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Local linear dynamics assignment in IDA-PBC for underactuated mechanical systems

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Abstract— **The problem of finding a set of design parameters in the well-known IDA-PBC approach for a class of nonlinear underactuated mechanical systems to realize desired time behavior of the closed loop Port-Hamiltonian system in a transparent way is considered. Using a local coordinate transformation, the effect of the homogeneous solution of the potential energy matching PDE is isolated. By comparison of desired local linear dynamics with the parametrized linearization of the closed loop dynamics a set of linear equations for the IDA-PBC design parameters is derived. Besides the possibility to assign predefined dynamics the definiteness check of the potential energy can be omitted. With an Acrobot-type mechanical system the design steps are illustrated and simulations validate the performance of the approach.**

Keywords: Nonlinear Control, Passivity Based Control, Underactuated Mechanical Systems.

I. Introduction

Underactuated mechanical systems are a challenging system class for nonlinear controller design. The fact that not all degrees of freedom are equipped with actuators results in restrictions with respect to achievable closed loop dynamics which are tackled in different ways. Besides techniques that depart from partial feedback linearization, see e.g. [11], [7], approaches which exploit and maintain the physical structure of the mechanical system are of major interest, e.g. the Controlled Lagrangian method [3] or its counterpart from the Hamiltonian viewpoint *Interconnection and Damping Assignment Passivity Based Control* (IDA-PBC) [12], [2].

In IDA-PBC the *structure* of the closed loop system is fixed to be Port-Hamiltonian, while the high number of unassigned design parameters and ansatz functions represents huge *design freedom*. This freedom is successively reduced by matching the original and the closed loop system with respect to potential and kinetic energy as well as dissipation. The remaining parameters are then used to adjust closed loop dynamic behavior, which due to the interdependence of different classes of design parameters is rather intransparent.

The *assignment of local linear dynamics* procedure has been presented, see e.g. [9], to simplify the application of IDA-PBC for input affine nonlinear systems. Desired closed loop dynamics *of the linearized system* is predefined by the designer. From the quadratic approximation of the achievable closed loop energy a system of linear equations is derived to fix the design parameters (while the solvability conditions of the occuring PDEs have to be respected). To sum up, properties of the linearization are exploited to find a parametrization of nonlinear IDA-PBC such that beyond local exponential stability a well defined estimate of the region of attraction (through the non-quadratic energy function) is achieved. Consider the recent work [5] where stabilizability of mechanical systems with IDA-PBC is related to the controllability of the linearization.

In Section II IDA-PBC for mechanical systems is reviewed and assumptions on the class of systems[1] are stated. The solution of the potential energy PDE is discussed applying an appropriate coordinate change. In Section III a linearized version of the transformation leads to an expression of the closed loop potential energy Hessian matrix which is used to match local linear dynamics. The result is a set of linear equations for the design parameters in the n DOF case which are solved successively. The stability statement is derived from a converse Lyapunov theorem for linear systems and standard IDA-PBC arguments. For 2 DOF systems an analytic expression of the nonlinear coordinate change is given. The exposition in Section III is a generalization of the preliminary results for 2 DOF systems [8] to the n DOF case in a more concise form. The controller design is illustrated for an Acrobot-type system in Section IV. Simulations underscore the quality of the nonlinear IDA-PBC controller with predefined local linear dynamics. Section V concludes the paper with an outlook to future work.

II. Preliminaries

A. IDA-PBC for mechanical systems

Some known facts and results are repeated to make the paper self-contained. A simple n DOF mechanical system in Hamiltonian formulation is described by

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & -R(q,p) \end{bmatrix} \begin{bmatrix} \nabla_q H(q,p) \\ \nabla_p H(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u \quad (1)$$

with $q \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$ the generalized coordinates and momenta, respectively, $H(q,p) = V(q) + \frac{1}{2} p^T \overline{M}(q) p$ the total energy (Hamiltonian), $M(q) \in \mathbb{R}^{n \times n}$ the positive definite mass matrix and $\overline{M}(q) = M^{-1}(q)$ its inverse[2], $R(q,p) \in \mathbb{R}^{n \times n}$ a positive semidefinite symmetric dissipation matrix, $G(q) \in \mathbb{R}^{n \times m}$ the input matrix and $u \in \mathbb{R}^m$ the vector of generalized forces/torques. We assume systems with underactuation degree one:

Assumption A.1: The input matrix has the form

$$G(q) = \begin{bmatrix} 0_{n-1}^T \\ G_\alpha(q) \end{bmatrix}, \qquad \text{rank } G_\alpha(q) = n-1, \quad (2)$$

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[1] Corresponding to the first class of systems discussed in [7]: rigid bodies connected in a tree structure with an unactuated pivot (an unactuated cyclic variable) while friction in actuated joints is neglected/compensated.

