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Dispatch of Distributed Generators Using a Local Replicator Equation

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Abstract— **The replicator dynamics model is an evolutionary game concept that describes the state of a population in a process inspired by natural selection. This model is used to analyze a resource allocation problem in distributed networked systems. The main properties of the replicator equation are analyzed to propose a novel technique based only on the available information of a system modeled as a connected graph. Likewise, we analyze the stability of the equilibrium points as a function of a certain class of fitness. Finally, an economic dispatch in distributed generation systems is presented in order to illustrate the theoretical results.**

Index Terms— **Distributed control, economic dispatch, evolutionary game theory, replicator dynamics, resource allocation.**

I. Introduction

Population dynamical methods (e.g., selection and mutation) can be defined with the concepts of evolutionary game theory (EGT) to model the behavior of a population of players, all of them using some strategy whose success depends on a certain payoff. In this sense, better strategies tend to spread within the population while the payoffs depend on others' decisions. The replicator equation [1] is a dynamic model where the individuals tend to switch to more successful strategies, and where at equilibrium, all individuals earn the same payoff (i.e., the same fitness). Given the simplicity and adaptability of this model, replicator dynamics have been widely studied in biology and economics [2], [3], [4]; as well as in specific engineering problems such as power systems [5] and control [6], [7], [8].

Dynamic resource allocation is one of the problems where the principles of EGT can be applied more naturally. This problem deals with the distribution of a fixed amount of resources to a given number of activities or agents to achieve an optimal result according to a certain criterion. For this reason, several techniques and algorithms have been studied for the solution of this problem in each application. For instance, the authors in [9] summarize the most common approaches to solve static problems, some extensions to nonlinear optimization are presented in [10], and different solutions for cases of utility maximization in networks are shown in [11].

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When the number of agents increases and the information flow is limited, centralized (or even hierarchical) architectures may be infeasible, non-scalable, or too expensive [11]. This is the case of power distribution systems with the inclusion of small scale generators connected directly to the distribution network (i.e., distributed generators (DGs)). These elements introduce new variables in the system that may affect the adequate performance of the grid if the constraints for power and performance of the units are not satisfied. For this reason, one of the challenges to the projected increase of distributed generation in electrical networks is the power dispatch of these units using economical and technical information of each generator [12]. To solve this problem, in [13], [14] a multi-agent system (MAS) where each agent has specific tasks is proposed. These tasks define the agent-coordination method, which is the more important field in the study of MAS and its applications. Although most of the control strategies for the coordination of agents accomplish their objective in specific situations, some of them (essentially based on heuristics) do not have the necessary elements to analyze the effects of the network topology and the convergence to desired steady states of the agents.

The main contribution of this work is to propose a novel agent-coordination method that we refer to as the *local replicator equation*. This approach uses the basic properties of replicator dynamics (i.e., the invariance of the population size and the final common payoff for all individuals) to deal only with local information in a MAS. For this purpose, we consider the multi-agent system as an undirected graph, where the agents are represented by the nodes and the information links by arcs. This allows the model to be implemented in systems with a large number of nodes, and facilitates the addition of new agents by means of single links to any node in the graph and specific dynamics including only the neighbors' states. In this way, the scalability property required in many applications of networked MAS is satisfied by the local replicator equation.

To summarize our work, first we define the resource allocation problem by means of replicator dynamics and the local replicator equation. In addition, we show that the equilibrium points achieved by means of the local replicator equation are asymptotically stable for a generic class of fitness. Finally, we solve a problem of dispatch in a defined network topology to show the applicability of the proposed technique.

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II. Replicator Dynamics

The replicator dynamics can be seen as an appropriate mechanism to model the behavior of a population whose individuals while in constant interaction seek habitats with different conditions (e.g., different density of food or mates) to feed or reproduce. The fundamental principle of this concept is that the animal population, after an evolutionary process, tends to reach an equilibrium point where all individuals achieve the same fitness (e.g., the same food intake rate). This process can be related easily with the resource allocation problem, where the fixed amount of resources is given by the total population, the agents among which the resources must be split are the patches in the environment, and the resources flow can be modeled by the population behavior in each habitat.

