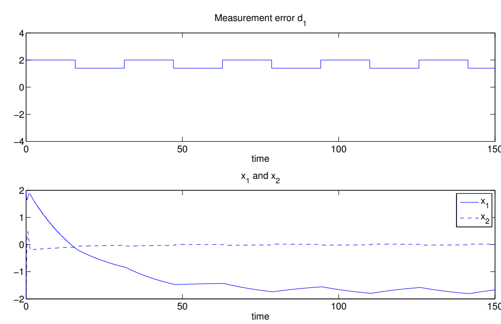

Fig. 4. The measurement disturbance and the system states.

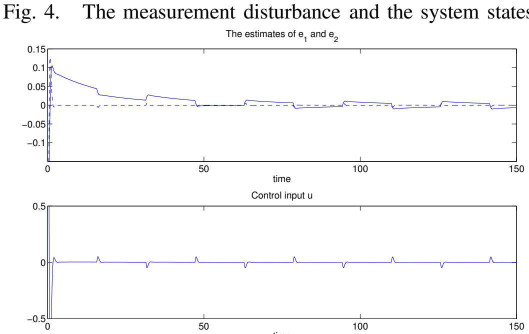

Fig. 5. The estimator states and the control input.

VII. Conclusions

This paper has presented a new dynamic state measurement feedback control strategy for nonlinear systems with discontinuous measurement errors, based on the cyclic-small-gain theorem. ISS-induced estimators are recursively constructed to estimate the disturbed state and the dynamic state-measurement control law is designed to transform the closed-loop system into an interconnection of ISS subsystems. ISS and IOS properties of the closed-loop system are guaranteed by the cyclic-small-gain theorem. In particular, the IOS gain from the measurement error of the system output to the system output can be designed to be arbitrarily close to the identity function.

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