[2] A bar indicates the inverse of a matrix in the following.

i.e. $m = n-1$. $q_\nu := q_1$ is the *unactuated* coordinate. For simplicity and without loss of generality $G_\alpha = I_{n-1}$ is considered. The greek letters α and ν indicate "actuated" and "unactuated" quantities, respectively.

The goal of IDA-PBC is to transform (1) by static state feedback into a closed loop system of more general structure

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & J_1(q)\\ -J_1^T(q) & J_2(q,p) - R_2(q,p)\end{bmatrix}\begin{bmatrix}\nabla_q H_d(q,p)\\ \nabla_p H_d(q,p)\end{bmatrix} \quad (3)$$

with a (virtual) closed loop energy $H_d(q,p) = V_d(q) + \frac{1}{2}p^T\overline{M}_d(q)p$ where $M_d(q) > 0$ in a (sufficiently large) neighborhood of the desired equilibrium $(q^*, 0)$ and

$$\nabla V_d(q)|_{q^*} = 0, \qquad Q_d := \partial^2_{q^2} V_d(q)\big|_{q^*} > 0 \quad (4)$$

ensures that q^* is a strict minimum of $V_d(q)$. By

$$J_1(q) = \overline{M}(q)M_d(q) \quad (5)$$

the relation $\dot{q} = \overline{M}(q)p$ between generalized velocities and momenta in the closed loop system is preserved. $J_2(q,p) = -J_2^T(q,p)$ and $R_2(q,p) = R_2^T(q,p) \geq 0$ remain free design parameters. Considering the second rows of (1) and (3), each multiplied by a full rank left hand annihilator $G^\perp(q) \in \mathbb{R}^{(n-m)\times n}$ such that $G^\perp(q)G(q) = 0$, one gets the matching PDE (omitting the arguments)

$$\begin{aligned} G^\perp(-\nabla V - \nabla_q(\tfrac{1}{2}p^T\overline{M}p) - R\overline{M}p) = \quad (6)\\ = G^\perp(-J_1^T\nabla V_d - J_1^T\nabla_q(\tfrac{1}{2}p^T\overline{M}_d p) + (J_2 - R_2)\overline{M}_d p).\end{aligned}$$

This single equation can be split into three equations, which need to be valid at the same time, concerning closed loop potential energy, kinetic energy and the dissipation:

$$G^\perp\nabla V = G^\perp M_d\overline{M}\nabla V_d \quad (7)$$

$$G^\perp\nabla_q(p^T\overline{M}p) = G^\perp M_d\overline{M}\nabla_q(p^T\overline{M}_d p) \quad (8)$$

$$-G^\perp R\overline{M}p = G^\perp(J_2 - R_2)\overline{M}_d p. \quad (9)$$

In the following the unity row vector $G^\perp = e_1^T = e_\nu^T$ is taken as the simplest left hand annihilator of (2). The potential energy PDE consequently takes the form

$$\partial_{q_\nu} V(q) = m_{d,\nu}^T(q)\overline{M}(q)\nabla V_d(q) \quad (10)$$

where $m_{d,\nu}^T(q)$ is the first (unactuated) row of the symmetric closed loop mass matrix according to

$$M_d(q) = \begin{bmatrix} m_{d,\nu}^T(q)\\ M_{d,\alpha}(q)\end{bmatrix}, \qquad \begin{matrix} m_{d,\nu}^T(q) \in \mathbb{R}^{1\times n}\\ M_{d,\alpha}(q) \in \mathbb{R}^{(n-1)\times n}.\end{matrix} \quad (11)$$

While the solution of the scalar linear PDE (10) for the potential energy $V_d(q)$ is rather down-to-earth, kinetic energy matching is much more challenging, see e.g. [2], [13]. We assume to have found a solution $M_d(q)$ which satisfies (8).

Assumption A.2: The kinetic energy matching PDE (8) admits a full rank solution (11).

Observe that the first part of condition (4) can only be fulfilled if $(q^*, 0)$ is an admissible equilibrium, i.e. $\partial_{q_\nu} V(q)|_{q^*} = 0$. It can further be shown [8] that $\nabla V_d(q)|_{q^*} = 0$ is then realized by appropriately shaping the first order terms of the homogeneous part of the solution of (10).