In order to mathematically model the replicator equation as an evolutionary game, let $\mathcal{H} = \{1,\ 2, \ldots,\ N\}$ be the set of pure strategies, and $x_i(t) \geq 0$ be the relative amount of individuals playing the strategy $i \in \mathcal{H}$. Let the vector $x(t) = [x_1(t), \ldots, x_N(t)]^\top$ be the *population state*, where x_i is also called *frequency* (or population share) of the i^{th} strategy, which is a normalized state variable. Therefore, $x(t) \in \Delta$ for all t, where

$$\Delta = \left\{ x \in \mathbb{R}_+^N : \sum_{i=1}^{N} x_i = 1 \right\}. \tag{1}$$

If we assume that the number of players in the population is large enough in order to approximate the amount of individuals playing a certain strategy as a continuous variable, the replicator equation is given by [1]

$$\dot{x}_i = x_i \left(f_i(x_i) - \bar{f}(x) \right), \quad \text{for all} \quad i = 1, \ldots, N, \tag{2}$$

where $f_i : \Delta \to \mathbb{R}$ represents the *fitness function* that the individuals perceive in the i^{th} habitat and $\bar{f}(x)$ is the average fitness defined as

$$\bar{f}(x) \triangleq \sum_{j=1}^{N} x_j f_j(x_j). \tag{3}$$

Therefore, under the replicator dynamics, the population share playing a strategy more (less) profitable than the average will increase (decrease). Moreover, with the selection of the average fitness (3), the set Δ is invariant under (2) [3], [4]. Hence, if the initial population state $x(0) \in \Delta$, all trajectories of the system remain in Δ, for all $t \geq 0$. Besides, the steady state of (2) is achieved when $x_i^* \left(f_i(x_i^*) - \bar{f}(x^*) \right) = 0$, where $x^* = [x_1^*\ x_2^*\ \ldots\ x_N^*]^\top \in \Delta$ is the equilibrium point. If $x_i^* > 0$, for all i, the equilibrium is satisfied by the condition

$$f_i(x_i^*) = \bar{f}(x^*) = \bar{f}^*, \quad \text{for all } i = 1, \ldots, N, \tag{4}$$

where $\bar{f}^*$ is the average fitness in equilibrium. The invariance of Δ and the definition of the equilibrium point of (2) are desirable properties when the replicator dynamics model is applied to resource allocation problems. In this case, the fixed amount of resources is held for all time while it is split dynamically among the agents to achieve a common fitness such as the set point in a distributed control system. However, the implementation of the replicator dynamics requires full information to calculate the average fitness (3), and when the number of agents in the process increases, this centralized technique may be inefficient since the process may require a high amount of information of different sources, possible synchronization among all agents, and high performance (and expensive) communication channels. To solve this problem, next we describe how to apply the advantageous properties of the replicator dynamics with only local information.

III. Local Replicator Equation

In order for a resource allocation problem to model the local-information exchange between agents, we describe the environment as a *connected* graph $\mathcal{G} \triangleq (\mathcal{H}, \mathcal{A})$, where $\mathcal{H} \triangleq \{1, \ldots, N\}$ is the set of nodes, and $\mathcal{A} \subset \mathcal{H} \times \mathcal{H}$ denotes the set of interconnections between nodes. Therefore, strategies, habitats, or agents are referred to the nodes in the graph, and the individuals of the population are referred to resources. To describe the interaction topology, we define that if $(i, j) \in \mathcal{A}$, node i has information about node j. Moreover, we consider that the pair $(i, j) \in \mathcal{A}$ is equal to the pair $(j, i) \in \mathcal{A}$, so the graph $\mathcal{G}$ is undirected. Additionally, each node must be connected to the graph to participate in the allocation process. Hence, for every $i \in \mathcal{H}$, there exists some $j \in \mathcal{H}$, such that $(i, j) \in \mathcal{A}$. In words, there exists a *path* between any two nodes in $\mathcal{G}$.

