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A Sliding Mode Observer for Fault Reconstruction under Output Sampling: A Time-Delay Approach

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***Abstract*— A sliding mode observer in the presence of sampled output information and its application to fault reconstruction is studied. The observer is designed by using the delayed continuous-time representation of the sampled-data system, for which a set of Linear Matrix Inequalities (LMIs) provide conditions for the ultimate boundedness. It is shown that an ideal sliding motion cannot be achieved in the observer when outputs are sampled. However, ultimately bounded solutions can be obtained provided the sampling frequency is fast enough. The bound on the solution is proportional to the sampling time and the magnitude of the switching gain. The proposed observer design is applied to the problem of fault reconstruction under sampled output. It is shown that, for a sufficiently small value of μ, a perturbation parameter, a transducer or sensor fault can be reconstructed reliably from the output error dynamics. An example of observer design for an inverted pendulum system is used to demonstrate the merit of the proposed methodology compared to existing sliding mode observer design approaches.**

I. INTRODUCTION

A sliding mode observer is a category of robust observer which facilitates the complete rejection of a class of uncertainty between the system and observer [21]. In most cases, the sliding surface is set to be the difference between the observer output and system output, which is therefore forced to zero [3], [22]. A discontinuous injection term is designed and applied to drive the observer so that the error between the output of the observer and the output of the plant will move onto this surface within the error space and then remain there. In the physical world, delays exist in many areas, for example those caused by transmission delay and computational delay. If performance levels are to be optimised in the presence of such delays it is necessary to consider the development of methodologies which incorporate knowledge of the delay in the design framework. There have been many contributions that investigate the effect of state delay on observer design [1], [2]. However very few contributions have considered the effect of delays in the output measurement on observer performance. In terms of work that considers the effect of time-delay in sliding mode observers, the literature is very sparse [15] and is strongly aligned to observer based control rather than fault detection and estimation with an emphasis on state delay rather than measurement delay [19], [20]. Since the switching term in a sliding mode observer depends on the output measurement, which may be subject to delay in practice, the resulting discontinuous injection applied to the observer has the potential to cause chattering of large amplitude and may limit the magnitude of the discontinuous signal that it is possible to apply to the observer.

There has been a great deal of interest in the application of sliding mode observers to the problem of model based fault detection and isolation [5], [6], [23]. The merit of the approach lies in the application of the so-called *equivalent output injection* to explicitly reconstruct fault signals. The results obtained to date mostly require that an ideal sliding motion is attained in finite time before the appearance of faults, and that no delay is present on the output measurement used to drive the observer. It is clear that in the presence of a sampled output, the ideal sliding mode cannot be achieved. Indeed the error dynamics in the observer can become unstable as the sampling frequency is reduced. Motivated by recent results in the area of relay delay control in [7], [13], this paper will consider the effects of sampled output measurements when designing sliding mode observers for fault reconstruction.

It has been shown in [9], [18] that a sampled output can be represented with fast varying delay where the derivative of the delay is equal to 1. From this representation, the main contribution in this paper is a general framework for sliding mode observer design and fault reconstruction under multiple sampled output. The error dynamics is forced to exhibit a bound proportional to the sampling period of the output and the magnitude of the discontinuous switching gain employed in the observer. The observer, which is designed using a singular perturbation approach, possesses a sufficiently small perturbation parameter μ such that faults are reliably constructed despite the presence of the sampled output. In section II, the problem of sliding mode observer design with sampled output is formulated in terms of a system representation with known fast varying delay. Section III develops a constructive observer design approach which guarantees ultimate boundedness of the error dynamics. Section IV highlights the advantages of the observer, using the singular perturbation method to achieve fault reconstruction. A linearized model of the inverted pendulum is used to demonstrate the efficiency of the results. Some preliminary results from this paper in the context of the input delay problem were presented in [12].

Notation: Throughout the paper, the superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$ means that P is symmetric and positive definite.

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The symmetric elements of the symmetric matrix are denoted by $*$. The symbol $\|\cdot\|_\infty$ stands for essential supremum.

II. PROBLEM STATEMENT

Consider the linear, time-invariant system with sampled outputs

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t)+Df_i(t) \\ y(t) &= Cx_d(t_k), \quad t_k \le t < t_{k+1} \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are the state and the input vector respectively and $y \in \mathbb{R}^p$ represents discrete-time output measurements generated by zero-order hold functions with a sequence of hold times $0 = t_0 < t_1 \cdots < t_k < \cdots$, where $\lim_{k\to\infty} t_k = \infty$. $f_i \in \mathbb{R}^q$ represents an unknown actuator fault which is assumed to be bounded by $\|f_i(t)\| \le \Delta$. It is assumed $q \le p < n$ and A, B, C, D are constant matrices of appropriate dimensions. Following the approach in [18], [9], system (1) with sampled output can be presented as a continuous-time system with a known output measurement delay

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t)+Df_i(t) \\ y(t) &= Cx(t-\tau(t)), \quad t\in[t_k,t_{k+1}), \quad \tau(t) = t-t_k \end{aligned} \tag{2}$$

Assume that $t_{k+1}-t_k \le h$, $\forall\ k \ge 0$, i.e. the time between any two sequential sampling times is not greater than some pre-chosen $h > 0$, then $\tau(t) \in (0,h]$ with $\dot{\tau}(t) = 1$ for $t \ne t_k$ is known. It is assumed that

1) rank $(CD) = q$;
2) any invariant zeros of (A,D,C) lie in the left half plane.

