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Robust $\mathcal{H}_\infty$ Filtering for Nonuniformly Sampled Systems

Ghulam Mustafa, Tongwen Chen and Hisaya Fujioka

***Abstract*— We consider the design of a robust $\mathcal{H}_\infty$ filter for sampled-data systems whose measurements are sampled at uncertain and nonuniform sampling instants. A discrete-time, fixed-structure filter is considered. The resulting error system is time-varying, which makes the filter design difficult. A procedure is presented to design the filter so that the error system remains asymptotically stable for all possible variations of the sampling period with an $\mathcal{H}_\infty$ performance level. The effectiveness of the proposed method is demonstrated through numerical examples.**

I. INTRODUCTION

The theory of sampled-data systems with uniform sampling has been well-developed during the last two decades. Many excellent references, e.g., [1], [2], [3], are available for the analysis and design of these systems.

An important assumption in the development of conventional sampled-data control and filtering theory is the periodic sampling of the measurements. However, there are situations where this assumption is not valid and the measurement sampling period uncertainly varies. This could happen, for example, in event-based control systems [4]. Another motivation is the wide spread use of networked/embedded control systems [5], [6]. In fact, networked control systems are the main cause of recent interest in nonuniformly sampled systems [7], [8], [9], [10]. Yet another motivation is the extension of the theory of sampled-data systems to the nonuniform sampling case.

Many controllers use state feedback to generate the control action; however, it is seldom the case that all the state variables of a plant are measured. A state estimator is designed to provide an estimate of the state variables using the measurements from the plant. A popular estimator, when the plant disturbances are not known, is the $\mathcal{H}_\infty$ filter. The problem of $\mathcal{H}_\infty$ filter design for discrete-time systems with uniform sampling has been well-studied, see, for instance, [11], [12]. For the nonuniform sampling case, the $\mathcal{H}_\infty$ filtering problem has been considered in [13], [14], [15]. In [13], the authors transform the nonuniformly sampled system into a continuous-time one with time-varying delay in the input; two types of $\mathcal{H}_\infty$ filters are designed to minimize a modified $\mathcal{H}_\infty$ performance criteria. But, the filters are time-varying in the discrete-time domain. In [14], a discrete-time, robust $\mathcal{H}_\infty$ filter is designed using a parameter-dependent Lyapunov function; the filter design, however, requires the solution of bilinear matrix inequalities. In [15], a class of nonuniformly sampled systems, where the measurement is sampled nonuniformly but at an integer multiples of the state estimation periods, is considered. A mode-dependent $\mathcal{H}_\infty$ filter is designed using a Markovian jump systems approach.

This paper studies the design of an $\mathcal{H}_\infty$ filter for nonuniformly sampled systems. The problem is addressed in the discrete-time domain and a linear constant-parameter filter is designed. It will be shown that the analysis and design of the filter requires the solution of infinite many matrix inequalities; because of the uncertainly varying sampling period. We use the stability robustness idea presented in [7] to generate a grid of finite sampling periods to solve the matrix inequalities for the sampling periods in the grid only. The grid is constructed in such a way that the resulting solution of the matrix inequalities is valid for all possible variations of the sampling period.

The rest of this article is organized as follows: In Section II we formulate the robust $\mathcal{H}_\infty$ filtering problem. Some preliminary results are discussed in Section III. The main results for the analysis and design of the $\mathcal{H}_\infty$ filter are presented in Section IV. Some extensions of the results are discussed in Section V which are followed by two numerical examples in Section VI to demonstrate the effectiveness of the proposed approach.

II. PROBLEM FORMULATION

Consider a stable, continuous linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \quad x(0) = 0 \\ y(t) &= Cx(t) + Dw(t), \\ z(t) &= Lx(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $w(t) \in \mathbb{R}^m$ is the disturbance, $y(t) \in \mathbb{R}^r$ is the measured output, and $z(t) \in \mathbb{R}^s$ is the signal to be estimated. A, B, C, D, and L are matrices of compatible dimensions.