An important concept in the local-information applications is the neighborhood of each node. To define this concept, let $\mathcal{N}_i$ be the set of adjacent nodes to the node i (the so-called neighborhood of i). Formally, $\mathcal{N}_i = \{j : (i, j) \in \mathcal{A}\}$ (note that $i \notin \mathcal{N}_i$). To obtain the local-information model for the resource allocation problem, the replicator dynamics with full information defined by (2) and (3) can be expressed as

$$\dot{x}_i = x_i \left(f_i(x_i) \sum_{j=1}^{N} x_j - \sum_{j=1}^{N} f_j(x_j) x_j \right). \tag{5}$$

Taking into account the summations in (5) only over the neighborhood of node i, we obtain the model for the *local replication equation* given by

$$\dot{x}_i = x_i \left(f_i(x_i) \sum_{j \in \mathcal{N}_i} x_j - \sum_{j \in \mathcal{N}_i} f_j(x_j) x_j \right), \tag{6}$$

for all $i \in \mathcal{H}$, where f_i is still the fitness function that describes the payoff for the i^{th} node.

In order to keep the most important characteristics of the original replicator dynamics in the local replication equation, we must show that: i) the simplex Δ is invariant under (6); ii) choosing appropriately the fitness functions, all individuals achieve the same payoff at equilibrium; and iii) the equilibrium point of (6) is asymptotically stable in Δ. The first condition guarantees that limited resources are preserved over time so that the local replicator equation is also an appropriate strategy for dynamic resource allocation. The accomplishment of the second condition allows the system to achieve different distributed control goals (e.g., equal agents' welfare). Finally, the stability of the equilibrium point guarantees the convergence of the proposed strategy.

A. Invariance of Δ Under the Local Replicator Equation

The next result shows that the invariance of the simplex holds for the local-information case.

Proposition 3.1: If $x(0) \in \Delta$, the set Δ is invariant under the local replicator equation given by Equation (6).

Proof: Let us define the adjacency matrix of the graph $\mathcal{G}$ as $A = [a_{ij}] \in \{0,1\}^{N\times N}$, where $a_{ij} = \begin{cases} 1 & \text{if } (i,j)\in\mathcal{A}, \\ 0 & \text{otherwise.}\end{cases}$
Note that given two vectors $x, y \in \mathbb{R}^N$,

$$y^\top Ax = \sum_{i\in\mathcal{H}}\left(y_i \sum_{j\in\mathcal{N}_i} x_j\right) = \sum_{i\in\mathcal{H}}\left(x_i \sum_{j\in\mathcal{N}_i} y_j\right) = x^\top Ay, \tag{7}$$

since $A = A^\top$ when the graph is undirected. Now, the sum over all $i \in \mathcal{H}$ in Equation (6) is given by[1]

$$\sum_{i\in\mathcal{H}} \dot{x}_i = \sum_{i\in\mathcal{H}}\left(x_i f_i \sum_{j\in\mathcal{N}_i} x_j\right) - \sum_{i\in\mathcal{H}}\left(x_i \sum_{j\in\mathcal{N}_i} x_j f_j\right).$$

If we define the vector $X_f \triangleq [x_1 f_1 \;\; x_2 f_2 \;\; \ldots \;\; x_N f_N]^\top$, the previous equation can be expressed as $\sum_{i\in\mathcal{H}} \dot{x}_i = X_f^\top Ax - x^\top AX_f$, and by the property in Equation (7), $\sum_{i\in\mathcal{H}} \dot{x}_i = 0$. Given that by assumption $\sum_{i\in\mathcal{H}} x_i(0) = 1$, the set Δ is invariant under (6). ■

B. Fitness Selection and Equilibrium Points

The choice of the fitness function is one of the most important issues in the behavior of the trajectories of (6). In evolutionary game theory, as well as in behavioral ecology [15], the fitness of the individuals depends on the frequencies of the strategies (number of individuals sharing some patch). In general, the more individuals sharing the same strategy, the smaller the fitness (which can be associated to the amount of resources in a habitat). In this way, the definition of $f_i : \Delta \mapsto \mathbb{R}$ as a Lipschitz continuous mapping in Δ, strictly decreasing, and $f_i(0) > 0$, for all i, allows us to make some analogies between the dynamics described by the replicator equation, and some natural processes as the population distribution along different habitats. For instance, the "truncation" behavior in nature is related to some habitats which become uninhabited since the supplies or mates in other habitats are large enough to provide a higher fitness to all individuals in the population. Thus in the local replicator equation, and since the unit simplex Δ is invariant (its interior (int(Δ)) and boundary (bd(Δ)) are also invariant), if $x(0) \in \text{int}(\Delta)$, some population shares may tend to zero only if $t \to \infty$. Then, in the limit some population shares may become extinct (i.e., $\lim_{t\to\infty} x(t) \in \text{bd}(\Delta)$, or $\lim_{t\to\infty} x_i(t) = 0$, for some $i \in \mathcal{H}$). This truncation is related to the behavior of the strategies whose associate fitnesses are smaller than the average fitness at equilibrium.