Under these assumptions and using the same linear change of coordinates as in [4], where $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T_0 x$, the system (2) can be transformed into:

$$\begin{aligned} \dot{x}_1(t) &= A_{11}x_1(t)+A_{12}x_2(t)+B_1u(t) \\ \dot{x}_2(t) &= A_{21}x_1(t)+A_{22}x_2(t)+B_2u(t)+D_1f_i(t) \\ y(t) &= Tx_2(t-\tau(t)) \end{aligned} \tag{3}$$

where $x_1 \in \mathbb{R}^{n-p}$, $x_2 \in \mathbb{R}^p$, $D_1 = \begin{bmatrix} 0 \\ \bar{D}_1 \end{bmatrix}$, $\bar{D}_1 \in \mathbb{R}^{q\times q}$, A_{11} has stable eigenvalues and T is an orthogonal matrix. An observer will be designed which, for sufficiently large t, induces motion in the $h\Delta$-neighbourhood of the surface

$$\mathscr{E} = \{x_2, \hat{x}_2 \in \mathbb{R}^p \ :\ s_e(t) = T\big(x_2(t-\tau(t)) - \hat{x}_2(t-\tau(t))\big) = 0\} \tag{4}$$

where $\hat{x}_2(t-\tau(t))$ is the corresponding component of the estimated states from an observer to be designed. An ideal sliding mode can be achieved with $h = 0$ under assumptions 1, 2.

III. OBSERVER DESIGN

Consider an observer of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t)+Bu(t)-G_l\bar{e}_2(t-\tau(t))+G_n v(t-\tau(t)) \\ \hat{y}(t) &= C\hat{x}_d(t_k), \quad t_k \le t < t_{k+1} \end{aligned} \tag{5}$$

where $G_l \in \mathbb{R}^{n\times p}$, $G_n \in \mathbb{R}^{n\times p}$ and $\bar{e}_2(t) = T\big(x_2(t)-\hat{x}_2(t)\big)$. The discontinuous injection term v is given by

$$v(t) = -(\|TD_1\|+\delta)\Delta[\operatorname{sign}\bar{e}_{2_1}(t),\ldots,\operatorname{sign}\bar{e}_{2_p}(t)]^T \tag{6}$$

where $\delta > 0$ is a positive number. Assuming there exists an $L \in \mathbb{R}^{(n-p)\times p}$ which has the form $L = \begin{bmatrix} \bar{L} & 0 \end{bmatrix}$ with $\bar{L} \in \mathbb{R}^{(n-p)\times(p-q)}$ such that the linear change of co-ordinates T_0 to the observer (5) has the form

$$\begin{aligned} &\dot{\hat{x}}_1(t) = A_{11}\hat{x}_1(t)+A_{12}\hat{x}_2(t)+B_1u(t) \\ &-(\tfrac{1}{\mu}L+A_{11}L)(x_2(t-\tau(t))-\hat{x}_2(t-\tau(t)))+LT^Tv(t-\tau(t)) \\ &\dot{\hat{x}}_2(t) = A_{21}\hat{x}_1(t)+A_{22}\hat{x}_2(t)+B_2u(t) \\ &-(A_{21}L-\tfrac{1}{\mu}I_p)(x_2(t-\tau(t))-\hat{x}_2(t-\tau(t))-T^Tv(t-\tau(t)) \\ &\hat{y}(t) = T\hat{x}_2(t-\tau(t)) \end{aligned} \tag{7}$$

where

$$G_l = T_0^{-1}\begin{bmatrix} \frac{1}{\mu}L+A_{11}L \\ A_{21}L-\frac{1}{\mu}I_p \end{bmatrix}, \quad G_n = T_0^{-1}\begin{bmatrix} LT^T \\ -T^T \end{bmatrix} \tag{8}$$

with $\mu > 0$. Defining the state estimation error as $e_1(t) = x_1(t)-\hat{x}_1(t)$ and $e_2(t) = x_2(t)-\hat{x}_2(t)$, it is obtained that

$$\begin{aligned} &\dot{e}_1(t) = A_{11}e_1(t)+A_{12}e_2(t) \\ &+L\big(\tfrac{1}{\mu}e_2(t-\tau(t))-T^Tv(t-\tau(t))\big)+A_{11}Le_2(t-\tau(t)) \\ &\dot{e}_2(t) = A_{21}e_1(t)+A_{22}e_2(t)+D_1f_i(t) \\ &+T^Tv(t-\tau(t))-(\tfrac{1}{\mu}I_p-A_{21}L)e_2(t-\tau(t)) \end{aligned} \tag{9}$$

A change of coordinates exists such that $\begin{bmatrix} \bar{e}_1(t) \\ \bar{e}_2(t) \end{bmatrix} = T_L\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$ with $T_L = \begin{bmatrix} I_{n-q} & L \\ 0 & T \end{bmatrix}$. Since $LD_1 = 0$, one obtains

$$\begin{aligned} \dot{\bar{e}}_1(t) &= (A_{11}+LA_{21})\bar{e}_1(t)-(A_{11}L+LA_{21}L-A_{12} \\ &-LA_{22})T^T\bar{e}_2(t)+(A_{11}+LA_{21})\bar{e}_2(t-\tau(t)) \end{aligned} \tag{10}$$

$$\begin{aligned} \dot{\bar{e}}_2(t) &= TA_{21}\bar{e}_1(t)-(TA_{21}LT^T-TA_{22}T^T)\bar{e}_2(t) \\ &+TA_{21}LT^T\bar{e}_2(t-\tau(t))-\tfrac{1}{\mu}\bar{e}_2(t-\tau(t)) \\ &+v(t-\tau(t))+TD_1f_i(t) \end{aligned} \tag{11}$$

with initial condition

$$\bar{e}(t_0) = \bar{e}_0, \quad \bar{e}(t) = 0, \quad t < t_0 \tag{12}$$

The dynamics of the switching manifold is governed by equation (10), where (A_{11}, A_{21}) is detectable from assumptions 1, 2.

