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On stability of continuous-time quantum filters

H. Amini M. Mirrahimi P. Rouchon

Abstract— **We prove that the fidelity between the quantum state governed by a continuous time stochastic master equation driven by a Wiener process and its associated quantum-filter state is a sub-martingale. This result is a generalization to non-pure quantum states where fidelity does not coincide in general with a simple Frobenius inner product. This result implies the stability of such filtering process but does not necessarily ensure the asymptotic convergence of such quantum-filters.**

I. INTRODUCTION

The quantum filtering theory provides a foundation of statistical inference inspired in e.g. quantum optical systems. These systems are described by continuous-time quantum stochastic differential equations. These stochastic master equations include the measurement back-action on the quantum-state. The quantum filtering theory has been developed by Davies in the 1960s [10], [11] and in its modern form by Belavkin in the 1980s [4], [5], [3].

To these stochastic master equations are attached so-called quantum filters providing, from the real-time measurements, estimations of the quantum states. Robustness and convergence of such estimation process has been investigated in many papers. For example, sufficient convergence conditions, related to observability issues, are given in [20] and [19]. As far as we know, general and verifiable necessary and sufficient convergence conditions do not exist yet. For links between quantum filtering and observers design on cones see [6]. In this paper, we generalize a stability result for pure states (see, e.g., [12]) to arbitrary mixed quantum states. More precisely, we prove that the fidelity between the quantum state (that could be a mixed state) and its associated quantum-filter state is a sub-martingale: this means that in average, the estimated state tends to be closer to the system state. This does not imply its asymptotic convergence for large times. To prove such convergence, more specific analysis depending on the precise structure of the Hamiltonian, Lindbladian and measurement operators defining the system model is required. This paper can also be seen as an extension to continuous-time evolution of [18].

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Mines ParisTech, Centre Automatique et Systèmes, Mathématiques et Systèmes, 60 Bd Saint Michel, 75272 Paris cedex 06, France, hadis.amini@mines-paristech.fr

INRIA Paris-Rocquencourt, Domaine de Voluceau, B.P. 105, 78153 Le Chesnay cedex, France, mazyar.mirrahimi@inria.fr

Mines ParisTech, Centre Automatique et Systèmes, Mathématiques et Systèmes, 60 Bd Saint Michel, 75272 Paris cedex 06, France, pierre.rouchon@mines-paristech.fr

This paper is organized as follows. In section II, we introduce the non linear stochastic master equations driven by Wiener processes and providing the evolutions of the quantum state and of the quantum-filter state and we state the main result (Theorem 2.1). Section III is devoted to the proof of this result: firstly we consider an approximation via stochastic master equations driven by Poisson processes (diffusion approximation); secondly, we prove the sub-martingale property via a time discretization. In final section, we suggest some possible extensions of this work.

II. MAIN RESULT

We will consider quantum systems of finite dimensions $1 < N < \infty$. The state space of such a system is given by the set of density matrices

$$\mathcal{D} := \{\rho \in \mathbb{C}^{N\times N} | \quad \rho = \rho^\dagger, \quad \mathrm{Tr}\,(\rho) = 1, \quad \rho \geq 0\}.$$

Formally the evolution of the real state $\rho \in \mathcal{D}$ is described by the following stochastic master equation (cf. [3], [7], [22])

$$d\rho_t = -\tfrac{i}{\hbar}[H, \rho_t]\, dt + \mathcal{L}(\rho_t)\, dt + \Lambda(\rho_t)\; dW_t\;, \qquad (1)$$

where

- the notation $[A, B]$ refers to $AB - BA$;
- $H = H^\dagger$ is a Hermitian operator which describes the action of external forces on the system ;
- dW_t is the Wiener process which is the following innovation

$$dW_t = dy_t - \;\mathrm{Tr}\left((L + L^\dagger)\,\rho_t\right)\, dt\;, \qquad (2)$$

  where y_t is a continuous semi-martingale with quadratic variation $\langle y, y\rangle_t = t$ (which is the observation process obtained from the system) and L is an arbitrary matrix which determines the measurement process (typically the coupling to the probe field for quantum optic systems) ;
- the super-operator $\mathcal{L}$ is the Lindblad operator,

$$\mathcal{L}(\rho) := -\tfrac{1}{2}\{L^\dagger L, \rho\} + L\rho L^\dagger,$$

  where the notation $\{A, B\}$ refers to $AB + BA$;
- the super-operator Λ is defined by

$$\Lambda(\rho) := L\rho + \rho L^\dagger - \mathrm{Tr}\left((L + L^\dagger)\rho\right)\rho.$$

All the developments remain valid when H and L are deterministic time-varying matrices. For clarity sake, we do not recall below such possible time dependence.