This situation may affect the connectivity of the graph in the local-information model due to the reduced number of connections. In this sense, the local replication model has a less robust topology (from the point of view of information interchange) than the ordinary replicator dynamics, especially in cases where there is truncation in nodes that may affect the connectivity of the graph. To analyze the connectivity, we define a *component* of $\mathcal{G}$ as a maximal connected subgraph of $\mathcal{G}$. Since we assume that $\mathcal{G}$ is connected (i.e., every pair of nodes are joined by a path), it has only one component. A *disconnected* graph has at least two components. A *cutpoint* of $\mathcal{G}$ is a node whose removal increases the number of components of the graph.

In the context of dynamic resource allocation, we consider a node removed from the graph when the associated population share is zero (i.e., a truncated node). Although physical communication channels may remain in the graph topology, when a node is truncated, no individuals inhabit this patch and the information flow is performed through the other nodes and their own paths. For this reason, the location of the truncated nodes may turn the connected graph into several components that can be seen as independent graphs with different resource allocation processes. Hence, to determine the equilibrium points of the system we consider three cases: i) when there is no truncation in the system, ii) when truncation is given in nodes which are not cutpoints, and iii) when there is truncation in cutpoints.

1) Equilibrium Points without Truncation: When there is no truncation in the local replicator equation, an equilibrium point must satisfy $x^* \in \text{int}(\Delta)$ since $x_i^* > 0$, for all $i \in \mathcal{H}$. Hence, a stationary point is achieved in (6) when

$$f_i(x_i^*) \sum_{j\in\mathcal{N}_i} x_j^* = \sum_{j\in\mathcal{N}_i} f_j(x_j^*) x_j^*, \quad \text{for all} \;\; i \in \mathcal{H}. \tag{8}$$

Condition (8) is satisfied if $f_i(x_i^*) = f_j(x_j^*)$, for all $j \in \mathcal{N}_i$ and all $i \in \mathcal{H}$. Given that the graph $\mathcal{G} = (\mathcal{H}, \mathcal{A})$ is assumed connected,

$$f_i(x_i^*) = f_j(x_j^*) = \bar{f}^*, \quad \text{for all} \quad i, j \in \mathcal{H}, \tag{9}$$

where $\bar{f}^*$ is the same equilibrium average fitness in (4). Then, at equilibrium all fitnesses are equal and all individuals earn the same payoff. Note that (9) is only a possible condition to satisfy (8). For this reason, the stability analysis of this equilibrium point is required to show the convergence from any $x(0) \in \text{int}(\Delta)$ to the point of equal social welfare.

2) Equilibrium Points with Truncation in Non-cutpoints: To better define the truncation states, we assume $f_i(0) = B_i$, $B_i > 0$, for all i. Given that $f_i(x_i)$ is assumed strictly decreasing, and that $x_i \geq 0$, B_i is the maximum value of the i^{th} fitness. The strategies in the set $\mathcal{H} = \{1, 2, \ldots, N\}$ may be ordered such that $B_1 \geq B_2 \geq \ldots \geq B_m > B_{m+1} \geq \ldots \geq B_N$, for any $m \leq N-1$. If the fitness functions conditions are such that the equilibrium average fitness

[1] In this proof, we omit the argument of $f_i(x_i)$ for simplicity in the notation.

satisfies $B_m \geq \bar{f}^* > B_{m+1}$, then the strategies $m+1, \dots, N$ will be extinct (i.e., $x_k^* = 0$, for $k = m+1, \dots, N$).

