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Error Probability Bounds for Balanced Binary Relay Trees

Zhenliang Zhang, Ali Pezeshki, William Moran, Stephen D. Howard, and Edwin K. P. Chong

***Abstract*— We study the detection error probability associated with a balanced binary relay tree, where the leaves of the tree correspond to N identical and independent sensors. The root of the tree represents a fusion center that makes the overall detection decision. Each of the other nodes in the tree are relay nodes that combine two binary messages to form a single output binary message. Only the leaves of the tree are sensors. In this way, the information from the sensors is aggregated into the fusion center via the intermediate relay nodes. In this context, we describe the evolution of Type I and Type II error probabilities of the binary data as it propagates from the leaves towards the root. Tight upper and lower bounds for the total error probability at the fusion center as functions of N are derived. These characterize how fast the total error probability converges to 0 with respect to N.**

I. Introduction

Consider a hypothesis testing problem under two scenarios: *centralized* and *decentralized*. Under the centralized network scenario, all sensors send their raw measurements to the fusion center which makes a decision based on these measurements. In the decentralized network introduced in [1], sensors send summaries of their measurements and observations to the fusion center. The fusion center then makes a decision. In a decentralized network, information is summarized into smaller messages. Evidently, the decentralized network cannot perform better than the centralized network. It gains because of its limited use of resources and bandwidth; through transmission of summarized information it is more practical and efficient.

The decentralized network in [1] involves a parallel architecture, also known as a *star* architecture [1]–[15],[31], in which all sensors directly connect to the fusion center. A typical result is that under the assumption of (conditionally) independence of the sensor observations, the decay rate of the error probability in a parallel network is exponential [6].

Several different sensor topologies have been studied under the assumption of conditional independence. The first configuration for such a fusion network considered was the tandem network [16]–[20],[31]. In such a network, each non-leaf node combines the information from its own sensor with the message it has received from the node at one level down, which is then transmitted to the node at the next level up. The decay rate of the error probability in this case is sub-exponential [20]. This sensor network represents a situation where the length of the network is the longest possible among all networks with N leaf nodes.

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Z. Zhang, A. Pezeshki, and E. K. P. Chong are with the Department of Electrical and Computer Engineering, Colorado State University, Fort Collins, CO 80523-1373, USA zhenliang.zhang, ali.pezeshki, edwin.chong@colostate.edu

W. Moran is with the Department of Electrical and Electronic Engineering, The University of Melbourne, Melbourne, VIC 3010, Australia wmoran@unimelb.edu.au

S. D. Howard is with the Defence Science and Technology Organisation, P.O. Box 1500, Edinburgh, South Australia 5111, Australia sdhoward@unimelb.edu.au

The asymptotic performance of single-rooted tree networks with bounded height is discussed in [21]–[29],[31]. Even though error probabilities in the parallel configuration decrease exponentially, in a practical implementation, the resources consumed in having each sensor transmit directly to the fusion center might be regarded as excessive. Energy consumption can be reduced by setting up a directed tree, rooted at the fusion center. In this tree structure, measurements are summarized by leaf sensor nodes and sent to their parent nodes, each of which fuses all the messages it receives with its own measurement (if any) and then forwards the new message to its parent node at the next level. This process takes place throughout the tree culminating in the fusion center, where a final decision is made. For a bounded-height tree, the error exponent is as good as that of the parallel configuration under certain conditions. For example, for a bounded-height tree network with $\lim_{\tau_N\to\infty} \ell_N/\tau_N = 1$, where τ_N denotes the total number of nodes and ℓ_N denotes the number of leaf nodes, the optimum error exponent is the same as that of the parallel configuration [22].

The variation of detection performance with increasing tree height is still largely unexplored. If only the leaf nodes have sensors making observations, and all other nodes simply fuse the messages received and forward the new messages to their parents, the tree network is known as a *relay tree*. The balanced binary relay tree has been addressed in [30], in which it is assumed that the leaf nodes are independent sensors with identical Type I error probability (also known as probability of false alarm, denoted by α_0) and identical Type II error probability (also known as probability of missed detection, denoted by β_0). It is shown there that if sensor error probabilities satisfy the condition $\alpha_0+\beta_0<1$, then both the Type I and Type II error probabilities at the fusion center both converge to 0 as the number of sensors N goes to infinity. If $\alpha_0+\beta_0>1$, then both Type I and Type II error probabilities converge to 1, which means that if we flip the decision at the fusion center, then the error probabilities converge to 0. Therefore, because of this symmetry, it suffices to consider the case where $\alpha_0+\beta_0<1$. If $\alpha_0+\beta_0=1$, then $\alpha_k+\beta_k=1$ for all k. Therefore, this case is not of interest.