The measurement $y(t)$ from the system is sampled when $t = \tau_k$ where $\{\tau_k : k \geq 0\}$ is a set of arbitrary sampling instants with properties

$$\tau_0 = 0, \quad \text{and} \quad 0 < h_l \leq \tau_{k+1} - \tau_k \leq h_u < \infty, \tag{2}$$

for given h_l and h_u. Note that (2) implies $\lim_{k\to\infty} \tau_k = \infty$.

Let h_k denote the k^{th} sampling period, namely, $h_k := \tau_{k+1} - \tau_k$, a discrete-time equivalent of (1) at the sampling instants τ_k is given as

$$\begin{aligned} x_{k+1} &= \Phi(h_k)x_k + \Gamma(h_k)w_k, \\ y_k &= Cx_k + Dw_k, \\ z_k &= Lx_k, \end{aligned} \tag{3}$$

G. Mustafa and T. Chen are with the Department of Electrical & Computer Engineering, University of Alberta, T6G 2V4 AB Edmonton, Canada [g.mustafa, tchen]@ece.ualberta.ca

H. Fujioka is with the Department of Applied Analysis and Complex Dynamical Systems, Graduate School of Informatics, Kyoto University Kyoto 606-8501, Japan fujioka@i.kyoto-u.ac.jp

where $x_k := x(\tau_k)$, $w_k := w(\tau_k)$, $y_k := y(\tau_k)$, $z_k := z(\tau_k)$, and

$$\Phi(h_k) := e^{h_k A}, \quad \Gamma(h_k) := \int_0^{h_k} e^{(h_k-\eta)A} \mathrm{d}\eta B.$$

Consider a discrete-time filter of the form

$$\begin{aligned} \hat{x}_{k+1} &= A_f \hat{x}_k + B_f y_k, \\ \hat{z}_k &= C_f \hat{x}_k, \end{aligned} \tag{4}$$

where $\hat{x}_k := \hat{x}(\tau_k)$ and $\hat{z}_k := \hat{z}(\tau_k)$.

Define $\bar{x}_k^T = [x_k^T \ \hat{x}_k^T]$; using (3) and (4), the error system can be written as

$$\begin{aligned} \bar{x}_{k+1} &= \bar{A}(h_k)\bar{x}_k + \bar{B}(h_k)w_k, \\ e_k &= \bar{C}\bar{x}_k, \end{aligned} \tag{5}$$

where

$$\bar{A}(h_k) = \begin{bmatrix} \Phi(h_k) & 0 \\ B_f C & A_f \end{bmatrix}, \quad \bar{B}(h_k) = \begin{bmatrix} \Gamma(h_k) \\ B_f D \end{bmatrix},$$
$$\bar{C} = \begin{bmatrix} L & -C_f \end{bmatrix}.$$

The goal is to design a filter of the form in (4) for the system in (1) such that the error system in (5) is asymptotically stable with an $\mathcal{H}_\infty$ performance level $\gamma > 0$.

Definition The error system in (5) is asymptotically stable if, for $w_k \equiv 0$ and $\bar{x}(0) \neq 0$, $\bar{x}_k \to 0$ as $k \to \infty$.

Definition For $w_k \neq 0$, the error system is said to have an $\mathcal{H}_\infty$ performance level $\gamma > 0$ if

$$\|e_k\|_2 \leq \gamma \|w_k\|_2.$$

III. Preliminaries

In order to analyze the $\mathcal{H}_\infty$ performance of the error system we give the following lemma.

Lemma 1: Given $0 < h_l < h_u < \infty$, $\gamma > 0$, and the filter parameters A_f, B_f, and C_f, the error system in (5) is asymptotically stable with an $\mathcal{H}_\infty$ performance level γ if there exists a symmetric matrix $\bar{P} > 0$ such that the following matrix inequality

$$\begin{bmatrix} -\bar{P} & 0 & \bar{A}(h_k)\bar{P} & \bar{B}(h_k) \\ * & -I & \bar{C}\bar{P} & 0 \\ * & * & -\bar{P} & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{6}$$

holds for all $h_k \in [h_l, h_u]$.