Lemma 1: Given scalars $\alpha > 0$, $b > 0$, if there exists an $(n-p)\times(n-p)$ matrix $P > 0$ and a matrix $Y \in \mathbb{R}^{(n-p)\times p}$ with last q columns zero, such that the LMI

$$\begin{bmatrix} PA_{11}+A_{11}^TP+YA_{21}+A_{21}^TY^T+\alpha P & -P \\ * & -bI \end{bmatrix} < 0 \tag{13}$$

holds, then the solution of (10) with $L = P^{-1}Y$ and with the initial condition (12) is bounded by

$$\begin{aligned} &\bar{e}_1^T(t)P\bar{e}_1(t) < e^{-\alpha(t-t_0)}\bar{e}_1^T(t_0)P\bar{e}_1(t_0)+\tfrac{b}{\alpha}\big(\|(A_{11}L+LA_{21}L \\ &-A_{12}-LA_{22})T^T\|^2+\|A_{11}+LA_{21}\|^2\big)\|\bar{e}_{2_{[t_0,t]}}(t)\|_\infty^2 \end{aligned} \tag{14}$$

The proof of the lemma follows analogously as in [10] by finding solutions for

$$\begin{aligned} &\dot{V}(t)+\alpha V(t)-b\big(\|(A_{11}L+LA_{21}L-A_{12}-LA_{22})T^T\|^2 \\ &\cdot\|\bar{e}_2(t)\|^2+\|A_{11}+LA_{21}\|^2\|\bar{e}_2(t-\tau(t))\|^2\big) < 0 \end{aligned}$$

where $V(t) = \bar{e}_1^T(t)P\bar{e}_1(t)$.

A. Input-to-state stability of the error dynamics: a singular perturbation approach

The closed-loop system (10), (11) can be expressed as

$$\dot{\bar{e}}_1(t) = \bar{A}_{11}\bar{e}_1(t) + \bar{A}_{12}\bar{e}_2(t) + \bar{A}_{11}LT^T\bar{e}_2(t-\mu\xi(t)) \tag{15}$$

$$\begin{aligned}\mu\dot{\bar{e}}_2(t) &= \mu\bar{A}_{21}\bar{e}_1(t) + \mu\bar{A}_{22}\bar{e}_2(t) + (\mu\bar{A}_{d22} - I_p)\\ &\cdot\bar{e}_2(t-\mu\xi(t)) + \mu\bar{f}_i(t)\end{aligned} \tag{16}$$

where $\bar{A}_{11} = A_{11} + LA_{21}$, $\bar{A}_{12} = -(A_{11}L + LA_{21}L - A_{12} - LA_{22})T^T$, $\bar{A}_{21} = TA_{21}$, $\bar{A}_{22} = -(TA_{21}LT^T - TA_{22}T^T)$, $\bar{A}_{d22} = TA_{21}LT^T$, $\mu\xi(t) = \tau(t)$, $\mu\bar{\xi} = h$, $0 \le \xi(t) \le \bar{\xi}$ and $\bar{f}_i(t) = v(t-\mu\xi(t)) + TD_1f_i(t)$, i.e. $\|\bar{f}_i(t)\| \le \big((\|TD_1\| + \delta)\sqrt{p} + \|TD_1\|\big)\Delta$. Let $P_\mu \in \mathbb{R}^{n\times n}$ be a positive definite matrix with the following structure [16]

$$P_\mu = \begin{bmatrix} P_1 & \mu P_3^T \\ * & \mu P_2 \end{bmatrix} > 0 \tag{17}$$

where $P_1 \in \mathbb{R}^{n-p}$, and choose the Lyapunov-Krasovskii functional designed for sampled data system [11]:

$$V(t) = \bar{e}(t)^T P_\mu \bar{e}(t) + (h - \mu\xi(t))\int_{t-\mu\xi(t)}^{t} e^{\bar{\alpha}(s-t)}\dot{\bar{e}}_2(s)U\dot{\bar{e}}_2(s)ds \tag{18}$$

with respect to the error dynamics (15), (16), where $U \in \mathbb{R}^p$ is a positive matrix, then following lemma can be stated:

Lemma 2: Given positive scalars μ, $\bar{\xi}$, $\bar{\alpha}$ and $\bar{b}$, let there exist a $n\times n$ matrix $P_\mu > 0$ in (17), $p\times p$ matrices $U > 0$, P_4, P_5 and $(n-p)\times(n-p)$ matrices P_6, P_7 such that the following LMIs

$$\Theta_{\mu 0} = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \mu\bar{A}_{21}^T P_5 + \mu P_3^T & 0 \\ * & \theta_{22} & \theta_{23} & \theta_{24} & P_4^T \\ * & * & -P_7^T & 0 & 0 \\ * & * & * & -\mu P_5^T + \mu\bar{\xi}U & P_5^T \\ * & * & * & * & -\bar{b}I \end{bmatrix} < 0 \tag{19}$$

$$\Theta_{\mu 1} = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \mu\bar{A}_{21}^T P_5 + \mu P_3^T & -\mu\bar{\xi}P_6^T\bar{A}_{11}LT^T & 0 \\ * & \theta_{22} & \theta_{23} & \theta_{24} & -\mu\bar{\xi}P_4^T(\mu\bar{A}_{d22} - I_p) & P_4^T \\ * & * & -P_7^T & 0 & -\mu\bar{\xi}P_7^T\bar{A}_{11}LT^T & 0 \\ * & * & * & -\mu P_5^T & -\mu\bar{\xi}P_5^T(\mu\bar{A}_{d22} - I_p) & P_5^T \\ * & * & * & * & -\mu\bar{\xi}e^{-\bar{\alpha}\mu\bar{\xi}}U & 0 \\ * & * & * & * & * & -\bar{b}I \end{bmatrix} < 0 \tag{20}$$