We consider the balanced binary relay tree configuration in this paper and describe the precise evolution of Type I and Type II error probabilities in this case. In addition, we provide upper and lower bounds for the total error probability at the fusion center as functions of N. These bounds characterize the decay rate of the total error probability.

II. Problem Formulation

We consider the problem of binary hypothesis testing between H_0 and H_1 in a balanced binary relay tree. Leaf nodes are sensors undertaking initial and independent detections of the same event in a scene. These measurements are summarized into binary messages and forwarded to nodes at the next level. Each non-leaf node with the exception of the root, the fusion center, is a relay node, which fuses two binary messages into one new binary message and forwards the new binary message to its parent node. This process takes place at each intermediate node culminating in the fusion center, at which the final decision is made based on the information received. Only the leaves are sensors in this tree architecture.

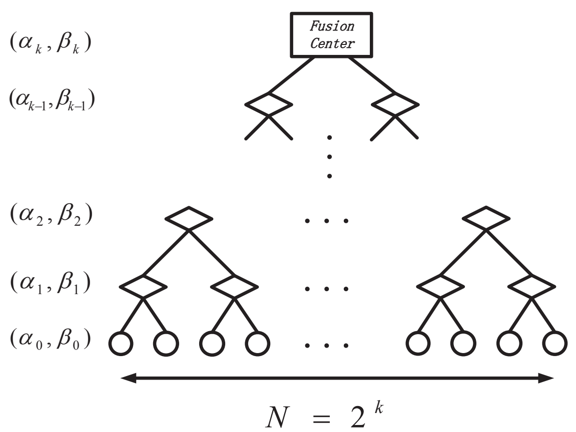

Fig. 1. A balanced binary relay tree with height k. Circles represent sensors making measurements. Diamonds represent relay nodes which fuse binary messages. The rectangle at the root represents the fusion center making an overall decision.

In this configuration, as shown in Fig. 1, the closest sensor to the fusion center is as far as it could be, in terms of the number of arcs in the path to the root. In this sense, this configuration is the worst case among all N sensor relay trees. Moreover, in contrast to the configuration in [22] discussed earlier, in our balanced binary tree we have $\lim_{\tau_N\to\infty} \ell_N/\tau_N = 1/2$ (as opposed to 1 in [22]). Hence, the number of times that information is aggregated here is essentially as large as the number of measurements (cf., [22], in which the number of measurements dominates the number of fusions). In addition, the height of the tree is $\log N$, which grows as the number of sensors increases. (Throughout this paper, $\log$ stands for the binary logarithm.)

We assume that all sensors are independent given each hypothesis, and that all sensors have identical Type I error probability α_0 and identical Type II error probability β_0. We apply the likelihood-ratio test [32] with threshold 1 as the fusion rule at the intermediate relay nodes and at the fusion center. This fusion rule is locally (but not necessarily globally) optimal in the case of equally likely hypotheses H_0 and H_1: it minimizes the total error probability locally at each fusion node. In the case where the hypotheses are not equally likely, the locally optimal fusion rule has a different threshold value, which is the ratio of the two hypothesis probabilities. However, this complicates the analysis without any additional insights. Therefore, for simplicity, we henceforth assume equally likely hypotheses in our analysis. We are interested in following questions:

- What are these Type I and Type II error probabilities as functions of N?
- Will they converge to 0 at the fusion center?
- If yes, how fast will they converge with respect to N?