Remark 1: Assuming underactuation degree *one* makes the solution of (10) quite tractable in contrast to the case $m < n-1$ where a *system* of linear PDEs with *non-constant* coefficients has to be examined for solvability and solved.

We consider kinematic chains which are only unactuated in the first link. For this class of systems the mass matrix $M(q)$ depends only on the actuated coordinates:

Assumption A.3: $M(q) = M(q_\alpha)$, $q_\alpha = [q_2, \ldots, q_n]^T$.

Under this assumption the kinetic energy PDE (8) becomes homogeneous, which allows for a construction of a solution $M_d(q_\alpha)$, see [13]. The kinetic energy PDE is trivially solved for $M_d = const.$, which will be assumed in the following.

Finally dissipation in the unactuated joints is excluded:

Assumption A.4: $e_1^T R(q,p) = 0^T$.

In this case equation (9) is met for $e_1^T(J_2(q,p) - R_2(q,p)) = 0^T$. We assume a block diagonal form with $n-1$ dimensional (skew-)symmetric matrices J_2' and R_2':

$$(J_2(q,p) - R_2(q,p)) = \text{diag}\{0, (J_2'(q,p) - R_2'(q,p))\}. \quad (12)$$

Remark 2: The presence of dissipation in the unactuated coordinates is a major obstacle in the application of IDA-PBC, see [6]. The lab example presented in this paper violates the dissipation condition therein for *arbitrary magnitudes* of damping in the unactuated joint. It can be shown that for the given configuration it is not possible to achieve (4) from the solution of (7) and at the same time $R_2 \geq 0$ such that the damping is matched according to (9) with $M_d(q) > 0$. There exist approaches to cope with dissipation in the unactuated coordinates like [1]. However the method proposed therein can not be applied to Acrobot-type systems like the one considered in Section IV.

B. Solution of the potential energy PDE

Under the above assumptions the potential energy PDE is

$$b(q) = \sum_{i=1}^{n} a_i(q_\alpha)\partial_{q_i} V_d(q) \quad (13)$$

with $b(q) = \partial_{q_\nu} V(q)$ and $a_i(q_\alpha) = m_{d,\nu}^T(q_\alpha)\overline{m}_i(q_\alpha)$, $i = 1, \ldots, n$. $\overline{m}_i$ denotes the i-th column of $\overline{M}$. This inhomogeneous first order linear PDE admits a solution of the form

$$V_d(q) = \Psi(q) + \Phi(z_\alpha) \quad (14)$$

where $\Psi(q)$ solves (13) and $\Phi(z_\alpha)$ is a solution of the homogeneous PDE

$$0 = a^T(q_\alpha)\nabla V_d(q), \quad (15)$$

with $a^T(q_\alpha) = [a_1(q_\alpha) \ \ldots \ a_n(q_\alpha)]$ the vector of coefficient functions of the PDE.

The *characteristic coordinates* z_α in which the energy (14) can be shaped freely are invariants of the corresponding system of *characteristic ODEs*

$$\dot{q}_i = a_i(q_\alpha), \quad i = 1, \ldots, n, \quad (16)$$

or $\dot{q} = a(q_\alpha)$, i.e.

$$\partial_q z_{\alpha,i}(q)a(q_\alpha) = 0, \quad i, \ldots, n-1 \quad (17)$$

holds. Together with an appropriate function $z_\nu(q)$ such that

$$\partial_q z_\nu(q) a(q_\alpha) = 1 \tag{18}$$

a coordinate transformation $z = [z_{\alpha,1} \dots z_{\alpha,n-1}\, z_\nu]^T = t(q)$ is defined (locally) rendering (13) an ODE

$$\tilde{b}(z) = \partial_{z_\nu} \tilde{V}_d(z), \tag{19}$$

which can be integrated. Determining the coordinate change $z = t(q)$ (i.e. solving the PDEs (17),(18)) demands in general the use of computer algebra. For 2 DOF $z = t(q)$ can be easily expressed, see Section III-E.

Having solved the matching PDE the control law is extracted comparing the actuated parts of (1) and (3).

C. Local linear dynamics assignment

In *local linear dynamics assignment* for input affine systems, see e.g. [9], [10], [8], the IDA-PBC design parameters are determined solving a (usually underdetermined) system of linear equations. For details we refer to the references as the procedure presented in this paper is self-contained.