The evolution of the quantum filter of state $\widehat{\rho}_t \in \mathcal{D}$ is described by the following stochastic master equation which depends on the time-continuous measurement y_t depending on the true quantum state ρ_t via (2) (see, e.g., [1]):

$$d\widehat{\rho}_t = -\tfrac{i}{\hbar}[H,\widehat{\rho}_t]dt + \mathcal{L}(\widehat{\rho}_t)\,dt + \Lambda(\widehat{\rho}_t)\big(dy_t - \operatorname{Tr}\big((L+L^\dagger)\widehat{\rho}_t\big)\,dt\big). \tag{3}$$

Replacing dy_t by its value given in (2), we obtain

$$d\widehat{\rho}_t = -\tfrac{i}{\hbar}[H,\widehat{\rho}_t]dt + \mathcal{L}(\widehat{\rho}_t)dt + \Lambda(\widehat{\rho}_t)\,dW_t + \Lambda(\widehat{\rho}_t)\Big(\operatorname{Tr}\big((L+L^\dagger)\rho_t\big) - \operatorname{Tr}\big((L+L^\dagger)\widehat{\rho}_t\big)\Big)dt.$$

A usual measurement of the difference between two quantum states ρ and σ, is given by the fidelity, a real number between 0 and 1. More precisely, the fidelity between ρ and σ in $\mathcal{D}$ is given by (see [16, chapter 9] for more details)

$$F(\rho,\sigma) = \operatorname{Tr}^2\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right). \tag{4}$$

Here $F(\rho,\sigma)=1$ means $\rho=\sigma$, and $F(\rho,\sigma)=0$ means that the support of ρ and σ are orthogonal. $F(\rho,\sigma)$ coincides with their inner product $\operatorname{Tr}(\rho\sigma)$ when at least one of the states ρ or σ is pure (i.e., orthogonal projector of rank one). It is well known that the stochastic master equations (1) and (3) leave the domain $\mathcal{D}$ positively invariant. This results form the fact that, using Ito rules, we have

$$\rho_{t+dt} = \frac{\left(\mathbb{I}-\frac{iH}{\hbar}dt-\frac{1}{2}L^\dagger Ldt+Ldy_t\right)\ \rho_t\ \left(\mathbb{I}-\frac{iH}{\hbar}dt-\frac{1}{2}L^\dagger Ldt+Ldy_t\right)^\dagger}{\operatorname{Tr}\left(\left(\mathbb{I}-\frac{iH}{\hbar}dt-\frac{1}{2}L^\dagger Ldt+Ldy_t\right)\ \rho_t\ \left(\mathbb{I}-\frac{iH}{\hbar}dt-\frac{1}{2}L^\dagger Ldt+Ldy_t\right)^\dagger\right)} \tag{5}$$

and

$$\widehat{\rho}_{t+dt} = \frac{\left(\mathbb{I}-\frac{iH}{\hbar}dt-\frac{1}{2}L^\dagger Ldt+Ldy_t\right)\ \widehat{\rho}_t\ \left(\mathbb{I}-\frac{iH}{\hbar}dt-\frac{1}{2}L^\dagger Ldt+Ldy_t\right)^\dagger}{\operatorname{Tr}\left(\left(\mathbb{I}-\frac{iH}{\hbar}dt-\frac{1}{2}L^\dagger Ldt+Ldy_t\right)\ \widehat{\rho}_t\ \left(\mathbb{I}-\frac{iH}{\hbar}dt-\frac{1}{2}L^\dagger Ldt+Ldy_t\right)^\dagger\right)} \tag{6}$$

where $dy_t = \operatorname{Tr}\big((L+L^\dagger)\,\rho_t\big)\ dt + dW_t$.

These alternative formulations imply then directly that, as soon as, ρ_0 and $\widehat{\rho}_0$ belong to $\mathcal{D}$, ρ_t and $\widehat{\rho}_t$ remain in $\mathcal{D}$ for all $t \geq 0$. Therefore the expression of fidelity given by (4) is well defined.

We are now in position to state the main result of this paper.

Theorem 2.1: Consider the Markov processes $(\rho_t,\widehat{\rho}_t)$ satisfying the stochastic master Equations (1) and (3) respectively with ρ_0, $\widehat{\rho}_0$ in $\mathcal{D}$. Then the fidelity $F(\rho_t,\widehat{\rho}_t)$, defined in Equation (4), is a submartingale, i.e. $\mathbb{E}\left(F(\rho_t,\widehat{\rho}_t)|(\rho_s,\widehat{\rho}_s)\right) \geq F(\rho_s,\widehat{\rho}_s)$, for all $t \geq s$.