When the truncated nodes are not cutpoints, connectivity of the graph is preserved with their elimination. Therefore, the condition in Equation (8) for the m untruncated nodes can be expressed as

$$f_i(x_i^*) \sum_{j \in \mathcal{N}_i} x_j^* = \sum_{j \in \mathcal{N}_i} f_j(x_j^*) x_j^*, \quad \text{for} \quad i = 1, \dots, m. \quad (10)$$

Notice that if a truncated node $k \in \mathcal{N}_i$, for some $i = 1, \dots, m$, Equation (10) is not altered since $x_k^* = 0$. Then, to satisfy condition (10) the equilibrium point is given by

$$\begin{array}{rcll} f_i(x_i^*) & = & \bar{f}^*, & \text{for} \quad i = 1, \dots, m \\ f_k(x_k^*) & = & B_k, & \text{for} \quad k = m+1, \dots, N. \end{array} \quad (11)$$

Recall that $\sum_{i=1}^m x_i^* = 1$ by invariance of Δ, and hence, all the individuals of the population also obtain the same fitness in equilibrium (as in the no truncation case).

3) Equilibrium Points with Truncated Cutpoints: Suppose that the removal of the $N-m$ truncated nodes splits the graph $\mathcal{G}$ into n different components. The resultant components noted as $\mathcal{C}_l \triangleq (\mathcal{H}_{\mathcal{C}_l}, \mathcal{A}_{\mathcal{C}_l})$, for $l = 1, \dots, n$, are connected subgraphs with a set $\mathcal{H}_{\mathcal{C}_l}$ of nodes and a set of arcs $\mathcal{A}_{\mathcal{C}_l}$. Hence, there are no paths between nodes i, j if $i \in \mathcal{H}_{\mathcal{C}_k}$, and $j \in \mathcal{H}_{\mathcal{C}_l}$. At equilibrium, each node of a certain component will achieve the same fitness, but in general, this value may differ between components, and it depends on the initial conditions and system transients. Given that individuals may achieve different equilibrium payoffs, this case could be unacceptable in applications such as in the maximization of a common utility for all agents [5], or in the achievement of a set point in a distributed system [7], [8].

C. Stability Analysis

In order to analyze the stability of the equilibrium points found in previous section, we use a Lipschitz continuous Lyapunov function. To establish the fitness functions conditions, we define $C_i \in \mathbb{R}_+$, for all $i \in \mathcal{H}$, as the point in which $f_i(C_i) = 0$. This value is called the *carriying capacity* [16], and it is used to determine the positiveness of the payoffs. Then, the conditions for the fitness functions are summarized as: $i)$ $f_i(x_i)$ is a scalar Lipschitz continuous mapping in Δ; $ii)$ $f_i(x_i)$ is strictly decreasing; $iii)$ $f_i(0) = B_i$, $B_i > 0$, for all $i \in \mathcal{H}$; and $iv)$ $\sum_{i \in \mathcal{H}} C_i \geq 1$. The next result shows that the equilibrium points where all individuals achieve the same fitness (i.e., no truncation and truncation in non-cutpoints cases) are asymptotically stable.

Theorem 3.2: Given the above conditions for f_i, and if $x(0) \in \text{int}(\Delta)$, the equilibrium point $x^* \in \Delta$ that satisfy (9) or (11) is asymptotically stable under the local replicator equation (6).

Proof: Note that given the conditions for f_i the vector field $\phi(x) = \dot{x}$ in Equation (6) is locally Lipschitz, and then the Lyapunov function defined by

$$V(x) = \max_{i \in \mathcal{H}} f_i(x_i) \quad (12)$$

is a locally Lipschitz continuous function in x, and continuous in t. In order to show that (12) is a valid Lyapunov function (i.e., $V(x) \geq 0$, for all $x \in \Delta$, and x^* is the global minimum of $V(x)$ in Δ), we use the carrying capacities constraint and the invariance of Δ when $x(0) \in \text{int}(\Delta)$.