III. Main result: Parameter choice for mechanical underactuated systems

Three groups of design parameters in the IDA-PBC setup are distinguished: (i) The elements of the closed loop mass matrix $M_d(q)$, (ii) the free parameters in $(J_2' - R_2')(q,p)$ and (iii) the free parameters – and also the ansatz of the function $\Phi(z_\alpha)$ – in the homogeneous solution of the potential energy PDE. The design parameters *as a whole* determine the closed loop behavior of the system and it is not transparent how individual parameters affect stability, the estimate of the domain of attraction and (transient) dynamics, respectively. As an example, consider the mass matrix $M_d(q)$. Clearly positive definiteness is necessary for the stability proof and the virtual kinetic energy $\frac{1}{2} p^T \overline{M}_d(q) p$ heavily influences the dynamics in the closed loop. However, also the shape of the potential energy and thus the estimate of the domain of attraction is affected by the "unactuated" part $m_{d,\nu}^T(q)$.

To realize desired closed loop dynamics – at least in its local approximation – and at the same time achieve a well defined estimate of the domain of attraction, in the sequel we follow the three steps:

- Establish a linearized version of the coordinate change that transforms the potential energy PDE into an ODE.
- Express the Hessian of the closed loop potential energy in new coordinates without solving the ODE/PDE.
- Replace the Hessian in the closed loop linearization and determine the IDA-PBC design parameters from comparison with predefined desired local linear dynamics.

A. Linearized coordinate change

We assume $a_n(q_\alpha) \equiv 1$, which can be easily achieved by dividing (13) by the last coefficient function.

Notation: For brevity the arguments of functions are sometimes omitted and a star indicates that the function is evaluated at the equilibrium $(q^*, 0)$. The vectors $\hat{a} = [a_1 \ \dots \ a_{n-1}]^T$ and $\hat{q} = [q_1 \ \dots \ q_{n-1}]^T$ denote the $n-1$ dimensional subvectors of a and q.

The coordinate change $z = [z_\alpha^T(q) \quad z_\nu(q)]^T = t(q)$ according to (17) and (18) renders the matching PDE (13) an ODE (19). Observe that arbitrary functions of $z_{\alpha,i}(q)$ (i.e. the characteristic coordinates) may be added to the coordinate function $z_\nu(q)$ without changing the validity of (18). This degree of freedom is used to achieve the local property

$$\partial_q z_\nu(q)\big|_{q^*} = e_n^T, \tag{20}$$

where e_n^T denotes the n-th unity row vector.

The linear approximation of the coordinate change is $\Delta z = T \Delta q$ with $T = \partial_q t(q)\big|_{q^*}$. The linearized versions of conditions (17), (18) and (20) are

$$T a^* = e_n \tag{21}$$

$$e_n^T T = e_n^T, \tag{22}$$

from which the structure of the transformation matrix T and its inverse follows:

$$T = \begin{bmatrix} \hat{T} & -\hat{T}\hat{a}^* \\ 0^T & 1 \end{bmatrix}, \quad \overline{T} = \begin{bmatrix} \hat{\overline{T}} & \hat{a}^* \\ 0^T & 1 \end{bmatrix}. \tag{23}$$

The $(n-1)$ square matrix $\hat{T}$ is the Jacobian of the (characteristic) coordinate functions $z_\alpha(q)$ w.r.t. $\hat{q}$ in q^*. The local linear transformation matrix T will be fruitfully used to derive the structure of the Hessian of the closed loop potential energy in the equilibrium.

B. Hessian of the closed loop potential energy

To derive an expression of the Hessian of V_d in q^* the Taylor series expansion of the potential energy PDE (13) is considered and the terms of order 0 and 1 are compared on both sides. The terms of order 0 are

$$(a^*)^T \nabla V_d^* = b^*, \tag{24}$$

which reflects that $(q^*, 0)$ must be an admissible equilibrium to enable $\nabla V_d(q)\big|_{q^*} = 0$, the first condition for a minimum of the potential energy. The first order terms of the power series, considering $\nabla V_d^* = 0$ are

$$(a^*)^T Q_d = \partial_q b^*. \tag{25}$$

Multiplying the equation by $\overline{T}$ from the right and expanding with $T^T \overline{T}^T$ we get

$$\underbrace{(a^*)^T T^T}_{=e_n^T} \underbrace{\overline{T}^T Q_d \overline{T}}_{=\tilde{Q}_d} = \partial_q b^* \overline{T}, \tag{26}$$

where $\tilde{Q}_d = \partial_{z^2}^2 \tilde{V}_d(z)\Big|_{z^*}$ is the Hessian of the energy in z-coordinates. Equation (26) determines the last row (and by symmetry the last column) of $\tilde{Q}_d$. Note that $\tilde{V}_d(z)$ can be shaped freely in the characteristic z_α-coordinates such that $\tilde{Q}_\alpha = \partial_{z_\alpha^2}^2 \tilde{V}_d(z)\Big|_{z^*}$ contains open design parameters of IDA-PBC. Putting the pieces together the Hessian of a solution of the potential energy PDE in z-coordinates has the form