We recall that the above theorem generalize the results of [12] to arbitrary purity of the real states and quantum filter. If ρ_0 is pure, then ρ_t remains pure for all $t>0$. In this case, $F(\rho_t,\widehat{\rho}_t)$ coincides with $\operatorname{Tr}(\rho_t\widehat{\rho}_t)$. It is proved in [12] that this Frobenius inner product is a sub-martingale for any initial value of $\widehat{\rho}_t$: $\frac{d}{dt}\mathbb{E}\left(\operatorname{Tr}(\rho_t\widehat{\rho}_t)\right) \geq 0$. The main idea of the proof in [12] consists in using Itô's formula to reduce the theorem to showing that $\mathbb{E}\left(\operatorname{Tr}(d\rho_t\widehat{\rho}_t + \rho_t d\widehat{\rho}_t + d\rho_t d\widehat{\rho}_t)\right) \geq 0$, and then using the shift invariance of the operator L in the dynamics (1) and (3) and choosing an appropriate value.

In the absence of any information on the purity of the real states and the quantum filter, the fidelity is given by (4), and the application of Itô's formula for the above expression becomes much more involved. In particular, the calculation of the cross derivatives was so complicated that it became hopeless to proceed this way. As the proof presented in the next section shows, we had to choose an undirect way to approach the theorem which allowed us to avoid the heavy calculations based on second order derivative of F.

III. Proof of Theorem 2.1

We proceed in two steps.

- In the first step, we describe briefly how we obtain the stochastic master equations (1) and (3) as the limits of the stochastic master equations with Poisson processes using the diffusive limits inspired from the physical homodyne detection model [2], [23].
- In the second step, we show that the fidelity between the real state and the quantum filter which are the solutions of stochastic master equations with Poisson processes is a submartingale.

Step 1. Take $\alpha>0$ a large real number and consider the evolution of the quantum state ρ_t^α described by the following stochastic master equation derived from homodyne detection scheme (see section 6.4 of [8] or [2], [23]) for more physical details):

$$d\rho_t^\alpha = -\tfrac{i}{\hbar}[H,\rho_t^\alpha]dt - \tfrac{1}{4}\Lambda_\alpha(\rho_t^\alpha)dt + \Upsilon_\alpha(\rho_t^\alpha)dN_1 - \tfrac{1}{4}\Lambda_{-\alpha}(\rho_t^\alpha)dt + \Upsilon_{-\alpha}(\rho_t^\alpha)dN_2\,, \tag{7}$$

where the super-operators Υ_α is defined as follows

$$\Upsilon_\alpha(\rho) := \frac{(L+\alpha)\rho(L^\dagger+\alpha)}{\operatorname{Tr}\left((L+\alpha)\rho(L^\dagger+\alpha)\right)} - \rho,$$

and the super-operator Λ_α is defined by

$$\Lambda_\alpha(\rho) := (L+\alpha)\rho + \rho(L^\dagger+\alpha) - \operatorname{Tr}\big((L+L^\dagger+2\alpha)\rho\big)\,\rho.$$

The super-operators $\Lambda_{-\alpha}$ and $\Upsilon_{-\alpha}$ are just obtained with replacing α by $-\alpha$ in the expressions given in above.

The two processes dN_1 and dN_2 are defined by

$$dN_1 := N_{t+dt}^1 - N_t^1 \quad \text{and} \quad dN_2 := N_{t+dt}^2 - N_t^2$$

where N^1 and N^2 are two Poisson processes. dN_1 and dN_2 take value 1 by probabilities $\frac{1}{2}\operatorname{Tr}\big((L^\dagger+\alpha)(L+\alpha)\rho_t^\alpha\big)\,dt$ and $\frac{1}{2}\operatorname{Tr}\big((L^\dagger-\alpha)(L-\alpha)\rho_t^\alpha\big)\,dt$, respectively, and take value 0 by the complementary probabilities.

Similarly, the following stochastic master equation describes the infinitesimal evolution of associated quantum filter of state $\widehat{\rho}_t^\alpha$ (see [1]):

$$d\widehat{\rho}_t^{\,\alpha} = -\tfrac{i}{\hbar}[H, \widehat{\rho}_t^{\,\alpha}]dt - \tfrac{1}{4}\Lambda_\alpha(\widehat{\rho}_t^{\,\alpha})dt + \Upsilon_\alpha(\widehat{\rho}_t^{\,\alpha})dN_1 - \tfrac{1}{4}\Lambda_{-\alpha}(\widehat{\rho}_t^{\,\alpha})dt + \Upsilon_{-\alpha}(\widehat{\rho}_t^{\,\alpha})dN_2. \quad (8)$$

The following diffusive limit is obtained by the central limit theorem when α tends to $+\infty$ for the semi-martingale processes applied to dN_q, $q = 1, 2$, (see [15] or [14] for more details)

$$dN_q \xrightarrow{\text{law}} \langle \tfrac{dN_q}{dt} \rangle \, dt + \sqrt{\langle \tfrac{dN_q}{dt} \rangle} \, dW_q \,, \quad (9)$$

where the notation $\langle A \rangle$ refers to the mean value of A. Here $\langle dN_1 \rangle = \frac{1}{2}\text{Tr}\left((L^\dagger + \alpha)(L+\alpha)\rho_t^\alpha\right) dt$ and $\langle dN_2 \rangle = \frac{1}{2}\text{Tr}\left((L^\dagger - \alpha)(L-\alpha)\rho_t^\alpha\right) dt$ and dW_1 and dW_2 are two independent Wiener processes and the convergence in (9) is in law.