where

$$\begin{aligned}
\theta_{11} &= P_6^T\bar{A}_{11} + \bar{A}_{11}^T P_6 + \bar{\alpha}P_1,\\
\theta_{12} &= \mu\bar{A}_{21}^T P_4 + \bar{\alpha}\mu P_3^T + P_6^T(\bar{A}_{12} + \bar{A}_{11}LT^T)\\
\theta_{13} &= -P_6^T + P_1 + \bar{A}_{11}^T P_7\\
\theta_{22} &= \mu P_4^T\bar{A}_{22} + \mu\bar{A}_{22}^T P_4 + P_4^T(\mu\bar{A}_{22} - I_p)\\
&\quad + (\mu\bar{A}_{22}^T - I_p)P_4 + \mu\bar{\alpha}P_2,\\
\theta_{23} &= \mu P_3 + (\bar{A}_{12} + \bar{A}_{11}LT^T)^T P_7\\
\theta_{24} &= \mu\bar{A}_{22}^T P_5 + (\mu\bar{A}_{22}^T - I_p)P_5 + \mu P_2 - \mu P_4^T
\end{aligned} \tag{21}$$

are feasible, then solutions of (10)-(11) with initial condition (12) satisfy the bound

$$\bar{e}^T(t)P_\mu\bar{e}(t) < e^{-\bar{\alpha}(t-t_0)}\bar{e}^T(t_0)P_\mu\bar{e}(t_0) + \tfrac{\mu^2\bar{b}}{\bar{\alpha}}\|\bar{f}_{i[t_0,t]}\|_\infty^2 \tag{22}$$

for all $\mu\xi(t) \in [0,\ h]$ with $\mu\dot{\xi}(t) = 1$, thus (10)-(11) is input-to-state stable.

Proof: The following inequality

$$W(t) = \tfrac{d}{dt}V(t) + \bar{\alpha}V(t) - \mu^2\bar{b}\bar{f}_i^T(t)\bar{f}_i(t) < 0 \tag{23}$$

along the trajectories of (10), (11) for $\|\bar{e}_0\|^2 + \|\bar{f}_{i[t_0,t]}\|_\infty^2 > 0$ guarantees (22) [10]. Differentiating V of the structure (17), (18) along (15), (16), and analogously to [11], it follows

$$\begin{aligned}
W(t) &\le 2\big(\bar{e}_1^T(t)P_1\dot{\bar{e}}_1(t) + \bar{e}_2^T(t)\mu P_3\dot{\bar{e}}_1(t) + \mu\bar{e}_1^T(t)P_3^T\dot{\bar{e}}_2(t)\\
&+ \mu\bar{e}_2^T(t)P_2\dot{\bar{e}}_2(t)\big) + \bar{\alpha}\big(\bar{e}_1^T(t)P_1\bar{e}_1(t) + \bar{e}_2^T(t)\mu P_3\bar{e}_1(t)\\
&+ \mu\bar{e}_1^T(t)P_3^T\bar{e}_2(t) + \mu\bar{e}_2^T(t)P_2\bar{e}_2(t)\big) - e^{-\bar{\alpha}h}v_1^T U v_1\\
&+ (h - \mu\xi(t))\dot{\bar{e}}_2^T(t)U\dot{\bar{e}}_2(t) - \mu^2\bar{b}\bar{f}_i^T(t)\bar{f}_i(t)
\end{aligned} \tag{24}$$

where $v_1 = \frac{1}{\mu\xi(t)}\int_{t-\mu\xi(t)}^{t}\dot{\bar{e}}_2(s)ds$. Apply the descriptor method [8], the right-hand sides of the expressions

$$\begin{aligned}
0 &= 2[\bar{e}_2^T(t)P_4^T + \dot{\bar{e}}_2^T(t)P_5^T][\mu\bar{A}_{21}\bar{e}_1(t) + \mu\bar{A}_{22}\bar{e}_2(t) + (\mu\bar{A}_{d22}\\
&- I_p)\bar{e}_2(t) - \mu\xi(t)(\mu\bar{A}_{d22} - I_p)v_1 + \mu\bar{f}_i(t) - \mu\dot{\bar{e}}_2(t)]\\
0 &= 2[\bar{e}_1^T(t)P_6^T + \dot{\bar{e}}_1^T(t)P_7^T][\bar{A}_{11}\bar{e}_1(t) + (\bar{A}_{12} + \bar{A}_{11}LT^T)\bar{e}_2(t)\\
&- \mu\xi(t)\bar{A}_{11}LT^T v_1 - \dot{\bar{e}}_1(t)]
\end{aligned} \tag{25}$$

with some $p\times p$-matrices P_4, P_5 and $(n-p)\times(n-p)$ matrices P_6, P_7 are added into the right-hand side of (24). Setting $\eta_1(t) = col\{\bar{e}_1(t),\ \bar{e}_2(t),\ \dot{\bar{e}}_1(t),\ \dot{\bar{e}}_2,\ v_1,\ \mu\bar{f}_i(t)\}$, it follows

$$\dot{V}(t) + \bar{\alpha}V(t) - \mu^2\bar{b}\bar{f}_i^T(t)\bar{f}_i(t) \le \eta_1^T(t)\Theta_\mu\eta_1(t) < 0 \tag{26}$$

if the following matrix inequality is feasible:

$$\Theta_\mu = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \mu\bar{A}_{21}^T P_5 + \mu P_3^T & -\mu\xi(t)P_6^T\bar{A}_{11}LT^T & 0 \\ * & \theta_{22} & \theta_{23} & \theta_{24} & -\mu\xi(t)P_4^T(\mu\bar{A}_{d22} - I_p) & P_4^T \\ * & * & -P_7^T & 0 & -\mu\xi(t)P_7^T\bar{A}_{11}LT^T & 0 \\ * & * & * & -\mu P_5^T + (h - \mu\xi(t))U & -\mu\xi(t)P_5^T(\mu\bar{A}_{d22} - I_p) & P_5^T \\ * & * & * & * & -\mu\xi(t)e^{-\bar{\alpha}h}U & 0 \\ * & * & * & * & * & -\bar{b}I \end{bmatrix} < 0 \tag{27}$$