Fusion at a single node receiving information from the two immediate child nodes where these have identical Type I error probabilities α and identical Type II error probabilities β provides a detection with Type I and Type II error probabilities denoted by (α', β'), and given by [30]:

$$(\alpha', \beta') = f(\alpha, \beta) := \begin{cases} (1-(1-\alpha)^2, \beta^2), & \alpha \le \beta, \\ (\alpha^2, 1-(1-\beta)^2), & \alpha > \beta. \end{cases} \tag{1}$$

Evidently, as all sensors have the same error probability pair (α_0, β_0), all relay nodes at level 1 will have the same error probability pair $(\alpha_1, \beta_1) = f(\alpha_0, \beta_0)$, and by recursion,

$$(\alpha_{k+1}, \beta_{k+1}) = f(\alpha_k, \beta_k), \qquad k = 0, 1, \ldots, \tag{2}$$

where (α_k, β_k) is the error probability pair of nodes at the kth level of the tree.

The recursive relation (2) allows us to consider the pair of Type I and II error probabilities as a discrete dynamic system. In [30], which focuses on the convergence issues for the total error probability, convergence was proved using Lyapunov methods. The analysis of the precise evolution of the sequence $\{(\alpha_k, \beta_k)\}$ and the total error probability decay rate remain open. In this paper, we will establish upper and lower bounds for the total error probability and deduce the precise decay rate of the total error probability.

To illustrate the ideas, consider first a single trajectory for the dynamic system given by equation (1), and starting at the initial state (α_0, β_0). This trajectory is shown in Fig. 2. It exhibits different behaviors depending on its distance from the $\beta = \alpha$ line. The trajectory approaches $\beta = \alpha$ very fast initially, but when (α_k, β_k) approaches within a certain neighborhood of the line $\beta = \alpha$, the next pair $(\alpha_{k+1}, \beta_{k+1})$ will appear on the other side of that line. In the next section, we will establish theorems that characterize the precise step-by-step behavior of the dynamic system (2).

III. Evolution of error probabilities

The relation (1) is symmetric about both of the lines $\alpha + \beta = 1$ and $\beta = \alpha$. Thus, it suffices to study the evolution of the dynamic system only in the region bounded by $\alpha+\beta < 1$ and $\beta \ge \alpha$. We denote

$$\mathcal{U} := \{(\alpha, \beta) \ge 0 | \alpha + \beta < 1 \text{ and } \beta \ge \alpha\}$$

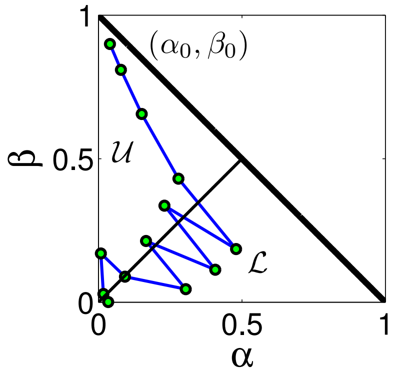

Fig. 2. A trajectory of the sequence $\{(\alpha_k, \beta_k)\}$ in the (α, β) plane.

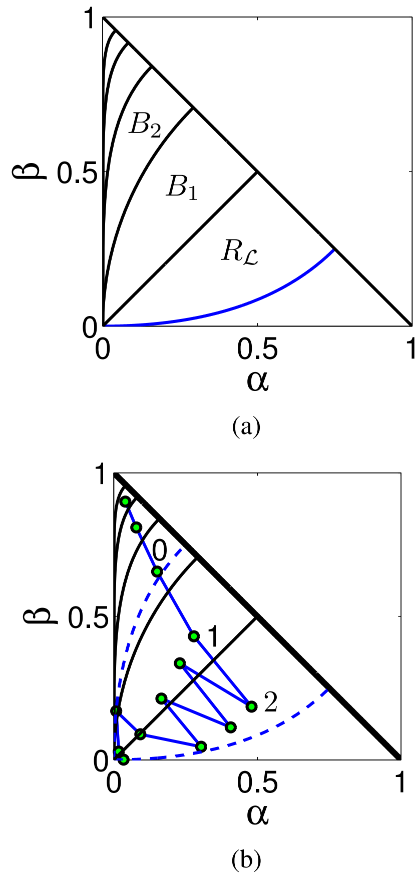

Fig. 3. (a) Regions B_1, B_2, and $R_{\mathcal{L}}$ in the (α, β) plane. (b) The trajectory in Fig. 2 superimposed on (a), where solid lines represent boundaries of B_m and dashed lines represent boundaries of R.

