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Automatic initialization of the Caputo fractional derivative

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Abstract— **Initialization of Riemann-Liouville and Caputo fractional derivatives remains an open research topic. These fractional derivatives are fundamentally related to fractional integration operators, so their initial conditions are the initial state vector of the associated fractional integrators. The paper is intended to propose an automatic initialization technique for practical calculation of the Caputo fractional derivative. Indirectly, this algorithm provides and efficient estimate of the initial state vector. Numerical simulations show the efficiency of the proposed initialization technique and provide interpretations of the long range memory phenomenon which is the main feature of fractional systems.**

I. INTRODUCTION

Fractional calculus is now considered as an efficient mathematical tool for solving many engineering problems, like the modeling of diffusive Partial Differential Equations or the design of robust control algorithms (refer to the proceedings of the four IFAC Workshops on Fractional Differentiation and Its Applications, Bordeaux 2004, Porto 2006, Ankara 2008 and Badajoz 2010). However, in spite of these renowned results, some theoretical problems have not yet received a satisfying solution. The mastery of initial conditions, either for Fractional Differential Equations (FDEs) or for the Caputo and Riemann-Liouville fractional derivatives, remains an open research domain. The solution of this fundamental problem, also related to the long range memory phenomenon, is certainly the necessary prerequisite for a satisfying definition of fractional systems controllability and observability.

Many contributions have been proposed to solve this problem, see for example [5], [2], [11], [6],[14]. Among these contributions, we have to notice the concept of the initialization function which has been introduced by Lorenzo and Hartley and applied to different initial condition problems.

Another approach is based on the infinite dimensional state vector of the fractional integrator which provides a straightforward interpretation of initial conditions. This new concept has been applied to the initialization of Fractional Differential Equations: it has been possible to estimate these initial conditions thanks to an observer and then to initialize correctly the corresponding FDE [18].

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In a recent paper [17] this concept has been generalized to the interpretation of the initial conditions of Riemann-Liouville and Caputo fractional derivatives. A new formulation of the initial conditions of the Laplace transforms of these derivatives has been proposed. The main result is the presence of the infinite dimensional state vector of the integrator in these initial conditions.

In this paper, our objective is to propose an automatic initialization technique for practical calculation of fractional derivatives. We also want to demonstrate that the integrator state vector is the right solution for the initialization of fractional derivatives. The main difficulty of the initialization problem is to estimate these initial conditions. We demonstrate that a feedback tracking system is able to provide an efficient estimate of the Caputo derivative and indirectly of the initialization state vector. Numerical simulations show the efficiency of the proposed initialization technique and provide interpretations of the long rang range memory phenomenon.

After a reminder of fractional integration and the definition of implicit fractional differentiation in section II, we present the fractional integration operator in section III and the definition of the Riemann-Liouville and Caputo fractional derivatives in section IV and of their initial conditions in section V. The proposed initialization technique is applied to the Caputo derivative in section VI and numerical simulations are presented in section VII.

II. FRACTIONAL INTEGRATION AND DIFFERENTIATION

A. Fractional integration

Consider the function $f(\mu)$ and its repeated integrals :

$$I_n(f(t)) = \int_0^t I_{n-1}(\mu)\,d\mu \qquad (1)$$

$$where \quad I_0 = f(t)$$

Using integration by parts , we get [7] [13]:

$$I_n(f(t)) = \int_0^t \frac{(t-\tau)^{n-1}}{(n-1)!} f(\tau)\,d\tau \qquad (2)$$

where n is an integer number.

Consider now that n is a real positive number : thus the factorial function $(n-1)!$ has to be replaced by the gamma function $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$ (3)

Then, the n[th] fractional order Riemann-Liouville integral (n real positive) of the function $f(t)$ is defined by the relation [8] [10] [12] :

$$I_n\big(f(t)\big)=\frac{1}{\Gamma(n)}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau \qquad (4)$$

Notice that $I_n\big(f(t)\big)$ is the convolution of the function $f(t)$ with the impulse response

$$h_n(t)=\frac{t^{n-1}}{\Gamma(n)} \qquad (5)$$

of the fractional integration operator whose Laplace transform is:

$$I_n(s)=L\{h_n(t)\}=\frac{1}{s^n} \qquad (6)$$

B. Implicit fractional differentiation

Fractional differentiation is the dual operation of the fractional integration.