$$\tilde{Q}_d = \begin{bmatrix} \tilde{Q}_\alpha & \hat{\overline{T}}^T \partial_{\hat{q}}^T b^* \\ \partial_{\hat{q}} b^* \hat{\overline{T}} & \partial_q b^* a^* \end{bmatrix}, \tag{27}$$

where $\tilde{Q}_\alpha$ is free, a^* and b^* depend on the elements of $m_{d,\nu}^T$, which enter the potential energy PDE as well as $\hat{\overline{T}}$ which results from the definition of the characteristic coordinates. The Hessian in original coordinates is calculated by

$$Q_d = T^T \tilde{Q}_d T \tag{28}$$

with T according to (23).

C. Matching of local linear dynamics

In order to determine the IDA-PBC design parameters such that desired local linear dynamics is realized the linearization of the closed loop Port-Hamiltonian dynamics (3)

$$\begin{bmatrix} \Delta\dot{q} \\ \Delta\dot{p} \end{bmatrix} = \begin{bmatrix} 0 & \overline{M}^* \\ -M_d^* \overline{M}^* Q_d & (J_2^* - R_2^*)\overline{M}_d^* \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta p \end{bmatrix} \tag{29}$$

is compared with desired dynamics that results e.g. from pole placement for the linearized original system ($A_{d,12} = \overline{M}^*$):

$$\begin{bmatrix} \Delta\dot{q} \\ \Delta\dot{p} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & A_{d,12} \\ A_{d,21} & A_{d,22} \end{bmatrix}}_{=A_d} \begin{bmatrix} \Delta q \\ \Delta p \end{bmatrix}. \tag{30}$$

The two matrix equations

$$-M_d^* \overline{M}^* Q_d = A_{d,21} \tag{31}$$
$$(J_2^* - R_2^*) = A_{d,22} M_d^* \tag{32}$$

are solved in four steps for the unknown values of the design parameters in the equilibrium:

(i) The Hessian matrix Q_d must be symmetric for any parametrization, such that from Eq. (31) the condition

$$A_{d,21}\overline{M}^* M_d^* - M_d^* \overline{M}^* A_{d,21}^T = 0 \tag{33}$$

can be derived. The condition represents $\frac{n(n-1)}{2}$ linear equations which have to be met by the elements of M_d^*.

(ii) To reflect the structure of $(J_2 - R_2)$ in (12) the first row and column of $A_{d,22}M_d^*$ must be zero. $e_1^T A_{d,22} M_d^* = 0^T$ is ensured by Assumption A.4 (no dissipation in the unactuated joint). As the first element is already zero it remains to ensure

$$\begin{bmatrix} 0 & I_{n-1} \end{bmatrix} A_{d,22} m_{d,\nu}^* = 0 \tag{34}$$

with $m_{d,\nu}$ the first column of the symmetric mass matrix M_d. The condition represents $n-1$ linear equations for the parameters in $m_{d,\nu}^*$. Together with (i) there are $\frac{n(n+1)}{2} - 1$ linear equations for the entries of M_d^* i.e. one less than the number of independent elements.

(iii) For a given parametrization of M_d the nonzero southestern submatrix of $J_2 - R_2$ in the equilibrium can be evaluated from (32):

$$(J_2' - R_2')^* = \begin{bmatrix} 0 & I_{n-1} \end{bmatrix} A_{d,22} (M_{d,\alpha}^*)^T. \tag{35}$$

(iv) Finally the values of the $\frac{n(n+1)}{2}$ open parameters in $\tilde{Q}_\alpha$ are deduced from Eq. (31), with Q_d according to (28) and rearranged for $\tilde{Q}_\alpha$:

$$\tilde{Q}_\alpha = -\begin{bmatrix} \hat{\overline{T}}^T & 0 \end{bmatrix} M^* \overline{M}_d^* A_{d,21} \begin{bmatrix} \hat{\overline{T}} \\ 0 \end{bmatrix}. \tag{36}$$

The following steps to determine the values of the IDA-PBC design parameters in the equilibrium for the considered class of underactuated mechanical systems are proposed:

- Solve the PDEs to determine the coordinate transformation (17), (18) and adjust $z_\nu(q)$ such that (20) holds.
- Determine the linear transformation submatrix $\hat{T}$.
- Successively evaluate the design parameters in $(q^*, 0)$ from the linear equations (33)–(36).