to be this triangular region. Similarly, define the complementary triangular region

$$\mathcal{L} := \{(\alpha, \beta) \geq 0 | \alpha + \beta < 1 \text{ and } \beta < \alpha\}.$$

We denote the following region by B_1:

$$B_1 := \{(\alpha, \beta) \in \mathcal{U} | (1-\alpha)^2 + \beta^2 \leq 1\}.$$

If $(\alpha_k, \beta_k) \in B_1$, then the next pair $(\alpha_{k+1}, \beta_{k+1}) = f(\alpha_k, \beta_k)$ crosses the line $\beta = \alpha$ to the opposite side from (α_k, β_k). More precisely, if $(\alpha_k, \beta_k) \in \mathcal{U}$, then $(\alpha_k, \beta_k) \in B_1$ if and only if $(\alpha_{k+1}, \beta_{k+1}) = f(\alpha_k, \beta_k) \in \mathcal{L}$. In other words, B_1 is the *inverse image* of $\mathcal{L}$ under f in $\mathcal{U}$. The set B_1 is shown in Fig. 3(a). Fig. 3(b) illustrates this behavior of the trajectory for the example in Fig. 2. For instance, as shown in Fig. 3(b), if the state is at point 1 in B_1, then it jumps to the next state point 2, on the other side of $\beta = \alpha$.

Denote the following region by B_2:

$$B_2 := \{(\alpha, \beta) \in \mathcal{U} | (1-\alpha)^2 + \beta^2 \geq 1 \\ \text{and } (1-\alpha)^4 + \beta^4 \leq 1\}.$$

It is easy to show that if $(\alpha_k, \beta_k) \in \mathcal{U}$, then $(\alpha_k, \beta_k) \in B_2$ if and only if $(\alpha_{k+1}, \beta_{k+1}) = f(\alpha_k, \beta_k) \in B_1$. In other words, B_2 is the inverse image of B_1 in $\mathcal{U}$ under f. The regions and the behavior of f is illustrated in the movement from 0 to point 1 in Fig. 3(b). The set B_2 is identified in Fig. 3(a), lying directly above B_1.

Now for an integer $m > 1$, recursively define B_m to be the inverse image of B_{m-1} under f, denoted by B_m. It is easy to see that

$$B_m := \{(\alpha, \beta) \in \mathcal{U} | (1-\alpha)^{2^{(m-1)}} + \beta^{2^{(m-1)}} \geq 1 \\ \text{and } (1-\alpha)^{2^m} + \beta^{2^m} \leq 1\}.$$

Notice that $\mathcal{U} = \bigcup_{m=1}^{\infty} B_m$. Hence, for any $(\alpha_0, \beta_0) \in \mathcal{U}$, there exists m such that $(\alpha_0, \beta_0) \in B_m$. This gives a complete description of how the dynamics of the system behaves in the upper triangular region $\mathcal{U}$. For instance, if the initial pair (α_0, β_0) lies in B_m, then the system evolves in the order

$$B_m \to B_{m-1} \to \cdots \to B_2 \to B_1.$$

Therefore, the system will enter B_1 after $m-1$ levels of fusion, i.e., $(\alpha_{m-1}, \beta_{m-1}) \in B_1$.

As the next stage, we consider the behavior of the system after it enters B_1. The *image* of B_1 under f, denoted by $R_{\mathcal{L}}$, is (see Fig. 3(a))

$$R_{\mathcal{L}} := \{(\alpha, \beta) \in \mathcal{L} | \sqrt{1-\alpha} + \sqrt{\beta} \geq 1\}.$$

We can define the reflection of B_m about the line $\beta = \alpha$ in the similar way for all m. Similarly, we denote by $R_{\mathcal{U}}$ the reflection of $R_{\mathcal{L}}$ about the line $\beta = \alpha$; i.e.,

$$R_{\mathcal{U}} := \{(\alpha, \beta) \in \mathcal{U} | \sqrt{1-\beta} + \sqrt{\alpha} \geq 1\}.$$

We denote the region $R_{\mathcal{U}} \cup R_{\mathcal{L}}$ by R. Below R is shown to be an *invariant region* in the sense that once the system enters R, it stays there. For example, as shown in Fig. 3(b), the system after point 1 stays inside R.