The latter matrix inequality for $\mu\xi(t) \to 0$ and $\mu\xi(t) \to h$, leads to the LMIs (19), (20). Setting $\eta_0(t) = col\{\bar{e}_1(t),\ \bar{e}_2(t),\ \dot{\bar{e}}_1(t),\ \dot{\bar{e}}_2,\ \mu\bar{f}_i(t)\}$, then the following holds

$$\frac{h - \tau(t)}{h}\eta_0^T\Theta_{\mu 0}\eta_0 + \frac{\tau(t)}{h}\eta_1^T\Theta_{\mu 1}\eta_1 = \eta_1^T\Theta_\mu\eta_1 < 0, \quad \forall\ \eta_1 \neq 0$$

■

B. LMIs for switching gain design

Conditions will now be derived that guarantee the following bound for the solutions of (11):

$$\begin{aligned}
\limsup_{t\to\infty}\left\|\begin{bmatrix}\bar{A}_{21} & \bar{A}_{22}\end{bmatrix}\bar{e}(t)\right\| &\le k_1\delta\Delta,\\
\limsup_{t\to\infty}\left\|\begin{bmatrix}0 & \bar{A}_{d22}\end{bmatrix}\bar{e}(t - \tau(t))\right\| &\le k_2\delta\Delta
\end{aligned} \tag{28}$$

with some $k_1, k_2 \ge 0$ such that $k_1 + k_2 = 1$. Taking into account (22) it can be concluded that (28) holds if the following inequalities are satisfied for $t \to \infty$:

$$\mu^2\bar{e}^T(t)[\bar{A}_{21}\ \bar{A}_{22}]^T[\bar{A}_{21}\ \bar{A}_{22}]\bar{e}(t) < \frac{\bar{\alpha}\bar{e}^T(t)P_\mu\bar{e}(t)k_1^2\delta^2}{\bar{b}\Big((\|TD_1\| + \delta)\sqrt{p} + \|TD_1\|\Big)^2}$$

$$\begin{aligned}
&\mu^2\bar{e}^T(t - \mu\xi(t))[0\ \bar{A}_{d22}]^T[0\ \bar{A}_{d22}]\bar{e}(t - \mu\xi(t))\\
&\quad < \frac{\bar{\alpha}\bar{e}^T(t - \mu\xi(t))P_\mu\bar{e}(t - \mu\xi(t))k_2^2\delta^2}{\bar{b}\Big((\|TD_1\| + \delta)\sqrt{p} + \|TD_1\|\Big)^2}
\end{aligned}$$

Hence, the inequalities

$$\begin{aligned}
&\begin{bmatrix} -k_1^2M_1P_1 & -\mu k_1^2M_1P_3^T & \mu\bar{A}_{21}^T \\ * & -\mu k_1^2M_1P_2 & \mu\bar{A}_{22}^T \\ * & * & -I_p \end{bmatrix} < 0\\
&\begin{bmatrix} -k_2^2M_1P_1 & -\mu k_2^2M_1P_3^T & 0 \\ * & -\mu k_2^2M_1P_2 & \mu\bar{A}_{d22}^T \\ * & * & -I_p \end{bmatrix} < 0
\end{aligned} \tag{29}$$

where $M_1 = \frac{\bar{\alpha}\delta^2}{\bar{b}\Big((\|TD_1\| + \delta)\sqrt{p} + \|TD_1\|\Big)^2}$, guarantee that the solutions of (10), (11) satisfy the bound (28).

C. Ultimate boundedness of the error dynamics

Let $\phi(t,t_0,\mu)$ be the fundamental solution of the equation

$$\mu\dot{z}(t) = -z(t-\mu\xi(t)), \quad z(t)\in\mathbb{R} \tag{30}$$

with $\phi(t_0,t_0,\mu)=1$ and $\phi(t,t_0,\mu)=0$ for $t<t_0$. It is shown in [11] that (30) remains exponentially stable for all variable delays $\mu\xi(t)\le 1.99$. Then the following bound holds

$$\|\phi(t,t_0,\mu)\| \le e^{-\frac{\alpha_2(t-t_0)}{\mu}} \tag{31}$$

for small enough $\alpha_2>0$ and $\forall\,\mu>0,\ \mu\xi(t)\le h,\ \mu\dot{\xi}(t)=1$. Main results may now be stated:

Theorem 1: Given positive constants $\mu,\ \bar{\xi},\ \bar{\alpha},\ \bar{b}$ and $k_1,\ k_2$, let there exist a $n\times n$-matrix $P_\mu>0$, positive $p\times p$-matrices $U>0$ and $p\times p$ matrices $P_4,\ P_5$, $(n-p)\times(n-p)$ matrices $P_6,\ P_7$ such that LMIs (17), (19), (20) and (29) are feasible. Let $\bar{e}(t)$ be a solution to (10), (11), then every component of $\bar{e}_2(t)$ satisfies the bound

$$\limsup_{t\to\infty}|\bar{e}_{2_i}(t)| \le 2M_0\mu\bar{\xi} \tag{32}$$

where $M_0 = 2(\delta+\|TD_1\|)\Delta$, $i=1,\dots,p$ denotes the i-th component of $\bar{e}_2$ for all $\mu\xi(t)\in[0,\ h]$ with $\mu\dot{\xi}(t)=1$.