D. Asymptotic stability

The very simple idea of local linear dynamics assignment is the comparison of the closed loop linearized dynamics with unassigned design parameters to a desired asymptotically stable state matrix A_d. If (29) and (30) match, asymptotic stability of the equilibrium follows immediately from Lyapunov's indirect method:

Fact 3: If the free parameters of M_d, $(J_2(q,p) - R_2(q,p))$ and $H_d(q,p)$ are determined such that the linearized matching equations (31) and (32) hold in the equilibrium $(q^*, 0)$, with A_d a matrix with eigenvalues only in the open left complex half plane, then the equilibrium is locally asymptotically stable.

Exploiting the Port-Hamiltonian structure, an *appropriate* solution of the linearized matching equations allows to extend the stability statement beyond the unspecified borders of local stability:

Proposition 4: If the solution of the linearized matching equations (31) and (32), with $A_{d,21}$ and $A_{d,22}$ submatrices of an asymptotically stable matrix A_d according to (30), yields $R_2^* \geq 0$ then $M_d^* > 0$ and $Q_d > 0$ hold.

Proof: The time derivative of the closed loop energy H_d is $\dot{H}_d = -p^T \overline{M}_d(q) R_2(q,p) \overline{M}_d p$ and its quadratic approximation $\Delta\dot{H}_d = -\Delta p^T \overline{M}_d^* R_2^* \overline{M}_d^* \Delta p$. On the other hand the quadratic approximation of the energy is $\Delta H_d = -\frac{1}{2}\Delta x^T P \Delta x$ with $\Delta x^T = [\Delta q^T \ \Delta p^T]$ and the block diagonal matrix $P = \mathrm{diag}\{Q_d, \overline{M}_d^*\}$. The time derivative, substituting matched linear dynamics $\Delta\dot{x} = A_d \Delta x$ is $\Delta\dot{H}_d = -\Delta x^T \left(A_d^T P + P A_d\right) \Delta x$. Setting equal both expressions for $\Delta\dot{H}_d$ results in a Lyapunov equation for $P = \mathrm{diag}\{Q_d, \overline{M}_d^*\}$ which has at least a positive semidefinite solution for A_d asymptotically stable and $R_2^* \geq 0$ [4]. Q_d and $\overline{M}_d^*$ (and M_d^*, respectively) are positive definite if in addition they are regular. For A_d asymptotically stable with the structure indicated in (30) the submatrix $A_{d,21}$ is regular. From Eq. (31), which is true when local linear dynamics is matched, follows regularity of Q_d and M_d^*. ■

If in addition $\nabla V_d(q)|_{q^*} = 0$ is ensured, then the closed loop potential energy has an isolated minimum in the desired equilibrium. Hence $V_d(q)$ (in the region where at the same time $R_2(q,p) \geq 0$ and $M_d > 0$) can be used to estimate the domain of attraction of the equilibrium along the standard IDA-PBC arguments: The estimate of the domain of attraction is enclosed by the closed contour surfaces of V_d with maximum value, where the above inequalities hold.

Two advantages of the procedure are obvious: (i) The definiteness check of the potential energy function is omitted. (ii) From linear asymptotic stability of the equilibrium, which

is achieved by matching the linearized dynamics, local exponential stability of $(q^*, 0)$ is deduced even if R_2 is only positive semidefinite (cf. Section 4.2 in [6] on a disussion of local exponential stability in the case of strong dissipation).

E. 2 DOF systems

For systems with 2 DOF and $M_d = const.$ the transformation into z-coordinates can be easily expressed analytically. The matching PDE for the closed loop potential energy is

$$m_{d,1}^T \overline{m}_1(q_2)\partial_{q_1} V_d(q) + m_{d,1}^T \overline{m}_2(q_2)\partial_{q_2} V_d(q) = \partial_{q_1} V(q), \tag{37}$$

where $\overline{m}_i$ denotes the i-th column of $\overline{M}$. Division by the second coefficient function yields

$$a_1(q_2)\partial_{q_1} V_d(q) + \partial_{q_2} V_d(q) = b(q) \tag{38}$$

with $a_1(q_2) = \frac{m_{d,1}^T \overline{m}_1(q_2)}{m_{d,1}^T \overline{m}_2(q_2)}$ and $b(q) = \frac{\partial_{q_1} V(q)}{m_{d,1}^T \overline{m}_2(q_2)}$. A solution of Eqs. (17) and (18) to determine the coordinate change is

$$z_\alpha = q_1 - \int_{q_2^*}^{q_2} a_1(s)ds, \qquad z_\nu = q_2, \tag{39}$$

which also satisifies (20). The inverse transformation is simply

$$q_1 = z_\alpha + \int_{z_\nu^*}^{z_\nu} a_1(s)ds, \qquad q_2 = z_\nu. \tag{40}$$

Local linear dynamics is matched as follows.