The stochastic master Equations (1) and (3) are obtained by replacing the processes dN_q for $q \in \{1, 2\}$ by their limits given in (9) in the master equations (7) and (8) and taking the limit when α goes to $+\infty$ and keeping only the lowest ordered terms in α^{-1}. Such a result is usually called diffusion approximation (see e.g [9]).

Notice that dW appearing in the stochastic master equations (1) and (3) is given in terms of its independent constituents by

$$dW = \sqrt{\tfrac{1}{2}}\left(dW_1 + dW_2\right),$$

and is thus itself a standard Wiener process.

The following theorem from [17] justifies the diffusion approximation described above.

Theorem 3.1 (Pellegrini-Petruccione [17]): The solutions of the stochastic master Equations (7) and (8) converge in law, when $\alpha \to +\infty$, to the solutions of the stochastic master Equations (1) and (3), respectively.

Step 2**.** We now prove that the fidelity between two arbitrary solutions of the stochastic master Equations (7) and (8) is a submartingale.

Proposition 3.1: Consider the Markov process $(\rho^\alpha, \widehat{\rho}^\alpha)$ which satisfy the stochastic master Equations (7) and (8). Then the fidelity defined in Equation (4) is a submartingale, i.e., for all $t \geq s$, we have

$$\mathbb{E}\left(F(\rho_t^\alpha, \widehat{\rho}_t^\alpha) | (\rho_s^\alpha, \widehat{\rho}_s^\alpha)\right) \geq F(\rho_s^\alpha, \widehat{\rho}_s^\alpha).$$

Proof: We consider approximations of the time-continuous Markov processes (7) and (8) by discrete-time Markov processes ξ_k and $\widehat{\xi}_k$:

$$\xi_{k+1} = \frac{M_{\mu_k}\xi_k M_{\mu_k}^\dagger}{\text{Tr}\left(M_{\mu_k}\xi_k M_{\mu_k}^\dagger\right)} \quad \text{and} \quad \widehat{\xi}_{k+1} = \frac{M_{\mu_k}\widehat{\xi}_k M_{\mu_k}^\dagger}{\text{Tr}\left(M_{\mu_k}\widehat{\xi}_k M_{\mu_k}^\dagger\right)}, \quad (10)$$

where

- $k \in \{0, \cdots, n\}$ for a fixed large n;
- initial condition $\xi_0 = \rho_s^\alpha$ and $\widehat{\xi}_0 = \widehat{\rho}_s^\alpha$;
- μ_k is a random variable taking values $\mu \in \{0, 1, 2\}$ with probability $P_{\mu,k} = \text{Tr}\left(M_\mu \xi_k M_\mu^\dagger\right)$;
- The operators M_0, M_1 and M_2 are defined as follows

$$M_0 := 1 - \tfrac{1}{4}(L^\dagger + \alpha)(L+\alpha)\epsilon_n - \tfrac{1}{4}(L^\dagger - \alpha)(L-\alpha)\epsilon_n - \tfrac{i}{\hbar}H\epsilon_n;$$

$$M_1 := (L+\alpha)\sqrt{\tfrac{1}{2}\epsilon_n};$$

and

$$M_2 := (L-\alpha)\sqrt{\tfrac{1}{2}\epsilon_n};$$

with $\epsilon_n = \frac{t-s}{n}$.

In the following lemma, we show that ξ_n and $\widehat{\xi}_n$ correspond to the Euler-Maruyama time discretization. Since (7) and (8) depend smoothly on ρ_t^α and $\widehat{\rho}_t^\alpha$, ξ_n and $\widehat{\xi}_n$ converge in law towards ρ_t^α and $\widehat{\rho}_t^\alpha$ when $n \mapsto +\infty$.

Lemma 3.1: The processes ξ_k and $\widehat{\xi}_k$ correspond up to second order terms in ϵ_n, to the Euler-Maruyama discretization scheme of (7) and (8) on $[s, t]$.

Proof: we regard the three following possible cases which arrive in according to the different values of μ_k . In each case, we show that ξ_k and $\widehat{\xi}_k$ for $k \in \{0, \cdots, n\}$ are the numerical solutions of the dynamics (7) and (8) respectively, with the following partition $s \leq s + \epsilon_n \leq \cdots \leq s + (n-1)\epsilon_n \leq t$, where the uniform step length ϵ_n is $\frac{t-s}{n}$.