In the worst case (i.e., $\sum_{i \in \mathcal{H}} C_i = 1$), the equilibrium point is given by $x_i^* = C_i$, and $V(x) = 0$ since $f_i(C_i) = f_j(C_j) = \bar{f}^* = 0$, for all $i, j \in \mathcal{H}$ (using condition (9)). However, any deviation of this equilibrium point, which may be represented for an increment in the i^{th} population share (i.e., $x_i > C_i$), is compensated by the reduction in another population (i.e., $x_j < C_j$, for some $j \neq i$). Therefore, in this case $f_j(x_j) > 0$, and $V(x) > 0$ since all fitness functions are strictly decreasing by assumption. When $\sum_{i \in \mathcal{H}} C_i > 1$, $\bar{f}^* > 0$, and any deviation of the equilibrium point increases some of the fitnesses. Then, x^* the minimum point of $V(x)$ in Δ.

Now, Equation (12) is differentiable almost everywhere (in the sense of Lebesgue measure), since it is locally Lipschitz [17]. However, there are points where the derivatives do not exist. In order to calculate $\dot{V}(x)$, let $\mathcal{H}_f = \{j : f_j(x_j) = V(x)\}$ be the set of indices for which there exists a point where the differentiability of $V(x)$ fails. Since the trajectories of (6) are continuous, and according to [16], [17], $\dot{V}(x)$ is given by

$$\dot{V}(x) = \left[\sum_{j \in \mathcal{H}_f} \lambda_j \nabla f_j^\top(x) \right] \dot{x} = \sum_{j \in \mathcal{H}_f} \lambda_j \frac{\partial f_j(x_j)}{\partial x_j} \dot{x}_j, \quad (13)$$

for all $\lambda_j > 0$ such that $\sum_{j \in \mathcal{H}_f} \lambda_j = 1$. Using the local replicator equation (6), Equation (13) can be expressed as

$$\dot{V}(x) = \sum_{j \in \mathcal{H}_f} \left\{ \lambda_j \frac{\partial f_j(x_j)}{\partial x_j} x_j \left[f_j(x_j) \sum_{l \in \mathcal{N}_j} x_l - \sum_{l \in \mathcal{N}_j} f_l(x_l) x_l \right] \right\} \quad (14)$$

In this equation, $f_j(x_j) \geq f_l(x_l)$ for all $l \in \mathcal{N}_j$ since $j \in \mathcal{H}_f$, and by the definition of $\mathcal{H}_f$, $f_j(x_j) = V(x) = \max_{i \in \mathcal{H}} f_i(x_i)$. Therefore, the expression inside the square brackets in (14) is greater or equal to zero. Moreover, $\lambda_j > 0$, $x_j > 0$, and $\partial f_j(x_j)/\partial x_j < 0$ for all $j \in \mathcal{H}_f$. Hence, $\dot{V}(x) \leq 0$. Additionally, $\dot{V}(x) = 0$ only when $f_j(x_j) = f_l(x_l)$, for all $j \in \mathcal{H}_f = \mathcal{H}$, which corresponds to the equilibrium point. Finally, with these results, the equilibrium point (9) achieved by means of the local replicator equation (6) is asymptotically stable.

Given that truncation is a steady state concept, the non-positivity analysis of $\dot{V}(x)$ above holds for the truncation case. In addition, if $x_j^* > 0$, for $j = 1, \dots, m$, and $x_k^* = 0$, for $k = m+1, \dots, N$, then, $k \notin \mathcal{H}_f$ since $f_k(x_k^*) < f_j(x_j^*)$. Therefore, by using Equation (14), $\dot{V}(x) = 0$ only at equilibrium. With these conditions, the truncation equilibrium point (11) is also asymptotically stable under the local replicator equation. ■

IV. Application to Economic Dispatch of Distributed Generators

The dispatch of distributed generators (DGs) is basically a resource allocation problem, where a total amount of power must be split among the available units according to technical and commercial aspects of each generator. In addition, the final goal of this process is the maximization of the general utility of all generators by providing periodic and planned supply of power to the system [18]. The basic optimization problem in a dispatch with N DGs can be specified by

$$\begin{aligned} \max \quad & u_{tot}(p) = \sum_{j=1}^{N} u_j(p_j) \\ \text{s.t.} \quad & \sum_{j=1}^{N} p_j = P_d \\ & 0 \le p_i \le P_{nom_i}, \quad \text{for} \quad i = 1, \ldots N, \end{aligned} \tag{15}$$