(i) and (ii): The symmetry condition on Q_d according to Eq. (33) boils down to

$$e_2^T A_{d,21} \overline{M}^* \begin{bmatrix} \frac{m_{d,11}}{m_{d,12}} \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & \frac{m_{d,22}}{m_{d,12}} \end{bmatrix} \overline{M}^* A_{d,21}^T e_1 = 0, \tag{41}$$

while (34) becomes

$$e_2^T A_{d,22} \begin{bmatrix} \frac{m_{d,11}}{m_{d,12}} \\ 1 \end{bmatrix} = 0. \tag{42}$$

Observe that both equations have been divided by $m_{d,12} \neq 0$ such that two linear equations can be solved for the two quantities $m_{d,11}/m_{d,12}$ and $m_{d,22}/m_{d,12}$.

(iii) In the 2 DOF case we have simply $(J_2' - R_2') = -r_2$ such that Eq. (35) for the given Matrix M_d is

$$r_2 = -e_2^T A_{d,22} \begin{bmatrix} 1 \\ \frac{m_{d,22}}{m_{d,12}} \end{bmatrix} m_{d,12}.$$

The sign of r_2 can be adjusted by the choice of $m_{d,12}$, i.e. $r_2 \geq 0$ is achievable and with this parametrization $M_d > 0$ is guaranteed according to Proposition 4.

Remark 5: In the case $n > 2$ ($m > 1$) positive semidefiniteness of R_2 will not be achievable by tuning the *single* remaining parameter in M_d. However, in the MIMO case the additional design freedom in the pole placement problem can be preassumably used to achieve $R_2 \geq 0$ (by means of the variation of $A_{d,22}$).

(iv) The second order partial derivative of the potential energy in the characteristic coordinate is calculated by

$$\tilde{q}_\alpha = -e_1^T M(q_2^*) \overline{M}_d A_{d,21} e_1,$$

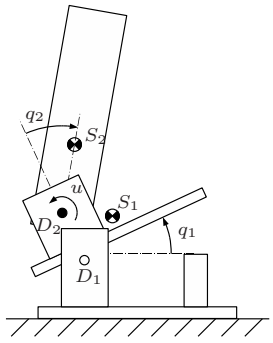

Fig. 1. Sketch of the experimental apparatus

TABLE I

SYMBOLS USED IN THE MODEL, ALL COORDINATES IN THE CORRESPONDING MOVING FRAME

(s_1, h_1)	Coordinates of the center of gravity in link 1
$(0, h_2)$	Coordinates of the center of gravity in link 2
$(0, d)$	Coordinates of the rotation axis for link 2 w.r.t. link 1
$m_{1,2}, m_D$	Masses of the links and the drive
$J_{1,2}, J_D$	Moments of inertia *around the rotation axis*

where $\hat{T} = \partial_{q_1} z_\alpha(q)|_{q^*} = 1$ has been used.

With the parameters of M_d the potential energy PDE is solved by computer algebra and $\tilde{q}_\alpha$ is realized by adding the homogeneous solution, e.g. $\Phi(z_\alpha) = \frac{1}{2}\tilde{q}_\alpha (z_\alpha - z_\alpha^*)^2$.

IV. EXAMPLE: TILTING ROBOT MODEL

The sketch of the lab experiment depicted in Fig. 1 represents a two dimensional 2 DOF underactuated mechanical system which – in an elementary way – mimics a tilting walking machine. A critical situation occurs when the robot has lost complete foot contact. The stabilizing controller initially tries to balance the underactuated robot before stable standing and hence full degree of actuation can be regained. Joint 1 is unactuated while joint 2 is driven by a DC gear motor with transmission ratio $i = \dot{q}_2^I / \dot{q}_2^{II}$. The example is from [8] with the controller derived using the systematic procedure presented in this paper.