Proof: It is an extension of the discrete, time-invariant $\mathcal{H}_\infty$ filtering lemma [11] to the time-varying case. ■

If the sampling period is fixed, i.e., $h_k = \tilde{h} \in [h_l, h_u]$, this is the classical discrete-time $\mathcal{H}_\infty$ filtering problem which has been discussed in [11], [12], [16]. However, when the sampling period is time-varying, the filter design problem becomes complicated as the matrix inequality in (6) has to hold for infinite many values of the sample period $h_k \in [h_l, h_u]$. The challenge is to convert the condition in (6) to a numerically tractable form.

One approach is to use a finite grid, such as $\mathcal{G} = \{h_1, h_2, \cdots, h_N\}$, and test the condition in (6) for the sampling periods in the grid. However, this approach does not imply, in general, that the matrix inequality in (6) holds for all possible variations of the sampling period. In [7], Fujioka proposed a stability robustness idea to construct a grid $\mathcal{G}$ such that if the matrix inequality in (6) holds for the finite number of sampling periods in the grid, it will hold for all sampling periods in $[h_l, h_u]$.

In this paper, we follow the idea given in [7] to test the condition in (6) for filter design. For this, we need the following lemma.

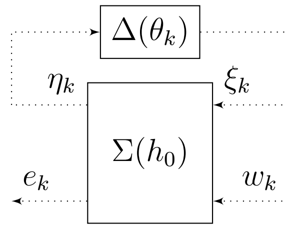

Fig. 1. LFT representation of error system

Lemma 2: The error system in (5) can be re-configured as shown in Fig. 1, where

$$\Sigma(h_0): \quad \begin{aligned} \bar{x}_{k+1} &= \bar{A}(h_0)\bar{x}_k + \bar{B}_1\xi_k + \bar{B}(h_0)w_k, \\ \eta_k &= \bar{C}_1(h_0)\bar{x}_k + \bar{D}_{12}(h_0)w_k, \\ e_k &= \bar{C}\bar{x}_k, \end{aligned} \tag{7}$$

$\xi_k = \Delta(\theta_k)\eta_k$, and $\Delta(\theta_k) = \int_0^{\theta_k} e^{\eta A} \mathrm{d}\eta$. The matrices in (7) are

$$\bar{A}(h_0) = \begin{bmatrix} \Phi(h_0) & 0 \\ B_f C & A_f \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \bar{B}(h_0) = \begin{bmatrix} \Gamma(h_0) \\ B_f D \end{bmatrix},$$
$$\bar{C}_1(h_0) = \begin{bmatrix} A\Phi(h_0) & 0 \end{bmatrix}, \quad \bar{D}_{12}(h_0) = A\Gamma(h_0) + B.$$

Proof: Fix $h_k = h_0 + \theta_k$, from (3) we can write

$$\Phi(h_0 + \theta_k) = e^{\theta_k A}\Phi(h_0) = (I + \Delta(\theta_k)A)\Phi(h_0)$$

and

$$\begin{aligned} \Gamma(h_0 + \theta_k) &= \int_0^{h_0} e^{(h_0+\theta_k-\eta)A} B \mathrm{d}\eta \\ &\quad + \int_{h_0}^{h_0+\theta_k} e^{(h_0+\theta_k-\eta)A} B \mathrm{d}\eta \\ &= e^{\theta_k A}\Gamma(h_0) + \Delta(\theta_k)B \\ &= (I + \Delta(\theta_k)A)\Gamma(h_0) + \Delta(\theta_k)B. \end{aligned}$$

Re-write the matrices in error system (5) as

$$\begin{aligned} \bar{A}(h_k) &= \begin{bmatrix} \Phi(h_0) + \Delta(\theta_k)A\Phi(h_0) & 0 \\ B_f C & A_f \end{bmatrix} \\ &= \begin{bmatrix} \Phi(h_0) & 0 \\ B_f C & A_f \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \Delta(\theta_k) \begin{bmatrix} A\Phi(h_0) & 0 \end{bmatrix}, \end{aligned}$$

$$\bar{B}(h_k) = \begin{bmatrix} \Gamma(h_0) \\ B_f D \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} \Delta(\theta_k) \begin{bmatrix} A\Gamma(h_0) + B \end{bmatrix}.$$