Proof: The i-th component of differential equation (11) with the initial condition (12) can be represented in the form of an integral equation [17]

$$\begin{aligned}&\bar{e}_{2_i}(t) = \phi(t,t_0,\mu)\bar{e}_{2_i}(t_0)+\textstyle\int_{t_0}^{t}\phi(t,s,\mu)\Big[[\bar{A}_{21_i}\ \bar{A}_{22_i}]\bar{e}(s)\\&+\bar{A}_{d22_i}\bar{e}_2(s-\mu\xi(t))+(TD_1)_i f_i(s)\\&-(\|TD_1\|+\delta)\Delta\,\mathrm{sign}\,\bar{e}_{2_i}(s-\mu\xi(s))\Big]ds\end{aligned} \tag{33}$$

The feasibility of (29) implies the bound (28), then the following inequality holds for $t\to\infty$:

$$\begin{aligned}&|[\bar{A}_{21_i}\ \bar{A}_{22_i}]\bar{e}(s)+\bar{A}_{d22_i}\bar{e}_2(s-\mu\xi(t))+(TD_1)_i f_i(s)\\&-(\|TD_1\|+\delta)\Delta\,\mathrm{sign}\,\bar{e}_{2_i}(s-\mu\xi(s))|<M_0\end{aligned} \tag{34}$$

Taking into account (31) and (34), it is established from (33) that for $t\to\infty$

$$\begin{aligned}&|\bar{e}_{2_i}(t+\theta)-\bar{e}_{2_i}(t)| \le \big|\textstyle\int_{t+\theta}^{t}\phi(t,s,\mu)\Big([\bar{A}_{21_i}\ \bar{A}_{22_i}]\bar{e}(s)\\&+\bar{A}_{d22_i}\bar{e}_2(s-\mu\xi(t))+(TD_1)_i f_i(s)\\&-(\|TD_1\|+\delta)\Delta\,\mathrm{sign}\,\bar{e}_{2_i}\big(s-\mu\xi(s)\big)\Big)ds\big|\\&< M_0\textstyle\int_{t+\theta}^{t}e^{\frac{\alpha_2(t-s)}{\mu}}ds < \mu M_0\frac{1-e^{2\alpha_2 h}}{\alpha_2}\\&\le 2M_0\mu\bar{\xi}\end{aligned}$$

where $\theta\in[-2\mu\bar{\xi},\ 0]$. Therefore,

$$\bar{e}_{2_i}(t)-2M_0\mu\bar{\xi} < \bar{e}_{2_i}(t+\theta) < \bar{e}_{2_i}(t)+2M_0\mu\bar{\xi} \tag{35}$$

for $t\to\infty$ and the following implication holds

$$|\bar{e}_{2_i}(t)|\ge 2M_0\mu\bar{\xi} \Rightarrow \mathrm{sign}\,\bar{e}_{2_i}(t+\theta)=\mathrm{sign}\,\bar{e}_{2_i}(t) \tag{36}$$

for large enough t. Thus, from (28), (34) and (36) for sufficiently large t the following holds:

$$\begin{aligned}&|\bar{e}_{2_i}(t)|\ge 2M_0\mu\bar{\xi}\Rightarrow\\&\bar{e}_{2_i}^T(t)\big[[\bar{A}_{21_i}\ \ \bar{A}_{22_i}]\bar{e}(t+\theta)+\bar{A}_{d22_i}\bar{e}_2(t-\mu\xi(t)+\theta)\\&+(TD_1)_i f_i(t+\theta)-(\|TD_1\|+\delta)\Delta\,\mathrm{sign}\,\bar{e}_{2_i}(t+\theta)\big]\\&<|\bar{e}_{2_i}(t)|\big(|[\bar{A}_{21_i}\ \ \bar{A}_{22_i}]\bar{e}(t+\theta)|+|[0\ \ \bar{A}_{d22_i}]\bar{e}(t-\mu\xi(t)+\theta)|\\&+\|TD_1\|\Delta\big)-(\|TD_1\|+\delta)\Delta|\bar{e}_{2_i}(t)|\\&<0\end{aligned} \tag{37}$$

It will be shown next that the $\bar{e}_{2_i}$-component of the solutions to (11) exponentially converges to the ball (32). Moreover, for sufficiently large t, whenever $\bar{e}_{2_i}(t)$ achieves the ball (32), it will never leave it. Taking into account (37), for sufficiently large t it follows that

$$\begin{aligned}&|\bar{e}_{2_i}(t)|\ge 2M_0\mu\bar{\xi}\Rightarrow\\&\tfrac{d}{dt}\mu\bar{e}_{2_i}^2(t)=2\mu\bar{e}_{2_i}(t)\dot{\bar{e}}_{2_i}(t)\\&=2\bar{e}_{2_i}(t)\big[-\bar{e}_{2_i}(t-\mu\xi(t))+\mu\big([\bar{A}_{21_i}\ \ \bar{A}_{22_i}]\bar{e}(t)+[0\ \ \bar{A}_{d22_i}]\\&\cdot\bar{e}(t-\mu\xi(t))+(TD_1)_i f_i(t)-(\|TD_1\|+\delta)\Delta\,\mathrm{sign}\,\bar{e}_{2_i}(t)\big)\big]\\&\le -2\bar{e}_{2_i}(t)\big(\bar{e}_{2_i}(t)-\textstyle\int_{t-\mu\xi(t)}^{t}\dot{\bar{e}}_{2_i}(s)ds\big)\\&=-2\bar{e}_{2_i}^2(t)+2\bar{e}_{2_i}(t)\textstyle\int_{t-\mu\xi(t)}^{t}\Big[-\frac{\bar{e}_{2_i}(s-\mu\xi(t))}{\mu}ds\\&+[\bar{A}_{21_i}\ \ \bar{A}_{22_i}]\bar{e}(s)+[0\ \ \bar{A}_{d22_i}]\bar{e}(s-\mu\xi(t))+(TD_1)_i f_i(s)\\&-(\|TD_1\|+\delta)\Delta\,\mathrm{sign}\,\bar{e}_{2_i}(s)\Big]ds\\&\le -2\bar{e}_{2_i}^2(t)-2\frac{\bar{e}_{2_i}(t)}{\mu}\textstyle\int_{t-\mu\xi(t)}^{t}\bar{e}_{2_i}(s-\mu\xi(t))ds\end{aligned}$$