Consider the fractional integration operator $I_n(s)$ whose input and output are respectively $x(t)$ and $y(t)$.

Then:

$$y(t)=I_n\big(x(t)\big) \quad (7) \qquad \text{or} \quad Y(s)=\frac{1}{s^n}X(s) \qquad (8)$$

Reciprocally, $x(t)$ is the n[th] order fractional derivative of $y(t)$ defined as:

$$x(t)=D_n\big(y(t)\big) \quad (9) \qquad \text{or} \quad X(s)=s^nY(s) \qquad (10)$$

where s^n represents the Laplace transform of the fractional differentiation operator (for initial conditions equal to zero).

Thus, this fractional derivative definition is based on the operator $I_n(s)$, without analytical formulation of $D_n\big(y(t)\big)$: so it is an implicit definition of the fractional derivative.

III. Fractional Integration Operator

A. Fractional integration operator

The fractional order integrator is an infinite dimensional system [3] [9] [15] [20] [21]. Its state-space model, provided by the inverse Laplace transform of $\frac{1}{s^n}$, is given by :

$$\begin{cases} \dfrac{\partial z(\omega,t)}{\partial t}=-\omega z(\omega,t)+v(t) \\ x(t)=\displaystyle\int_0^\infty \mu_n(\omega)z(\omega,t)d\omega \end{cases} \qquad (11)$$

$$\mu_n(\omega)=\frac{\sin(n\pi)}{\pi}\omega^{-n} \qquad (12)$$

where: $v(t)$: input, $x(t)$: output

$z(\omega,t)$: continuously distributed state

$x(t)$ is the weighted sum of the $z(\omega,t)$ internal state variables of the integrator.

B. State of the fractional integrator

The state $z(\omega,t)$ of the operator $I_n(s)$ is an infinite dimensional distributed state. Let $z(\omega,t_0)$ be the frequency distributed state at the instant t_0. This state represents the initial condition (or initialization function) of the integrator: it summarizes all the past behavior for $t<t_0$.

The solution of system (11) excited by $v(t)$ for $t\geq t_0$, with the initial condition $z(\omega,t_0)$ is given by [14]:

$$z(\omega,t)=z(\omega,t_0)\,e^{-\omega(t-t_0)}+\int_{t_0}^t e^{-\omega(t-\tau)}\,v(\tau)\,d\tau \qquad (13)$$

Consequently, the output $x(t)$ of the fractional integrator (11) is composed of a free response term caused by the initial condition $z(\omega,t_0)$ and of a forced response term caused by $v(t)$, like all linear systems [4].

Remark : For an integer order integrator $I(s)=\frac{1}{s}$, we have

$$\mu_1(\omega)=\delta(\omega)\text{, i.e.} \qquad x(t)=z(t) \qquad (14)$$

which means that $x(t)$ and $z(t)$ are the same variable and that the output of the integer order integrator is also the state variable of the integer order integrator, located in $\omega=0$.

At the opposite, for $I_n(s)$, the output $x(t)$, which is the integral of $z(\omega,t)$ weighted by the function $\mu_n(\omega)$, is only a pseudo state variable: this means that $x(t_0)$ is unable to summarize the past behavior of $I_n(s)$ for $t<t_0$. Thus , the initialization function $z(\omega,t_0)$ is equivalent to the initial state $x(t_0)$ of the integer order integrator $I(s)$.

C. Discrete approximation of the operator

The continuously distributed integrator model is not directly suited to practical applications, like simulation. A discrete frequency approximation of this operator has been proposed in [15] [16]. J+1 cells, ranging from 0 to J, provide a modal state space model of the integrator. See these references, particularly for the definition of the different modes ω_j and of their weights c_j.

$$\underline{Z}(t)^T=[z_0 \quad z_1 \quad .... \quad z_J\,] \qquad (15)$$

$$\begin{aligned} &\frac{d}{dt}\underline{Z}(t)=A_I\,\underline{Z}(t)+\underline{B}_I\,v(t) \\ &x(t)=\underline{C}^T{}_I\,\underline{Z}(t) \end{aligned} \qquad (16)$$

with:

$$A_I = \begin{bmatrix} 0 & & & 0 \\ & -\omega_1 & & \\ & & \ddots & \\ 0 & & & -\omega_J \end{bmatrix} ; \; \underline{B}_I = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\underline{C}^T{}_I = [c_0 \quad c_1 \quad \cdots \quad c_J] \qquad (17)$$

This discrete model has been used to initialize successfully a FDE (refer to [18] for more details).