Define $\bar{B}_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and plug-in these expression in (5) to get (7). ■

Define

$$\Sigma_{11} = \bar{C}_1(h_0)(zI - \bar{A}(h_0))^{-1}\bar{B}_1,$$
$$\Sigma_{12} = \bar{D}_{12}(h_0) + \bar{C}_1(h_0)(zI - \bar{A}(h_0))^{-1}\bar{B}(h_0),$$
$$\Sigma_{21} = \bar{C}(zI - \bar{A}(h_0))^{-1}\bar{B}_1,$$
$$\Sigma_{22} = \bar{C}(zI - \bar{A}(h_0))^{-1}\bar{B}(h_0).$$

The mapping from $w_k \to e_k$ is

$$e_k = (\Sigma_{22} + \Sigma_{12}\Delta(\theta_k)(I - \Sigma_{11}\Delta(\theta_k))^{-1}\Sigma_{21})w_k$$

To ensure stability in the presence of variations of the sampling period, it is required that $\{\Sigma_{i,j}|\, i,j = 1,2\}$ be stable and

$$\|\Sigma_{11}\Delta(\theta_k)\|_\infty < 1. \tag{8}$$

One can easily find a scalar α such that $\alpha > \|\Sigma_{11}\|_\infty$. Therefore, the system will be robustly stable as long as

$$\|\Delta(\theta_k)\|_\infty \leq \frac{1}{\alpha}.$$

Thus, in order to ensure robust stability, we need to bound $\|\Delta(\theta_k)\|$. We can define many different bounds for $\Delta(\theta_k)$, one such bounds is given in the following lemma.

Lemma 3 ([17]): For a given matrix $A \in R^{n\times n}$ and $t \geq 0$, we have

$$\|e^{At}\| \leq e^{\mu(A)t},$$

where $\mu(A)$ is the logarithmic norm of A associated with the 2-norm, and is given by

$$\mu(A) = \lambda_{\max}\left(\frac{A + A^*}{2}\right).$$

IV. Main Results

A. Analysis

In this section, we state the main theorem to analyze the robust stability and $\mathcal{H}_\infty$ performance of the error system.

Theorem 1: Given $h_0 > 0$, $\gamma > 0$, and the filter parameters A_f, B_f, and C_f, the error system in (5) will be robustly stable for all $h_k \in \mathcal{H}(h_0, \alpha)$ if there exists a symmetric matrix $\bar{P} > 0$ such that (6) and (8) hold. Here $\alpha = \|\Sigma_{11}\|_\infty$ and the interval $\mathcal{H}(h_0, \alpha)$ is defined as

$$\mathcal{H}(h_0, \alpha) := (\underline{h}, \overline{h}) \cap (0, \infty), \tag{9}$$

where $\underline{h}$ and $\overline{h}$ are given as follows:

$L1)$ if $\mu(-A) = 0, \quad \underline{h} = h_0 - \alpha^{-1},$

$L2)$ elseif $\mu(-A) \leq -\alpha, \quad \underline{h} = -\infty,$

$L3)$ else $\quad \underline{h} = h_0 - \dfrac{1}{\mu(-A)}\log(1 + \alpha^{-1}\mu(-A)).$

$U1)$ if $\mu(A) = 0, \quad \overline{h} = h_0 + \alpha^{-1},$

$U2)$ elseif $\mu(A) \leq -\alpha, \quad \overline{h} = \infty$

$U3)$ else $\quad \overline{h} = h_0 + \dfrac{1}{\mu(A)}\log(1 + \alpha^{-1}\mu(-A)).$

Proof: Assume minimal realizations of the system in (1) and the filter in (4). If the matrix inequality (6) is satisfied, this means there exists a symmetric and positive-definite matrix $\bar{P}$ such that $\rho(\bar{A}(h_0)) < 1$. Since, $\{\Sigma_{ij}|\, i,j = 1,2\}$ have the same state matrix, they are stable.