Proposition 1: If $(\alpha_{k_0}, \beta_{k_0}) \in R$ for some k_0, then $(\alpha_k, \beta_k) \in R$ for all $k \geq k_0$.

Proof: First we show that $B_1 \subset R_{\mathcal{U}} \subset B_1 \cup B_2$.

Notice that B_1, $R_{\mathcal{U}}$, and $B_1 \cup B_2$ share the same lower boundary $\beta = \alpha$. It suffices to show that the upper boundary of $R_{\mathcal{U}}$ lies between the upper boundary of B_2 and that of B_1 (see Fig. 4).

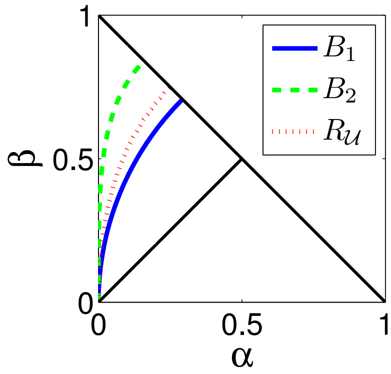

Fig. 4. Upper boundaries for $B_1, B_2,$ and $R_{\mathcal{U}}$.

First, we show that the upper boundary of $R_{\mathcal{U}}$ lies above the upper boundary of B_1. We have

$$\begin{aligned} & 1-(1-\sqrt{\alpha})^2 \geq \sqrt{1-(1-\alpha)^2} \\ \iff\ & 2\sqrt{\alpha}-\alpha \geq \sqrt{2\alpha-\alpha^2} \\ \iff\ & \alpha^2+\alpha-2\alpha^{\frac{3}{2}} \geq 0, \end{aligned}$$

which holds for all α in $[0,1)$. Thus, $B_1 \subset R_{\mathcal{U}}$.

Now we prove that the upper boundary of $R_{\mathcal{U}}$ lies below that of B_2. We have

$$\begin{aligned} & (1-(1-\alpha)^4)^{\frac{1}{4}} \geq 1-(1-\sqrt{\alpha})^2 \\ \iff\ & 1-(1-\alpha)^4 \geq (2\sqrt{\alpha}-\alpha)^4 \\ \iff\ & -2(\sqrt{\alpha}-1)^2\alpha(-\alpha^{\frac{3}{2}}+\alpha(\sqrt{\alpha}-1) \\ & +4\sqrt{\alpha}(\sqrt{\alpha}-1)+\alpha-2) \geq 0, \end{aligned}$$

which holds for all α in $[0,1)$ as well. Hence, $R_{\mathcal{U}} \subset B_1 \cup B_2$.

Without loss of generality, we assume that $(\alpha_{k_0}, \beta_{k_0}) \in R_{\mathcal{U}}$. That means $(\alpha_{k_0}, \beta_{k_0}) \in B_1$ or $(\alpha_{k_0}, \beta_{k_0}) \in B_2 \cap R_{\mathcal{U}}$. If $(\alpha_{k_0}, \beta_{k_0}) \in B_1$, then the next pair $(\alpha_{k_0+1}, \beta_{k_0+1})$ is in $R_{\mathcal{L}}$. If $(\alpha_{k_0}, \beta_{k_0}) \in B_2 \cap R_{\mathcal{U}}$, then $(\alpha_{k_0+1}, \beta_{k_0+1}) \in B_1 \subset R_{\mathcal{U}}$ and $(\alpha_{k_0+2}, \beta_{k_0+2}) \in R_{\mathcal{L}}$. By symmetry considerations, it follows that the system stays inside R for all $k \geq k_0$. ∎

So far we have studied the precise evolution of the sequence $\{(\alpha_k, \beta_k)\}$ in the (α, β) plane. In the next section, we will consider the step-wise reduction in the total error probability while the system lies inside the invariant region and deduce upper and lower bounds for it.