IV. Caputo and Riemann-Liouville Fractional Derivatives

A. Differentiation and convolution

The relation

$$I_n(s)\, D_n(s) = \frac{1}{s^n} s^n = 1 \qquad (18)$$

corresponds in the time domain to the convolution relation :

$$h_n(t) * d_n(t) = \delta(t) \qquad (19)$$

where $d_n(t)$, impulse response of the fractional differentiator, is the convolution inverse of $h_n(t)$ [7].

So we get : $d_n(t) = \dfrac{t^{-n-1}}{\Gamma(-n)}$ where $n > 0$ (20)

B. Explicit formulation of the fractional derivatives

Assume that the fractional order n is situated between the two integer numbers $N-1$ and N : $N-1 < n \le N$

We can write $D_n(s) = s^n = \dfrac{1}{s^N} s^n s^N = \dfrac{1}{s^{N-n}} s^N$ (21)

where $\dfrac{1}{s^{N-n}}$ represents the fractional integration $I_{N-n}()$ and s^N the integer order differentiation $\dfrac{d^N()}{dt^N}$.

Then $L\{D_n(f)\} = D_n(s)\, F(s) = \dfrac{1}{s^{N-n}} s^N F(s)$ (22)

(with zero initial conditions)

Using the inverse Laplace transform , we get two expressions for $D_n(f)$:

The first one corresponds to :

$$D_n(f) = h_{N-n}(t) * \frac{d^N f(t)}{dt^N} \qquad (23)$$

and the second one to :

$$D_n(f) = \frac{d^N}{dt^N}\left(h_{N-n}(t) * f(t)\right) \qquad (24)$$

This first expression is known as the Caputo derivative [1] :

$$D_n(f(t)) = \int_0^t \frac{(t-\tau)^{N-n-1}}{\Gamma(N-n)} \frac{d^N f(\tau)}{dt^N} d\tau \qquad (25)$$

while the second one is the Riemann-Liouville derivative [13] :

$$D_n(f(t)) = \frac{d^N}{dt^N}\left\{\int_0^t \frac{(t-\tau)^{N-n-1}}{\Gamma(N-n)} f(\tau)\, d\tau\right\} \qquad (26)$$

V. Initial Conditions of Fractional Derivatives

A. An introductory example

Consider the sine function $f(t) = \sin t$ (27)

whose theoretical fractional derivative [10] is:

$$D_n(f) = \sin(t + n\frac{\pi}{2}) \qquad (28)$$

Practically, we calculate the Caputo derivative using (25):

$$x(t) = I_{1-n}\left(\frac{df(t)}{dt}\right) \quad for \quad 0 < n < 1 \qquad (29)$$

with the distributed integrator :

$$\begin{cases} \dfrac{\partial z(\omega,t)}{\partial t} = -\omega\, z(\omega,t) + \dfrac{d\, f(t)}{d\, t} \\ x(t) = \displaystyle\int_0^\infty \mu_{1-n}(\omega)\, z(\omega,t)\, d\omega \end{cases} \qquad (30)$$

We have performed this differentiation with no initialization of the $z(\omega,t)$ state function, at two different instants, $t_0 = 0$ and $t_0 = 1.57\, s$.

The corresponding graphs of $f(t)$, $D_n(f)$ and $x(t)$ are represented on figure n°1. On figure n°2, we have represented the difference between the exact derivative and $x(t)$ for the two instants t_0, with a longer time scale. For $t_0 = 0$, we notice that $x(t)$ converges to $D_n(f)$ after a slow transient. For $t_0 = 1.57s$, there is also convergence, but more slowly than previously.