The interval in (9) can be determined using (8) and Lemma 3. See [7, Proof of Theorem 1]. ■

As pointed out in [7], a direct use of Theorem 1 could be conservative because of the small-gain type condition in (8). This conservatism can be reduced by using a multi-model representation of the error system.

Theorem 2: Given $h_i > 0$ $(i = 1, 2, \cdots, N)$, $\gamma > 0$, and the filter parameters A_f, B_f, and C_f, if there exists a symmetric matrix $\bar{P} > 0$ such that (6) and (8) hold, then the error system in (5) will be robustly stable for all

$$h_k \in \bigcup_{i=1}^{N} \mathcal{H}(h_i, \alpha_i)$$

Proof: Consider the case $i = 1$. Using similar arguments as in Theorem 1 if (6) and (8) hold, there exists a symmetric and positive-definite matrix $\bar{P}$ when $h_k = h_1$. Continuing the discussion for $i = 2, 3, .., N$, we can prove there exists a $\bar{P}$ if (6) and (8) hold for all h_i. ■

B. Design

In this section, we discuss the $\mathcal{H}_\infty$ filter design.

Theorem 3: Given $h_i > 0$ $(i = 1, 2, \cdots, N)$ and $\gamma > 0$, if there exist symmetric and positive-definite matrices $Z \in R^{n\times n}$, $Y \in R^{n\times n}$, and matrices $F \in \mathcal{R}^{r\times n}$, $G \in \mathcal{R}^{s\times n}$, and $Q \in \mathcal{R}^{n\times n}$ such that the small-gain condition in (8) and the LMIs

$$\begin{bmatrix} -Z & -Z & 0 & Z\Phi(h_i) & Z\Phi(h_i) & Z\Gamma(h_i) \\ * & -Y & 0 & Y\Phi(h_i) + FC + Q & Y\Phi(h_i) + FC & Y\Gamma(h_i) + FD \\ * & * & -I & L - G & L & 0 \\ * & * & * & -Z & -Z & 0 \\ * & * & * & * & -Y & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0, \tag{10}$$

hold for all $h_i > 0$, then the error system in (5) is robustly stable for all

$$h_k \in \bigcup_{i=1}^{N} \mathcal{H}(h_i, \alpha_i)$$

with $\mathcal{H}_\infty$ performance level γ for all h_i $(i = 1, 2, \cdots, N)$ with filter parameters

$$A_f = -Y^{-1}Q(I - Y^{-1}Z)^{-1}, \quad B_f = -Y^{-1}F,$$
$$C_f = G(I - Y^{-1}Z)^{-1}. \tag{11}$$

Proof: The LMI in (10) is obtained through a congruence transformation on the matrix inequality (6). For that, we take

$$\bar{P} := \begin{bmatrix} X & U \\ * & \hat{X} \end{bmatrix}, \quad \bar{P}^{-1} := \begin{bmatrix} Y & V \\ * & \hat{Y} \end{bmatrix},$$

Algorithm 1 Robust $\mathcal{H}_\infty$ Optimal Filter Design for Nonuniformly Sampled Systems

Given $0 < h_l < h_u < \infty$, and a large positive integer N_0

0. Initialization: $\mathcal{G} \leftarrow \{(h_l + h_u)/2\}$

1. if $\#\mathcal{G} \geq N_0$, stop without obtaining a filter.

2. Minimize $\delta = \gamma^2$ subject to (10) for all $h_i's$ where h_i is the i^{th} smallest element in $\mathcal{G}$.

3. If

$$[h_l, h_u] \subseteq \bigcup_{i=1}^{\#\mathcal{G}} \mathcal{H}(h_i, \alpha_i)$$

The error system in (5) will be robustly stable with $\mathcal{H}_\infty$ performance $\gamma = \sqrt{\delta}$ with the filter parameters given by (11). Stop. Here

$$\alpha_i := \|\Sigma_{11}(h_i)\|_\infty.$$

4. Update $\mathcal{G}$ by

$$\mathcal{G} \leftarrow \mathcal{G} \bigcup \{(L_j + U_j)/2\}$$

for all j where L_j and U_j are determined so that

$$\bigcup_{j=1}^{M} (L_j, U_j) = (h_l, h_u) \bigcup_{i=1}^{\#\mathcal{G}} \mathcal{H}(h_i, \sqrt{\alpha_i}),$$
$$L_1 < U_1 < L_2 < U_2 < \cdots < L_M < U_M$$

are satisfied. Where $M \leq \#\mathcal{G} + 1$. Go to step **1**.