IV. Error Probability Bounds

Because of our assumption that the hypotheses are equally likely, the total error probability for a node with (α_k, β_k) is $(\alpha_k+\beta_k)/2$. Let $L_k = \alpha_k + \beta_k$, namely, twice the total error probability. Analysis of the total error probability will result from consideration of the sequence $\{L_k\}$. In fact, we will derive bounds on $\log L_k^{-1}$, whose growth rate is related to the rate of convergence of L_k to 0.

If $(\alpha_0, \beta_0) \in B_m$ for some $m \neq 1$, then $(\alpha_{m-1}, \beta_{m-1}) \in B_1$. The system afterward stays inside the invariant region R (but not necessarily inside B_1). Hence, the decay rate of the total error probability in the invariant region R determines the asymptotic decay rate. We have the following proposition.

Proposition 2: Suppose that $(\alpha_k, \beta_k) \in R$. Then,

$$1 \leq \frac{L_{k+2}}{L_k^2} \leq 2.$$

Proof: Because of symmetry, we only have to prove the case where (α_k, β_k) lies in $R_{\mathcal{U}}$. We consider two cases: $(\alpha_k, \beta_k) \in B_1$ and $(\alpha_k, \beta_k) \in B_2 \cap R_{\mathcal{U}}$.

In the first case,

$$\frac{L_{k+2}}{L_k^2} = \frac{(1-(1-\alpha_k)^2)^2+1-(1-\beta_k^2)^2}{(\alpha_k+\beta_k)^2}.$$

To prove the lower bound of the ratio, it suffices to show that

$$\begin{aligned} L_{k+2}-L_k^2 &= (\alpha_k+\beta_k-1)(-(\alpha_k-\beta_k)^2-2\alpha_k^2 \\ &\quad +(\alpha_k-\beta_k)^3+2\alpha_k\beta_k(\alpha_k-\beta_k)) \geq 0. \end{aligned}$$

We have $\alpha_k+\beta_k < 1$ and $\alpha_k \leq \beta_k$ for all $(\alpha_k, \beta_k) \in B_1$. Therefore, the above inequality holds.

To prove the upper bound of the ratio, it suffices to show that

$$L_{k+2}-2L_k^2 = \alpha_k^4-4\alpha_k^3+2\alpha_k^2-4\alpha_k\beta_k-\beta_k^4 \leq 0.$$

The partial derivative with respect to β_k is

$$\frac{\partial(L_{k+2}-2L_k^2)}{\partial\beta_k} = -4\alpha_k-4\beta_k^3 \leq 0$$

which is non-positive. Therefore, it suffices to consider its values on the curve $\beta_k = \alpha_k$, on which $L_{k+2}-2L_k^2$ is clearly non-positive. See Fig. 5(a) for a plot of values of L_{k+2}/L_k^2 in B_1.

Now we consider the second case, namely $(\alpha_k, \beta_k) \in B_2 \cap R_{\mathcal{U}}$, which gives

$$\frac{L_{k+2}}{L_k^2} = \frac{1-(1-\alpha_k)^4+\beta_k^4}{(\alpha_k+\beta_k)^2}.$$

To prove the lower bound of the ratio, it suffices to show that

$$\begin{aligned} L_{k+2}-L_k^2 &= (1-(1-\alpha_k)^4)+\beta_k^4-(\alpha_k+\beta_k)^2 \\ &= (1-\alpha_k-\beta_k)(\alpha_k^3-\alpha_k^2\beta_k-3\alpha_k^2 \\ &\quad +\alpha_k\beta_k^2+2\alpha_k\beta_k-\beta_k^3-\beta_k^2+4\alpha_k) \geq 0. \end{aligned}$$

Therefore, it suffices to show that

$$\begin{aligned} \phi(\alpha_k,\beta_k) =& \alpha_k^3-\alpha_k^2\beta_k-3\alpha_k^2+\alpha_k\beta_k^2 \\ & +2\alpha_k\beta_k-\beta_k^3-\beta_k^2+4\alpha_k \geq 0. \end{aligned}$$

The partial derivative with respect to β_k is

$$\frac{\partial\phi}{\partial\beta_k} = -(\alpha_k-\beta_k)^2-2\beta_k^2+2(\alpha_k-\beta_k) \leq 0.$$

Thus, it is enough to consider the values on the upper boundaries $\sqrt{1-\beta_k}+\sqrt{\alpha_k}=1$ and $\alpha_k+\beta_k=1$.