Therefore, given (36) holds for large enough t, it follows that

$$-\int_{t-\mu\xi(t)}^{t}\bar{e}_{2_i}(t)\bar{e}_{2_i}(s-\mu\xi(t))ds\le 0$$

Hence

$$|\bar{e}_{2_i}(t)|\ge 2M_0\mu\bar{\xi}\Rightarrow\frac{d}{dt}\mu\bar{e}_{2_i}^2(t)\le -2\bar{e}_{2_i}^2(t) \tag{38}$$

Assume now that for large enough t_1 the $\bar{e}_{2_i}$ component of the solution to (1) is outside the ball (32). Then from (38) it follows that for all $t\ge t_1$ such that $|\bar{e}_{2_i}(t)|\ge 2M_0\mu\bar{\xi}$ then

$$\bar{e}_{2_i}^2(t)\le e^{-\frac{2}{\mu}(t-t_1)}\bar{e}_{2_i}^2(t_1) \tag{39}$$

i.e. $\bar{e}_{2_i}$ exponentially converges to the ball (32). Let $t_2>t_1$ is the time when $|\bar{e}_{2_i}(t_2)|=2M_0\mu\bar{\xi}$. Then due to (38) $\bar{e}_{2_i}^2(t_2^+)<\bar{e}_{2_i}^2(t_2)$. Therefore, whenever $\bar{e}_{2_i}(t)$ attains the ball (32), it will never leave it. ■

IV. Reconstruction of the input and output fault signals

The fault reconstruction properties of the observer designed above are now considered. Effectively this extends the presentation in [5] to consider the effect of the sampled output. For sufficiently small μ, (16) becomes

$$0\approx -\frac{1}{\mu}\bar{e}_2(t-\tau(t))+v(t-\tau(t))+TD_1 f_i(t) \tag{40}$$

Since $rank(D_1)=q$ it follows from (40) that

$$f_i(t)\approx\big((TD_1)^T(TD_1)\big)^{-1}(TD_1)^T\big(\frac{1}{\mu}\bar{e}_2(t-\tau(t))-v(t-\tau(t))\tag{41}$$

To reconstruct the fault signal f_i, [5] proposed to replace the discontinuous component $v(t)$ by the continuous approximation

$$v_r = -(\|TD_1\| + \delta)\Delta[\frac{\bar{e}_{2_1}}{|\bar{e}_{2_1}| + r}, \ldots, \frac{\bar{e}_{2_p}}{|\bar{e}_{2_p}| + r}]^T \tag{42}$$

where $r \geq 0$ is chosen to be small enough. Now consider the case when $f_i = 0$ and consider the effect of a fault $f_0(t)$ at the output. In this situation, x_2 is replaced by $x_2 \rightarrow x_2 + f_0$ and $e_y = e_2 + f_0$. For sufficiently small μ, it can be obtained from (9) that

$$\begin{aligned} &e_1(t) \approx -A_{11}^{-1}(\tfrac{L}{\mu} + A_{11}L)f_0 \\ &\dot{e}_y(t) = A_{21}e_1(t) + A_{22}e_y(t) - A_{22}f_0 \\ &+(A_{21}L - \tfrac{I_p}{\mu})e_y(t-\tau(t)) + T^T v(t-\tau(t)) + \dot{f}_0 \end{aligned} \tag{43}$$

The fault can be approximated by

$$f_0 \approx W^{-1}\left((A_{22}L - \tfrac{I_p}{\mu})e_y(t-\tau(t)) + T^T v(t-\tau(t))\right) \tag{44}$$

if $W = A_{21}A_{11}^{-1}(\frac{L}{\mu} + A_{11}L) + A_{22}$ is invertible.

Remark 1: Fault reconstruction using sliding mode technique usually requires an ideal sliding motion to be attained in finite time [5], [24]. Practically, due to model uncertainties and sampled output effects for example, an ideal sliding motion in the observer does not usually appear. Instead, motion is bounded within a region of the sliding surface. This paper uses a singular perturbation approach for fault reconstruction under sampled outputs for which, by choosing a sufficiently small μ, the fault can be approximated depending only on the output error.

V. Example

An inverted pendulum system is considered as in [5] which is linearized about the equilibrium at the origin

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1.9333 & -1.9872 & 0.0091 \\ 0 & 36.9771 & 6.2589 & -0.1738 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ 0 \\ 0.3205 \\ -1.0095 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned} \tag{45}$$

A compensator approach from [14] is designed to stabilize the pendulum. It is assumed that $D = B$ and an input fault is bounded by $\|f_i\| \leq \Delta = 2$. The sampled data outputs are implemented in the simulation using the zero-order-hold function. In the LMI (13), $L = [0\ 1.526\ 0]$ is obtained. LMIs (19) and (20) are feasible with $\bar{\alpha} = 8$, $\mu = 0.019$, $\bar{\xi} = 0.524$, i.e. the sampling period is given by $\mu\bar{\xi} = 0.01s$. LMI (29) is feasible with $\delta = 77$ and $k_1 = 0.8$, $k_2 = 0.2$. Hence the observer (5) with gains in (8) and (6) has been chosen which ensures the error variable is bounded in the range $|e_{2_i}(t)| \leq 6.2$ according to the estimate (32). Figure 1 is plotted using the sign function where every error variable is stabilized into a bound $|e_{2_i}| \leq 1.1$, which is within the estimate. Note that the high degree of switching is acceptable for an observer error signal; this is not present in the reconstruction of the fault signals.