Case 1. We first consider the case where $\mu_k = 0$ which arrives with probability $P_{0,k} = \text{Tr}\left(M_0 \xi_k M_0^\dagger\right)$. Note that

$$M_0\xi_k M_0^\dagger = \xi_k - \tfrac{1}{4}\{(L^\dagger + \alpha)(L+\alpha), \xi_k\}\,\epsilon_n - \tfrac{1}{4}\{(L^\dagger - \alpha)(L-\alpha), \xi_k\}\,\epsilon_n - \tfrac{i}{\hbar}[H, \xi_k]\,\epsilon_n + \mathcal{O}(\epsilon_n^2).$$

Therefore

$$\text{Tr}\left(M_0\xi_k M_0^\dagger\right) = 1 - \tfrac{1}{2}\text{Tr}\left((L^\dagger + \alpha)(L+\alpha)\xi_k\right)\,\epsilon_n - \tfrac{1}{2}\text{Tr}\left((L^\dagger - \alpha)(L-\alpha)\xi_k\right)\,\epsilon_n + \mathcal{O}((\epsilon_n)^2)$$

and

$$\left(\text{Tr}\left(M_0\xi_k M_0^\dagger\right)\right)^{-1} \approx 1 + \tfrac{1}{2}\text{Tr}\left((L^\dagger + \alpha)(L+\alpha)\xi_k\right)\,\epsilon_n + \tfrac{1}{2}\text{Tr}\left((L^\dagger - \alpha)(L-\alpha)\xi_k\right)\,\epsilon_n + \mathcal{O}((\epsilon_n)^2).$$

Therefore, we find the following dynamics

$$\xi_{k+1} \approx \xi_k - \tfrac{1}{4}\{(L^\dagger + \alpha)(L+\alpha), \xi_k\}\,\epsilon_n - \tfrac{1}{4}\{(L^\dagger - \alpha)(L-\alpha), \xi_k\}\,\epsilon_n + \tfrac{1}{2}\text{Tr}\left((L^\dagger + \alpha)(L+\alpha)\xi_k\right)\,\xi_k\,\epsilon_n + \tfrac{1}{2}\text{Tr}\left((L^\dagger - \alpha)(L-\alpha)\xi_k\right)\,\xi_k\,\epsilon_n + \mathcal{O}(\epsilon_n^2).$$

This can also be written as follows

$$\xi_{k+1} - \xi_k \approx -\tfrac{1}{4}\Lambda_\alpha(\xi_k)\,\epsilon_n - \tfrac{1}{4}\Lambda_{-\alpha}(\xi_k)\,\epsilon_n + \mathcal{O}(\epsilon_n^2). \quad (11)$$

Obviously, this dynamics in the first order of ϵ_n is equivalent to the dynamics of the numerical solution of the stochastic master Equation (7) with the partition $s \leq s+\epsilon_n \leq \cdots \leq s+(n-1)\epsilon_n \leq t$, when

$$N^1_{s+(k+1)\epsilon_n} - N^1_{s+k\epsilon_n} = 0 \quad \text{and} \quad N^2_{s+(k+1)\epsilon_n} - N^2_{s+k\epsilon_n} = 0,$$

which arrives with probability

$$\left(1 - \tfrac{1}{2}\mathrm{Tr}\left((L+\alpha)(L^\dagger+\alpha)\,\xi_k\right)\,\epsilon_n\right)\cdots$$
$$\cdots\left(1 - \tfrac{1}{2}\mathrm{Tr}\left((L-\alpha)(L^\dagger-\alpha)\,\xi_k\right)\,\epsilon_n\right).$$

This probability, in the first order of ϵ_n is equal to $\mathrm{Tr}\left(M_0\xi_k M_0^\dagger\right)$.

Case 2. The second case corresponds to $\mu_k = 1$ which arrives with probability $\mathrm{Tr}\left(M_1\xi_k M_1^\dagger\right)$. We find the following dynamics

$$\xi_{k+1} = \tfrac{(L+\alpha)\xi_k(L^\dagger+\alpha)}{\mathrm{Tr}((L+\alpha)\xi_k(L^\dagger+\alpha))} = \Upsilon[L+\alpha]\,\xi_k + \xi_k.$$

We observe that the numerical solution of the stochastic master Equation (7) follows also the same dynamics when

$$N^1_{s+(k+1)\epsilon_n} - N^1_{s+k\epsilon_n} = 1 \quad \text{and} \quad N^2_{s+(k+1)\epsilon_n} - N^2_{s+k\epsilon_n} = 0,$$

which arrives with probability

$$\left(\tfrac{1}{2}\mathrm{Tr}\left((L+\alpha)(L^\dagger+\alpha)\,\xi_k\right)\,\epsilon_n\right)\cdots$$
$$\cdots\left(1 - \tfrac{1}{2}\mathrm{Tr}\left((L-\alpha)(L^\dagger-\alpha)\,\xi_k\right)\,\epsilon_n\right).$$

This is equal to $\mathrm{Tr}\left(M_1\xi_k M_1^\dagger\right)$, in the first order of ϵ_n.