If $\alpha_k+\beta_k=1$, then the inequality is trivial, and if $\sqrt{1-\beta_k}+\sqrt{\alpha_k}=1$,

$$L_{k+2}-L_k^2 = 2\alpha_k^2(1-2\sqrt{\alpha_k})(2\alpha_k-6\sqrt{\alpha_k}+5)$$

and the inequality holds because $\alpha_k \le \frac{1}{4}$ in region $B_2 \cap R_{\mathcal{U}}$.

The claimed upper bound for the ratio L_{k+2}/L_k^2 can be written as

$$\begin{aligned} L_{k+2} - 2L_k^2 &= -\alpha_k^4 + 4\alpha_k^3 - 8\alpha_k^2 + 4\alpha_k \\ &\quad -4\alpha_k\beta_k + \beta_k^4 - 2\beta_k^2 \le 0. \end{aligned}$$

The partial derivative with respect to β_k is

$$\frac{\partial(L_{k+2} - 2L_k^2)}{\partial \beta_k} = -4\alpha_k + 4\beta_k^3 - 4\beta_k \le 0.$$

Again, it is sufficient to consider values on the upper boundary of B_1. Therefore,

$$L_{k+2} - 2L_k^2 = 2\beta_k^2 - 2(\alpha_k + \beta_k)^2 \le 0.$$

The reader is referred to Fig. 5(b) for a plot of values of L_{k+2}/L_k^2 in $B_2 \cap R_{\mathcal{U}}$.

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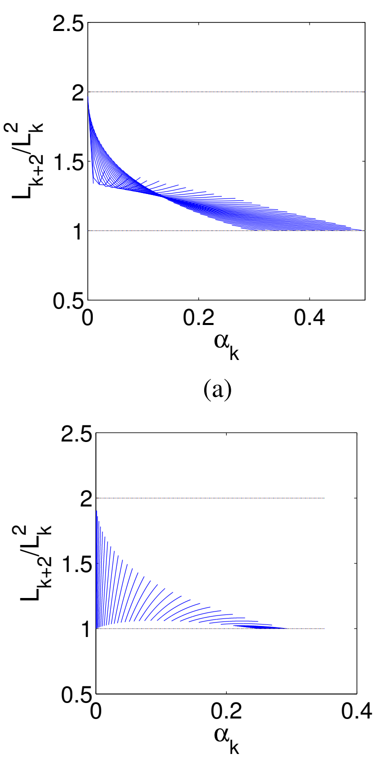

Fig. 5. (a) Ratio L_{k+2}/L_k^2 in region B_1. (b) Ratio L_{k+2}/L_k^2 in region $B_2 \cap R_{\mathcal{U}}$. Each line depicts the ratio versus α_k for a fixed β_k.

Suppose that the balanced binary relay tree has N leaf nodes. Then, the height of the fusion center is $\log N$. For convenience, let $P_N = L_{\log N}$ be (twice) the total error probability at the fusion center. Proposition 2 gives bounds on the relationship between L_k and L_{k+2} in the invariant region R. Hence, in the special case of trees with even height, that is, when $\log N$ is an even integer, it is easy to bound P_N in terms of L_0. In fact, we will bound $\log P_N^{-1}$ which in turn provides bounds for P_N.

Theorem 1: If (α_0, β_0) is in the invariant region R and $\log N$ is even, then

$$\sqrt{N}\left(\log L_0^{-1} - 1\right) \le \log P_N^{-1} \le \sqrt{N} \log L_0^{-1}.$$