where $p \triangleq [p_1 \quad p_2 \ \ldots \ p_N]^\top$ is the vector of dispatched powers to the N generators, $u_i(p_i)$ and $u_{tot}(p)$ are the utility functions for the i^{th} DG and for the whole system, respectively; P_d is the desired (total) power to be dispatched; and P_{nom_i} is the nominal power of each unit. Note that the constraints define the interval of possible generation of each DG and the complete distribution of the total power among the DGs. If we define the utility $u_i : \mathbb{R}_+ \mapsto \mathbb{R}_+$ to be a strictly concave function, the problem (15) is separable and has a unique optimal solution. This optimal point is obtained when all the marginal utilities are equal, i.e.,

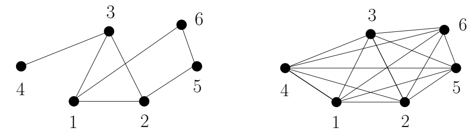
$$\left.\frac{\partial u_1}{\partial p_1}\right|_{p_1=p_1^*} = \ldots \left.\frac{\partial u_i}{\partial p_i}\right|_{p_i=p_i^*} = \ldots \left.\frac{\partial u_N}{\partial p_N}\right|_{p_N=p_N^*} = d \tag{16}$$

for some $d > 0$ such that $\sum_{j=1}^{N} p_j^* = P_d$. This problem may be solved by market based techniques [14], centralized Lagrange optimization methods, or decomposition techniques with heuristic or iterative algorithmic procedures [11]. However, when the number of generators increases and the information about all nodes is not available, centralized algorithms are not effective, and decomposition techniques may not converge since they need some type of coordination between decentralized subsystems. For these reasons, and given that distributed generation dispatches must be performed in short intervals of time due to the changing demand [13], reliable and fast methods must be used to calculate the optimal power for each DG.

In order to solve the dispatch process using replicator dynamics and the local replicator equation, let $p_i = P_d x_i$ be the dispatched power to the i^{th} generator, where P_d is the total power to be allocated. With this definition, and replacing $x_i = p_i/P_d$ in Equations (2), (3), and (6), we obtain the dynamic equations to solve the dispatch problem with full and partial information. On the other hand, by using condition (16), appropriate fitness functions can be specified by

$$f_i(p_i) = \beta \frac{\partial u_i}{\partial p_i}, \quad \text{for all} \quad i \in \mathcal{H}, \tag{17}$$

where $\beta \in \mathbb{R}_+$ does not affect the equilibrium or the stability of the replicator equations [4]. Hence, given that $f_i(p_i^*) = f_j(p_j^*)$, for all i, j (when there is no truncation), condition (16) is satisfied at equilibrium for full and local information techniques. In the truncation case, the non-negativity constraints are active for some generators, and therefore, the equilibrium point (11) solves optimally the dispatch problem. In the next example, we show the application of the replicator equations for a simple system with a specified topology. Let us consider a system as the one shown in Figure 1 (i.e., $N = 6$), formed by DGs with different characteristics defined by a relative generation cost factor (c_i), and the nominal power (P_{nom_i}) of each unit. With these economic and technical parameters, we can define a general utility function of the form



(a) Connected graph. (b) Complete graph.

Fig. 1. Example of graph structures for dispatch processes with (a) local and (b) full information.

$$u_i(p_i) = \frac{-p_i}{c_i P_{nom_i}} \left(p_i - 2P_{nom_i}\right), \text{ for } i = 1, \ldots, N,$$

where the possible maximum utility is achieved when each unit generates its nominal power (i.e., $u_i(P_{nom_i}) = P_{nom_i}/c_i$). This is a valid utility function since quadratic expressions are generally used to describe the profit in generation units, and $u_i(P_{nom_i})$ satisfies the expected relationship between an economic utility, the nominal power, and the costs of generation of the DGs. According to (17), the fitness of each DG in the replicator equations process can be given by

$$f_i(p_i) = \frac{1}{c_i}\left(1 - \frac{p_i}{P_{nom_i}}\right), \text{ for } i = 1, \ldots, N. \tag{18}$$