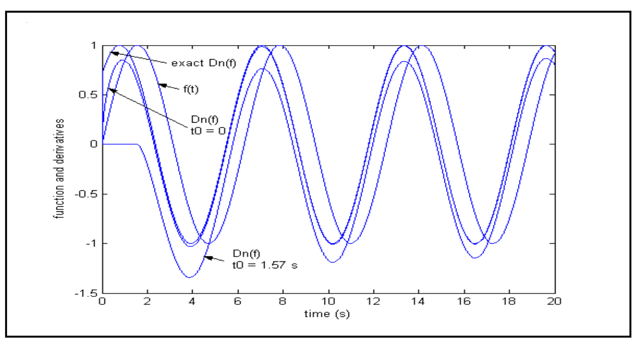

Figure n°1: Calculation of Caputo derivative beginning at two instants t_0

These two examples exhibit the influence of initial conditions, also known as long range memory phenomenon. Notice that this phenomenon depends on the initial instant t_0.

In this paper, our objective is to give a theoretical explanation of this phenomenon in the present section and to propose an initialization technique for the Caputo derivative in section VI, in order to master the influence of initial conditions.

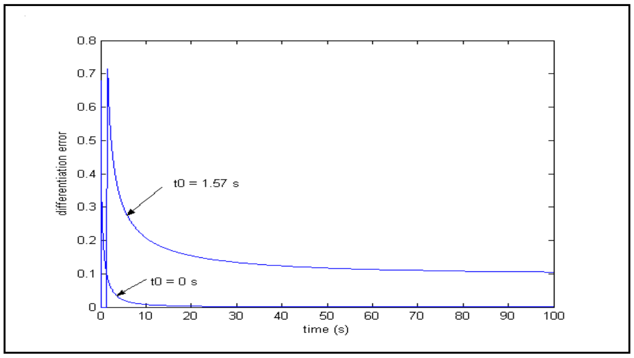

Figure n°2: Differentiation error at the two instants t_0

B. *Initial conditions for* $0 < n < 1$

Refer to [17] for a complete presentation of the initial conditions problem, mainly for the general case.

B.1. Implicit derivative

Consider the integrator $I_n(s)$, with initial condition $z_I(\omega,0)$.

Because : $L\left\{\frac{\partial z(\omega,t)}{\partial t}\right\} = s\ Z(w,s) - z_I(w,0)$ (31)

the Laplace transform of equation (11) is :

$$Z(\omega,s) = \frac{V(s) + z_I(\omega,0)}{s+\omega} \qquad (32)$$

Because $\int_0^\infty \frac{\mu_n(\omega)}{s+\omega} d\omega = \frac{1}{s^n}$ (33)

we get finally the Laplace transform of the implicit fractional derivative :

$$L\{D_n(f)\} = s^n\ F(s) - s^n \int_0^\infty \frac{\mu_n(\omega)\, z_I(\omega,0)}{s+\omega} d\omega \qquad (34)$$

where the second term is based on the distributed initial condition $z_I(\omega,0)$ (with $x(t) = f(t)$ and $v(t) = D_n(f(t))$).

B.2. Caputo derivative

Expression (23) can be written as:

$$D_n(f) = h_{1-n}(t) * \frac{df(t)}{dt} = I_{1-n}\left(\frac{df(t)}{dt}\right) \qquad (35)$$

Consider the Laplace transform of the Caputo derivative, which is defined as :

$$L\{D_n(f(t))\} = L\left\{I_{1-n}\left(\frac{df(t)}{dt}\right)\right\} \qquad (36)$$

Then, using the results related to the implicit derivative (34), and replacing n by $1-n$, we get :

$$L\{D_n(f)\} = L\left\{I_{1-n}\left(\frac{df}{dt}\right)\right\} = \frac{1}{s^{1-n}} L\left\{\frac{df}{dt}\right\} + \int_0^\infty \frac{\mu_{1-n}(\omega) z_C(\omega,0)}{s+\omega} d\omega \qquad (37)$$

where $z_C(\omega,0)$ is the initial state of integrator $I_{1-n}(s)$.

Moreover : $L\left\{\frac{df}{dt}\right\} = s\,F(s) - f(0)$ (38)

where $f(0)$ is the initial state of integer order integrator $I(s)$.