where X, $\hat{X}$, Y, and $\hat{Y}$ are symmetric and positive-definite matrices. Define

$$J_1 := \begin{bmatrix} X^{-1} & Y \\ 0 & V^T \end{bmatrix}$$

and perform a congruence transformation on (6) with $J = \mathrm{diag}(J_1, I, J_1, I)$. From the definitions of $\bar{P}$ and $\bar{P}^{-1}$, we note that $XY + UV^T = I$ and $XV + U\hat{Y} = 0$. Using these relations, defining $Z := X^{-1}$, $F := VB_f$, $Q := VA_fU^TZ$, $G := C_fU^TZ$, and replacing h_k with h_i, we get (10). ■
The difficulty in applying Theorem 3 is the selection of the sampling periods $\{h_i | \, i = 1, \cdots, N\}$ such that

$$[h_l, h_u] \subseteq \bigcup_{i=1}^{N} \mathcal{H}(h_i, \alpha_i)$$

We present Algorithm 1 to systematically generate the grid. The step 2 in the algorithm is introduced to avoid numerical issues which could happen when $\#\mathcal{G}$ is too large.

V. Some Extensions

In this section, we discuss some extensions of the preceding theory.

A. Robust Performance

The Algorithm 1 gives a filter that is robustly stable but has nominal $\mathcal{H}_\infty$ performance; the condition in (10) guarantees the $\mathcal{H}_\infty$ performance for the sampling periods in the grid $\mathcal{G}$ only. If we are interested in robust $\mathcal{H}_\infty$ performance, it is required that, in addition to (8), the following condition

$$\|\Sigma_{22} + \Sigma_{12}\Delta(\theta_k)(I - \Sigma_{11}\Delta(\theta_k))^{-1}\Sigma_{21}\|_\infty \leq \gamma \qquad (12)$$

holds [18] for all $h_k \in [h_l, h_u]$. In the present framework, this can be achieved by replacing the LMI in (10) with the following LMI:

$$\begin{bmatrix} -Z & -Z & 0 & 0 & Z\Phi(h_i) & Z\Phi(h_i) & Z & Z\Gamma(h_i) \\ * & -Y & 0 & 0 & Y\Phi(h_i) + FC + Q & Y\Phi(h_i) + FC & Y & Y\Gamma(h_i) + FD \\ * & * & -dI & 0 & A\Phi(h_i) & A\Phi(h_i) & 0 & A\Gamma_(h_i) + B \\ * & * & * & -I & L - G & L & 0 & 0 \\ * & * & * & * & -Z & -Z & 0 & 0 \\ * & * & * & * & * & -Y & 0 & 0 \\ * & * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0. \qquad (13)$$

The condition in (13) is based on minimizing the $\mathcal{H}_\infty$ norm of the nominal system in (7) for both channels. It also applies a fixed D-scaling for the uncertainty channel. It should be noted that the attempt to achieve the same $\mathcal{H}_\infty$ performance γ for both channels may lead to conservative results. A trade-off can be made between the range of variations in the sampling period and the $\mathcal{H}_\infty$ performance by selecting different performance measures for both channels.

B. Sampled-Data Approach to $\mathcal{H}_\infty$ Filtering

In this paper, the $\mathcal{H}_\infty$ performance is measured in discrete time. Another approach could be to measure the performance in continuous time. In the present framework, this can be achieved by holding the state estimate using a zero-order-hold. However, formulating the problem in this way may not be very insightful; as there will always be an estimation error. A more appropriate approach will be to consider a continuous-time filter structure with sampled inputs. This type of filter structure was considered in [13] and design was presented using an input-delay approach.

VI. Numerical Examples

In this section, we use two numerical examples to demonstrate the applicability and effectiveness of the proposed approach.