Case 3. Now we consider the last case $\mu_k = 2$ which arrives with probability $\mathrm{Tr}\left(M_2\xi_k M_2^\dagger\right)$. Therefore, we have

$$\xi_{k+1} = \tfrac{(L-\alpha)\xi_k(L^\dagger-\alpha)}{\mathrm{Tr}((L-\alpha)\xi_k(L^\dagger-\alpha))} = \Upsilon_{-\alpha}(\xi_k) + \xi_k.$$

Which can also be written by the stochastic master equation (7) with taking ξ_k as the numerical solution and

$$N^1_{s+(k+1)\epsilon_n} - N^1_{s+k\epsilon_n} = 0 \quad \text{and} \quad N^2_{s+(k+1)\epsilon_n} - N^2_{s+k\epsilon_n} = 1,$$

which arrives with probability

$$\left(1 - \tfrac{1}{2}\mathrm{Tr}\left((L+\alpha)(L^\dagger+\alpha)\,\xi_k\right)\,\epsilon_n\right)\cdots$$
$$\cdots\left(\tfrac{1}{2}\mathrm{Tr}\left((L-\alpha)(L^\dagger-\alpha)\,\xi_k\right)\,\epsilon_N\right).$$

Where in the first order of ϵ_n, this probability is equal to $\mathrm{Tr}\left(M_2\xi_k M_2^\dagger\right)$.

Remark that, if we neglect the terms in the order of ϵ_n^2, The probability of $N^1_{s+(k+1)\epsilon_n} - N^1_{s+k\epsilon_n} = 1$ and $N^2_{s+(k+1)\epsilon_n} - N^2_{s+k\epsilon_n} = 1$ is negligible. Now it is clear that ξ_k and similarly $\widehat{\xi}_k$ are respectively the numerical solutions of the stochastic master Equations (7) and (8) obtained by Euler-Maruyama method. As the right hand side of the stochastic master Equations (7) and (8) are smooth with respect to ρ and $\widehat{\rho}$, we can use the result of [13, Theorem 1] to conclude the convergence in law of ξ_n and $\widehat{\xi}_n$ to ρ_t^α and $\widehat{\rho}_t^\alpha$ for large n.

■

Now we notice that

$$M_0^\dagger M_0 + M_1^\dagger M_1 + M_2^\dagger M_2 = \mathbb{I} + \mathcal{O}(\epsilon_n^2) := A,$$

Take $\widetilde{M_r} := (\sqrt{A})^{-1} M_r$ for $r = 0, 1, 2$ which satisfy necessarily

$$\widetilde{M_0}^\dagger \widetilde{M_0} + \widetilde{M_1}^\dagger \widetilde{M_1} + \widetilde{M_2}^\dagger \widetilde{M_2} = \mathbb{I}. \qquad (12)$$

Now we define the following Markov processes χ_k and $\widehat{\chi}_k$ by

$$\chi_{k+1} = \frac{\widetilde{M_{\mu_k}}\chi_k \widetilde{M_{\mu_k}}^\dagger}{\mathrm{Tr}\left(\widetilde{M_{\mu_k}}\chi_k \widetilde{M_{\mu_k}}^\dagger\right)} \qquad (13)$$

and

$$\widehat{\chi}_{k+1} = \frac{\widetilde{M_{\mu_k}}\widehat{\chi}_k \widetilde{M_{\mu_k}}^\dagger}{\mathrm{Tr}\left(\widetilde{M_{\mu_k}}\widehat{\chi}_k \widetilde{M_{\mu_k}}^\dagger\right)}, \qquad (14)$$

where

- $k \in \{0, \cdots, n\}$ for a fixed large n;
- $\chi_0 = \rho_s^\alpha$ and $\widehat{\chi}_0 = \widehat{\rho}_s^\alpha$;
- μ_k is a random variable taking values $\mu \in \{0, 1, 2\}$ with probability $P_{\mu,k} = \mathrm{Tr}\left(\widetilde{M}_\mu \chi_k \widetilde{M}_\mu^\dagger\right)$.

Clearly χ_k and $\widehat{\chi}_k$ can also be seen as the numerical solutions of the stochastic master Equations (7) and (8), since $(\sqrt{A})^{-1} = \mathbb{I} - \mathcal{O}(\epsilon_n^2)$, therefore in the first order of ϵ_n, the solutions ξ_k and $\widehat{\xi}_k$ are equal to χ_k and $\widehat{\chi}_k$, respectively. But, the advantage of using χ_k and $\widehat{\chi}_k$ instead of ξ_k and $\widehat{\xi}_k$ is that the operators $\widetilde{M_r}$ are Kraus operators since they satisfy Equality (12). Thus we can apply Theorem 1 in [18], which proves that $F(\chi_k, \widehat{\chi}_k)$ is a sub-martingale.