A. Modeling

The model of the 2 DOF system has been derived using the Euler-Lagrange formalism for both links (index 1 and 2) and the drive (index D). The mass matrix results to be

$$M(q_2) = \begin{bmatrix} c_1 + c_{20} + 2c_3 \cos q_2 & c_{21} + c_3 \cos q_2 \\ c_{21} + c_3 \cos q_2 & c_{22} \end{bmatrix} \tag{43}$$

with constants $c_1 = J_1 + (m_2 + m_D)d^2$, $c_{2k} = J_2 + J_D i^k$, $k = 0, 1, 2$ and $c_3 = m_2 d h_2$. The potential energy whose gradient appears in the equations of motion is

$$V(q) = c_4 \cos(q_1 + q_2) + c_5 \cos q_1 + c_6 \sin q_1 \tag{44}$$

with $c_4 = m_2 h_2 g$, $c_5 = ((m_2 + m_D)d + m_1 h_1)g$ and $c_6 = m_1 s_1 g$. All symbols are explained in Table I, g denotes the gravitational acceleration.

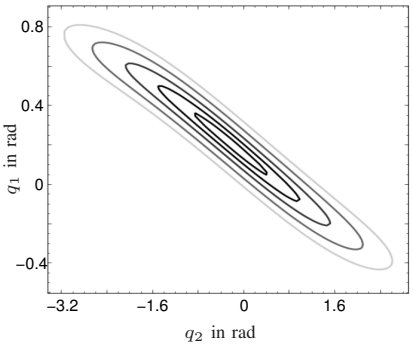

Fig. 2. Contour lines of the closed loop potential energy

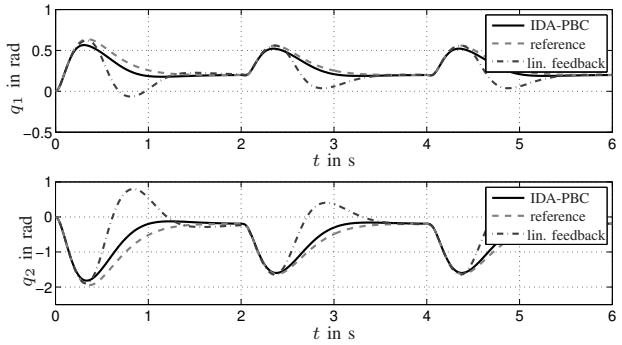

Fig. 3. Transients: IDA-PBC vs. reference system vs. linear state feedback

B. Controller design

The IDA-PBC controller (neglecting friction in q_1) is parametrized according to the procedure presented in Section III-E with double eigenvalues of the predefined matrix A_d at -6 an -7. The equilibrium to be stabilized is chosen such that link 2 is perpendicular to the ground, i.e. $q_2^* = -q_1^*$. The ansatz of the homogeneous solution of the potential energy PDE is quadratic in $z_\alpha - z_\alpha^*$. The design steps and the solution of the potential energy PDE are executed with a computer algebra system. Some contour lines of the closed loop potential energy around q^* are depicted in Fig. 2.

C. Simulation

The nonlinear IDA-PBC controller is compared to the corresponding linear state feedback and the linear reference system (i.e. the simulation of the linear target dynamics (30)), see Fig. 3. The simulations start from an initial position $q(0) = 0 \neq q^*$. From $t = 2$s on rectangular disturbance torques are applied to the unactuated link which leads to a deviation of the state from the equilibrium. Obviously the IDA-PBC controller, which accounts for the nonlinear nature of the plant, forces the state to stay closer to the equilibrium. This on the one hand leads to the better accordance with the reference, on the other gives evidence of improved robustness of stability.

Remark 6: Unfortunately the simulations could not be validated in the experiment so far due to the friction in the unactuated coordinate of the experimenal apparatus. The mainly viscous friction violates (independent of its magnitude!) the dissipation condition for controller redesign in [6]. Until now we have not succeeded in accounting for this friction in a Port-Hamiltonian structure of the closed loop system.

V. Conclusions and Outlook

This contribution presents the adaptation of *local linear dynamics assignment* in IDA-PBC to mechanical systems with underactuation degree one. For the considered class of systems the kinetic energy PDE can be trivially solved with a constant virtual mass matrix. To derive a parametrization of the IDA-PBC approach the solution of the potential energy PDE is discussed. Based on a local linear coordinate transformation the structure of the achievable potential energy Hessian matrix is revealed. As a consequence a four step procedure is presented to calculate the design parameters such that linearized closed loop dynamics is matched with a predefined linear target system. As in the case of general input affine systems an explicite definiteness check of the closed loop energy Hessian is not required.

The problem of matching the damping of the unactuated coordinate in a closed loop Hamiltonian structure – which to the best of our knowledge is unsolved so far for the considered class of systems – gives rise to future work. A possibility could be to allow cross terms of coordinates and momenta in the Hamiltonian.

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