In the dispatch process, in every negotiation period a desired power $P_d \le \sum_{i=1}^{N} P_{nom_i}$ is programmed to be distributed among the generators in the topologies shown in Figure 1 for the full and local information techniques. In this example, we propose two cases for different parameters to illustrate the theoretical results: i) truncation of a non-cutpoint (i.e., node 2); and ii) truncation of cutpoints (i.e., nodes 1 and 2). Figure 2 shows the simulation results for both of the cases with the full and local information replicator equations with the parameter $\beta = 0.5$ and $\beta = 7.5$, respectively, for the fitness functions in (17). Although the transient behavior is different, in the first case the equilibrium points are the same for both techniques and correspond to the optimal resource allocation. Moreover, convergence time depends mainly on the complexity of the system and on β. Hence, this parameter can be adjusted to obtain a fast enough response according to the length of each negotiation period. In the second case with local information, the rightmost figure shows the formation of the two components. Here, the fitnesses of each node of

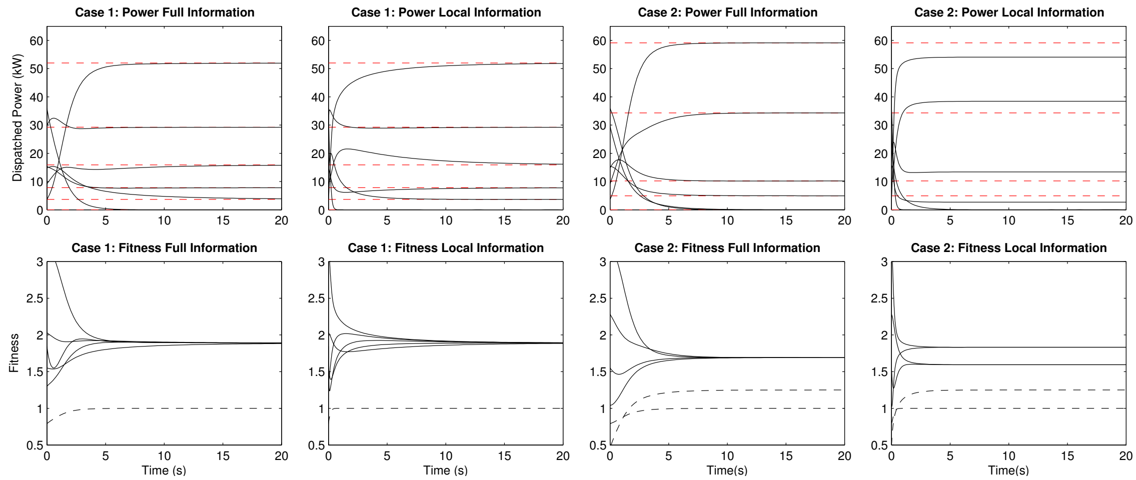


Fig. 2. Results of the dispatch process for a negotiation period with full and local information techniques with $\beta = 0.5$ and $\beta = 7.5$, respectively. Power (upper graphics) and fitnesses (lower graphics) are shown for each of the two cases. The power graphics show the behavior of the six dispatched powers (solid lines) and the theoretical optimal values (dashed line). The fitness of the truncated nodes are shown in dashed lines in the bottom figures.

the two components are equal, but the equilibrium point is different from the optimal. This deviation depends on the initial conditions and on the convergence time of the truncated nodes. Although non-optimal results are obtained with the formation of separate components, generally the network structure of the power distribution systems allows the more suitable nodes (the more active generators) to be connected with a larger number of weak nodes (possible truncated generators). Hence, the connectedness of the graph is strong and the possible deviations from the optimal equilibrium due to truncated cutpoints are avoided.

V. Conclusions

We present a novel technique based on evolutionary game theory for the solution of resource allocation problems in distributed systems with defined topologies. The local replicator equation uses the main concepts of the replicator dynamics model to reach a common fitness in a multi-agent system, where the dynamics of each component are related only with the information of the neighborhood in a connected graph. The analysis of the stability of the achieved equilibrium points and the simplicity of the model allow the local-information technique to be applied in a general distributed environment with an appropriate choice of the fitness function. For instance, the application to an economic dispatch of DGs shows that the optimal equilibrium point is obtained with certain constraints in the connectivity of the system.

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