Proof: If $(\alpha_0, \beta_0) \in R$, then we have $(\alpha_k, \beta_k) \in R$ for $k = 0, 1, \ldots, \log N - 2$. From Proposition 2, we have

$$L_{k+2} = a_k L_k^2$$

for $k = 0, 1, \ldots, \log N - 2$ and some $a_k \in [1, 2]$. Therefore, for $k = 2, 4, \ldots, \log N$, we have

$$L_k = a_{(k-2)/2} \cdot a_{(k-4)/2}^2 \cdots a_0^{2^{(k-2)/2}} L_0^{2^{k/2}},$$

where $a_i \in [1, 2]$. Substituting $k = \log N$, we have

$$\begin{aligned} P_N &= a_{(k-2)/2} \cdot a_{(k-4)/2}^2 \cdots a_0^{2^{(k-2)/2}} L_0^{2^{\log \sqrt{N}}} \\ &= a_{(k-2)/2} \cdot a_{(k-4)/2}^2 \cdots a_0^{\sqrt{N}/2} L_0^{\sqrt{N}}. \end{aligned}$$

Hence,

$$\begin{aligned} \log P_N^{-1} &= -\log a_{(k-2)/2} - 2\log a_{(k-4)/2} \\ &\quad - \ldots - \frac{\sqrt{N}}{2} \log a_0 + \sqrt{N} \log L_0^{-1}. \end{aligned}$$

Notice that $\log L_0^{-1} > 0$ and, for each i, $0 \le \log a_i \le 1$. Thus,

$$\log P_N^{-1} \le \sqrt{N} \log L_0^{-1}.$$

Finally,

$$\begin{aligned} \log P_N^{-1} &\ge -1 - 2 - \ldots - \frac{\sqrt{N}}{4} - \frac{\sqrt{N}}{2} + \sqrt{N} \log L_0^{-1} \\ &\ge -\sqrt{N} + \sqrt{N} \log L_0^{-1} \\ &= \sqrt{N}\left(\log L_0^{-1} - 1\right). \end{aligned}$$

■

We have derived error probability bounds for balanced binary relay trees with even height. See [33] for error probability bounds for trees with odd height. In the next section, we will use these bounds to study the asymptotic rate of convergence.

V. Asymptotic Rates

In this section, we will use the following notation. Suppose that $f(N) > 0$ and $g(N) > 0$ are two functions defined on positive integers N. If $c_1 g(N) \le f(N) \le c_2 g(N)$ for some positive c_1 and c_2 for sufficiently large N, then we write $f(N) = \Theta(g(N))$.

Notice that as N becomes large, the sequence $\{(\alpha_k, \beta_k)\}$ will eventually move into the invariant region R at some level and stays inside from that point. Therefore, it suffices to consider the decay rate in the invariant region R.

Proposition 3: If $L_0 = \alpha_0 + \beta_0$ is fixed, then

$$\log P_N^{-1} = \Theta(\sqrt{N}).$$

Proof: The convergence of P_N has been proved in [30]. Therefore, for fixed L_0, we have $L_k < 1/2$ for sufficiently large k. Hence, without loss of generality, we may assume that $L_0 < 1/2$. In this case, the bounds in Theorem 1 imply that

$$\log P_N^{-1} = \Theta(\sqrt{N}).$$

■

This implies that the convergence of the total error probability is sub-exponential; more precisely, the decay exponent is essentially $\sqrt{N}$.

Suppose that we wish to determine how many sensors we need to have so that $P_N \le \varepsilon$. Without loss of generality, we assume that L_0 is fixed ($L_0 < 1/2$) and $\varepsilon \in (0, 1)$. The solution is simply to find an N (e.g., the smallest) satisfying the inequality

$$\sqrt{N}\left(\log L_0^{-1} - 1\right) \ge -\log \varepsilon.$$

The smallest N grows like $\Theta((\log \varepsilon)^2)$ (cf., [30], in which the smallest N has a larger growth rate).

VI. Conclusion

We have studied the detection performance of balanced binary relay trees. We precisely describe the evolution of error probabilities in the (α, β) plane as we move up the tree. This allows us to deduce error probability bounds at the fusion center as functions of N. These bounds imply that the total error probability converges to 0 sub-exponentially, with a decay exponent that is essentially $\sqrt{N}$. All our results apply not only to the fusion center, but also to any other node in the tree network. In other words, we can similarly analyze a sub-tree inside the original tree network.

Needless to say, our conclusions are subject to our particular architecture and assumptions. Several questions follow: Considering balanced binary relay trees with sensor and/or connection failures, how would the error probability behave? More generally, what can we say about unbalanced relay trees? In addition, how would the performance change if all the relay nodes make their own measurements?

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