A. Example 1

We consider the same parameters as in [13] for the plant in (1):

$$A = \begin{bmatrix} 0 & 1 \\ -16 & -4.8 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 16 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.1, \quad L = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

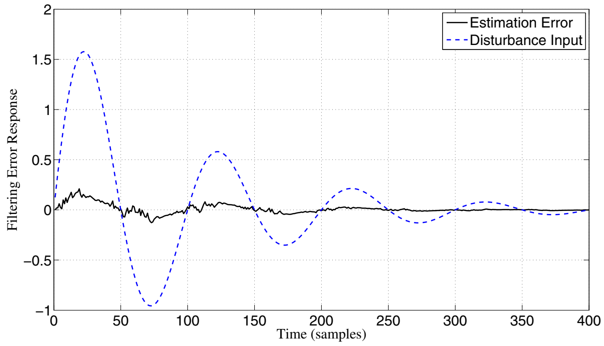

Fig. 2. Estimation error response

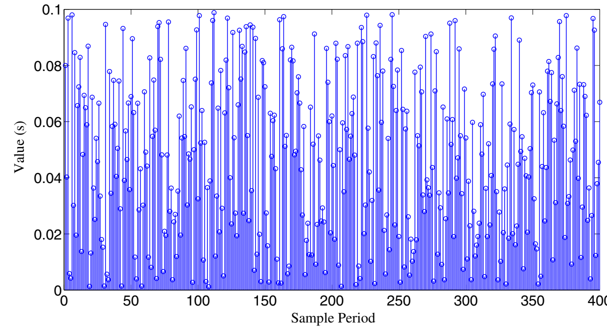

Fig. 3. Variation of the measurement sampling interval

We take $h_l = 0.01$ and $h_u = \pi/25$ and apply the filtering design procedure given in Algorithm 1. We can find a robust $\mathcal{H}_\infty$ filter with $\gamma = 0.1017$ and with only one element in the grid $\mathcal{G}$.

For the same system parameters, the authors in [13] achieved $\mathcal{H}_\infty$ performance $\gamma = 0.4647$ for the type 1 filter and $\gamma = 0.4876$ for the type 2 filter. It is remarked that the performance index in [13] is in continuous time whereas the performance index in this paper is in discrete time.

B. Example 2

Consider the following matrices for the plant in (1):

$$A = \begin{bmatrix} -b/J & K_T/J \\ -K_\theta/L_a & -R_a/L_a \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0, \quad L = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The values of the constants are $b = 0.1\text{Nms}$, $J = 0.01\text{kgm}^2/\text{s}^2$, $K_T = K_q = 0.01\text{Nm/A}$, $R_a = 1\Omega$, and $L_a = 0.5\text{H}$. This system was considered in [14] where the authors designed a robust $\mathcal{H}_\infty$ filter for $h_l = 0.001$ and $h_u = 0.099$ with $\mathcal{H}_\infty$ performance $\gamma = 1.8174$.

Following Algorithm 1, we can find a filter with $\gamma = 0.0952$ and grid $\mathcal{G} = \{0.05\}$. It should be noted that this value of γ is based on the nominal performance condition. The parameters of the filter are

$$A_f = \begin{bmatrix} -0.0440 & 0.0373 \\ -1.0674 & 0.9048 \end{bmatrix}, \quad B_f = \begin{bmatrix} 0.6505 \\ 1.0666 \end{bmatrix},$$
$$C_f = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Let $w_k = 2\exp(-0.01k)\sin(0.02\pi k)$ and $\bar{x}(0) = 0$. Fig. 2 shows a plot of the estimation error and disturbance input. We observe that the disturbance is effectively attenuated. The corresponding measurement sampling intervals are shown in Fig. 3.

VII. Conclusions

This paper presents a discrete-time, robust $\mathcal{H}_\infty$ filter design procedure for systems whose sampling period varies between a lower and upper bounds. The designed filter ensures robust stability of the error system for all possible variations of the sampling period with an $\mathcal{H}_\infty$ performance level. The effectiveness of the approach is demonstrated using numerical examples.

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