Suppose the input fault is $f_i(t) = 2sin(t)$, while the output fault $f_o = 0$. The fault is reconstructed in Figure 2(a) according to (41). From the fault distribution structure

$$W = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -4.77 & -3.15 \\ 0 & -1.97 & -2.01 \end{bmatrix}$$

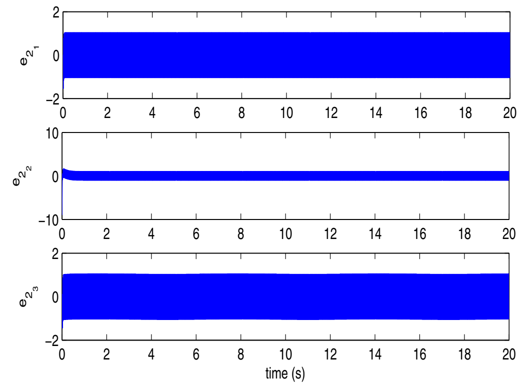


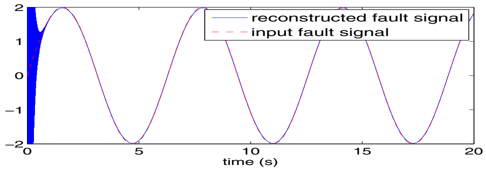
Fig. 1. Error response e_2 with sampling period $h = 0.01s$ at outputs.

in (44), only the third output fault can be reconstructed reliably despite the appearance of the input fault and other output faults. It can be approximated as the equivalent injection signal at the first channel of $(A_{22}L - \frac{I_p}{\mu})e_y(t-\tau(t)) + T^T v(t-\tau(t))$. Suppose the third output fault $f_{0_3} = 5sin(t)$, the fault signal is reconstructed accurately in Figure 2(b) under the sampled output with sampling period $h = 0.01s$. The observer preserves the construction accuracy even when it is operating under a larger sampling time $h = 0.03s$ as seen in Figure 4(a).

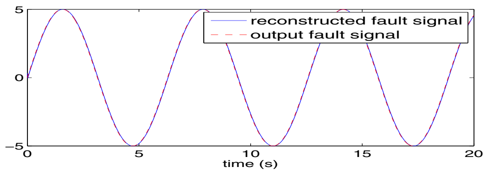
The observer in [5], which did not incorporate the effect of output sampling in the design, will now be compared to benchmark the results obtained in this paper. It is seen that the corresponding fault reconstruction shown in Figure 3 is achieved with lower accuracy in this case. Increase in the sampling period produces a decrease in the accuracy of the fault reconstruction. This can be demonstrated in Figure 4(b) where the input fault is reconstructed with sampling time $h = 0.03s$. The proposed method of observer design is shown to have significant advantages when compared with the classical approach if the output is sampled. It should be noted that the $\bar{e}_2$ term in (41) is pertinent to the reconstruction accuracy. For the observer in [5], the equivalent term is assumed to be zero.

VI. Conclusion

This paper develops an observer design framework for systems with multiple outputs where the outputs are sampled and thus the output error signal used to drive the observer is subject to delay. A singular perturbation approach is employed for the analysis which guarantees the ultimate bound on the error dynamics is proportional to the sampling time and the switching gain. A corresponding fault reconstruction technique is proposed which finds a sufficiently small value of the singular perturbation parameter, μ, for which the fault can be reconstructed reliably when the measured output is subject to sampling. The results are obtained by presenting the sampled output system in the form of continuous delayed system, and thus many existing advances in the continuous time domain can be employed to solve problems relating to model uncertainties and related robustness problems within

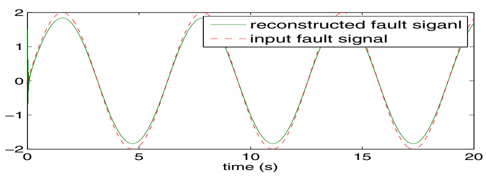


(a) Input fault reconstruction

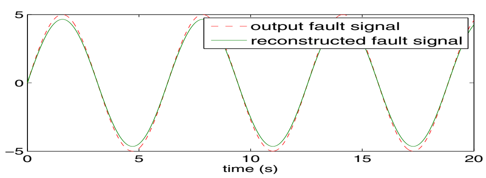


(b) Output fault reconstruction in the third component of the output vector

Fig. 2. Fault reconstruction using the proposed observer scheme with output sampling period $h = 0.01s$, as assumed for the observer design

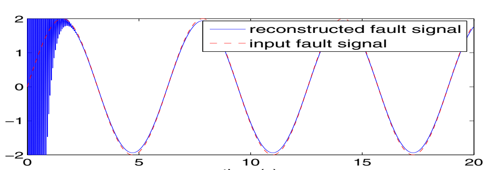


(a) Input fault reconstruction

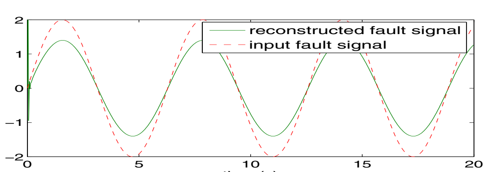


(b) Output fault reconstruction in the third component of the output vector

Fig. 3. Fault reconstruction using a classical sliding mode observer [5]; implemented with output sampling period $h = 0.01s$



(a) Fault reconstruction using the proposed observer approach; the output sampling period is greater than that assumed in the design.



(b) Fault reconstruction using the classical observer designed with no a priori knowledge of output sampling characteristics [5]

Fig. 4. Input fault reconstruction with output sampling period $h = 0.03s$

the observer design. This could be a promising direction for future work.

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