Theorem 3.2 ([18]): Consider the Markov chain $(\chi_k, \widehat{\chi}_k)$ satisfying (13) and (14). Then $F(\chi_k, \widehat{\chi}_k)$ is a sub-martingale: $\mathbb{E}\left(F(\chi_{k+1}, \widehat{\chi}_{k+1}) | (\chi_k, \widehat{\chi}_k)\right) \geq F(\chi_k, \widehat{\chi}_k)$.

Thus we have

$$\mathbb{E}\left(F(\chi_n, \widehat{\chi}_n) \mid \chi_0, \widehat{\chi}_0\right) \geq F(\chi_0, \widehat{\chi}_0) = F(\rho_s^\alpha, \widehat{\rho}_s^\alpha)$$

Therefore by Lemma 3.1, we have necessarily

$$\mathbb{E}\left(F(\rho_t^\alpha, \widehat{\rho}_t^\alpha) | \rho_s^\alpha, \widehat{\rho}_s^\alpha)\right) \geq F(\rho_s^\alpha, \widehat{\rho}_s^\alpha),$$

for all $t \geq s$, since we have (convergence in law) $\rho_t^\alpha = \lim_{n\longrightarrow\infty} \chi_n$, $\widehat{\rho}_t^\alpha = \lim_{n\longrightarrow\infty} \widehat{\chi}_n$, $\chi_0 = \rho_s^\alpha$ and $\widehat{\chi}_0 = \widehat{\rho}_s^\alpha$.

■

We now apply Theorem 3.1 and we use the fact that the function F is bounded by one and continuous with respect to ρ and $\widehat{\rho}$:

$$\mathbb{E}\left(F(\rho_t, \widehat{\rho}_t) | (\rho_s, \widehat{\rho}_s)\right) \geq F(\rho_s, \widehat{\rho}_s),$$

for all $t \geq s$, which ends the proof of Theorem 2.1.

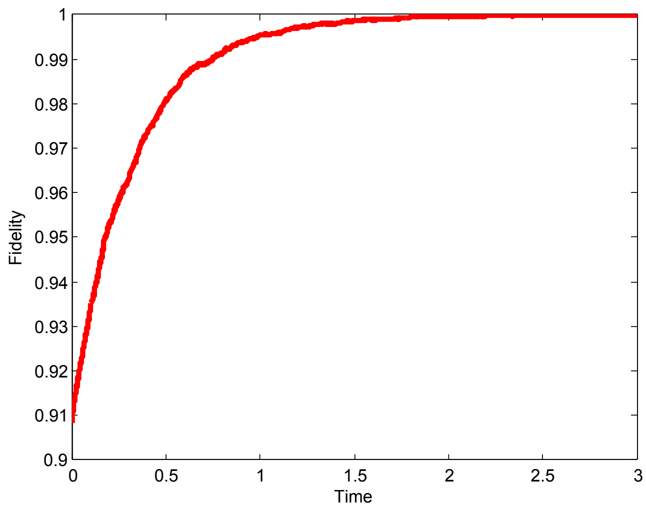

Fig. 1. The average fidelity between the Markov processes ρ and $\widehat{\rho}$, over 500 realizations, time t from 0 to $T = 3$ with discretization time step $dt = 10^{-4}$.

IV. Numerical Test

In this section, we test the result of Theorem 2.1 through numerical simulations. Considering the two-level system of [21], we take the following Hamiltonian and measurement operators:

$$H = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad L = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The simulations of figure 1 illustrates the fidelity for 500 random trajectories starting at

$$\rho_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \widehat{\rho}_0 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}.$$

In particular, we note that both initial states are mixed ones. As it can be seen the average fidelity is monotonically increasing. Here, the fidelity converges to one indicating the convergence of the filter towards the physical state. An interesting direction here is to characterize the situations where this convergence is ensured.

Here in order to simulate the Equations (1) and (3), we have considered the alternative formulations (5) and (6) and the resulting discretization scheme ($k \in \mathbb{N}$ and time step $0 < dt \ll 1$)

$$\rho_{(k+1)dt} = \frac{\mathcal{M}_k \rho_{(kdt)} \mathcal{M}_k^\dagger}{\mathrm{Tr}\left(\mathcal{M}_k \rho_{(kdt)} \mathcal{M}_k^\dagger\right)}, \quad \widehat{\rho}_{(k+1)dt} = \frac{\mathcal{M}_k \widehat{\rho}_{(kdt)} \mathcal{M}_k^\dagger}{\mathrm{Tr}\left(\mathcal{M}_k \widehat{\rho}_{(kdt)} \mathcal{M}_k^\dagger\right)},$$

where $\mathcal{M}_k = \mathbb{I} - \frac{iH}{\hbar} dt - \frac{1}{2} L^\dagger L dt + L dy_{(kdt)}$ and $dy_{(kdt)} = \mathrm{Tr}\left((L + L^\dagger)\, \rho_{(kdt)}\right) dt + dW_{(kdt)}$. For each k, the Wiener increment $dW_{(kdt)}$ is a centered Gaussian random variable of standard deviation $\sqrt{dt}$. The major interest of such discretization is to guaranty that, if $\rho_0, \widehat{\rho}_0 \in \mathcal{D}$, then ρ_k and $\widehat{\rho}_k$ also remain in $\mathcal{D}$ for any $k \geq 0$.

V. Concluding Remarks

The fact that the fidelity between the real quantum state and the quantum-filter state increases in average remains valid for more general stochastic master equations where other Lindbald terms are added to $\mathcal{L}(\rho)$ appearing in (1). In this case the dynamics (1) and (3) become

$$d\rho_t = -\tfrac{i}{\hbar}[H, \rho_t]\, dt + \sum_{\nu=1}^{m'} \mathcal{L}'_\nu(\rho_t)\, dt + \sum_{\mu=1}^{m} \mathcal{L}_\mu(\rho_t)\, dt + \sum_{\mu=1}^{m} \Lambda_\mu(\rho_t) dW_t^\mu$$

and

$$d\widehat{\rho}_t = -\tfrac{i}{\hbar}[H, \widehat{\rho}_t]\, dt + \sum_{\nu=1}^{m'} \mathcal{L}'_\nu(\widehat{\rho}_t)\, dt + \sum_{\mu=1}^{m} \mathcal{L}_\mu(\widehat{\rho}_t)\, dt + \sum_{\mu=1}^{m} \Lambda_\mu(\widehat{\rho}_t)\Big(dy_t^\mu - \mathrm{Tr}\left((L_\mu + L_\mu^\dagger)\widehat{\rho}_t\right) dt\Big).$$

where dW_t^μ are independent Wiener processes,

$$\mathcal{L}_\mu(\rho) := -\tfrac{1}{2}\{L_\mu^\dagger L_\mu, \rho\} + L_\mu \rho L_\mu^\dagger,$$

$$\mathcal{L}'_\nu(\rho) := -\tfrac{1}{2}\{{L'_\nu}^\dagger L'_\nu, \rho\} + L'_\nu \rho {L'_\nu}^\dagger,$$

and $\Lambda_\mu(\rho) := L_\mu \rho + \rho L_\mu^\dagger - \mathrm{Tr}\left((L_\mu + L_\mu^\dagger)\rho\right) \rho$.

Here $m, m' \geq 1$, and $(L'_\nu)_{1 \leq \nu \leq m'}$ and $(L_\mu)_{1 \leq \mu \leq m}$ are arbitrary operators. The special case considered here corresponds to $m = 1$ and $m' = 1$ with $L_1 = L$ and $L'_1 = 0$. The formulations analogue to (5) and (6) read then

$$\rho_{t+dt} = \frac{(\mathbb{I} - dM_t)\rho_t(\mathbb{I} - dM_t^\dagger) + \sum_{\nu=1}^{m'} L'_\nu \rho_t {L'_\nu}^\dagger dt}{\mathrm{Tr}\left((\mathbb{I} - dM_t)\rho_t(\mathbb{I} - dM_t^\dagger) + \sum_{\nu=1}^{m'} L'_\nu \rho_t {L'_\nu}^\dagger dt\right)}$$

and

$$\widehat{\rho}_{t+dt} = \frac{(\mathbb{I} - dM_t)\widehat{\rho}_t(\mathbb{I} - dM_t^\dagger) + \sum_{\nu=1}^{m'} L'_\nu \widehat{\rho}_t {L'_\nu}^\dagger dt}{\mathrm{Tr}\left((\mathbb{I} - dM_t)\widehat{\rho}_t(\mathbb{I} - dM_t^\dagger) + \sum_{\nu=1}^{m'} L'_\nu \widehat{\rho}_t {L'_\nu}^\dagger dt\right)}$$

where, denoting $dy_t^\mu = \mathrm{Tr}\left((L_\mu + L_\mu^\dagger)\rho_t\right) dt + dW_t^\mu$,

$$dM_t = \tfrac{iH}{\hbar} dt + \tfrac{1}{2} \sum_{\nu=1}^{m'} {L'_\nu}^\dagger L'_\nu dt + \tfrac{1}{2} \sum_{\mu=1}^{m} {L_\mu}^\dagger L_\mu dt - \sum_{\mu=1}^{m} L_\mu dy_t^\mu.$$

For this general case, the proof of Theorem 2.1 should follow the same lines: first step still relies on Theorem 3.1; second step relies now on [18, Theorem 2].

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