Finally, we get the Laplace transform of the Caputo derivative with initial conditions $f(0)$ and $z_C(\omega,0)$:

$$L\{D_n(f)\} = s^n F(s) - s^{n-1} f(0) + \int_0^\infty \frac{\mu_{1-n}(\omega) z_C(\omega,0)}{s+\omega} d\omega \qquad (39)$$

B.3. Riemann-Liouville derivative

Expression (24) can be written as :

$$D_n(f) = \frac{d}{dt}[h_{1-n}(t) * f(t)] = \frac{d}{dt}[I_{1-n}(f(t))] \qquad (40)$$

Using the same technique as previously, it is staightforward to get the Lalace transform of the Riemann-Liouville derivative, with initial conditions $(I_{1-n}(f))_0$ and $z_{RL}(\omega,0)$):

$$L\{D_n(f)\} = s^n F(s) - (I_{1-n}(f))_0 + s\int_0^\infty \frac{\mu_{1-n}(\omega)\, z_{RL}(\omega,0)}{s+\omega} d\omega$$

(41)

VI. Initialization of the Caputo derivative

A. *Introduction*

Our objective is to propose a general initialization technique for the Caputo derivative. This technique requires only the knowledge of the function $f(t)$ (in a previous paper [19], we had assumed the knowledge of $f(t)$ and $D_n(f)$). Practically, because we want to validate this algorithm, we will use the knowledge of the theoretical fractional derivative, but only to verify that we get an accurate estimate of $D_n(f)$ after a short transient. Thus , we will use the sine function as in the introductory example of section V. A .

Because the main cause of the long range memory phenomenon is the internal state function $z(\omega,t)$, we will consider only the case $0< n <1$.

B. An initialization technique

We consider two fractional integrators :

$$I_{1-n}(s)=\frac{1}{s^{1-n}} \text{ and } I_n(s)=\frac{1}{s^n}. \tag{42}$$

Notice that $I_{1-n}(s)\, I_n(s)=I_1(s)=\frac{1}{s}$ (43)

These two integrators are connected according to the fractional system :

$$\begin{cases} \dfrac{\partial z_1(\omega,t)}{\partial t}=-\omega\, z_1(\omega,t)+v_1(t) \\ x_1(t)=\displaystyle\int_0^\infty \mu_{1-n}(\omega)\, z_1(\omega,t)\, d\omega \end{cases} \tag{44}$$

$$\begin{cases} \dfrac{\partial z_2(\omega,t)}{\partial t}=-\omega\, z_2(\omega,t)+v_2(t) \\ x_2(t)=\displaystyle\int_0^\infty \mu_n(\omega)\, z_2(\omega,t)\, d\omega \end{cases} \tag{45}$$

with $v_1(t)=\dfrac{d\,f(t)}{d\,t}$ and $v_2(t)=x_1(t)$. (46)

Because these two integrators are connected in cascade,

$$x_2(t)=\int_0^t \frac{d\,f(\tau)}{d\,\tau}\, d\tau \tag{47}$$

(so $x_2(t)$ is an estimate of $f(t)$) and

$$x_1(t)=I_{1-n}\left(\frac{df(t)}{dt}\right) \tag{48}$$

is an estimate of $D_n(f)$.

But, because we have no knowledge of the required initial functions $z_1(\omega,t_0)$ and $z_2(\omega,t_0)$ at $t=t_0$, these two estimates will converge slowly respectively to $D_n(f)$ and $f(t)$.

In order to accelerate the convergence, we propose to modify the input $v_1(t)$ to introduce a feedback component in such a way that :

$$v_1(t)=\frac{d\,f(t)}{d\,t}+K(f(t)-x_2(t)) \tag{49}$$

Notice that when $x_2(t)\to f(t)$, the feedback term decreases and $v_1(t) \to \dfrac{d\,f(t)}{dt}$.

Because of (43), this feedback system is unconditionally stable for $K>0$; practically, we have to limit K to a maximum value because of small time delays introduced by the numerical simulation of the two fractional integrators.

Remark : There are two main reasons which motivate this tracking system.

First, if $x_2(t)\to f(t)$, then $v_1(t)\to\dfrac{d\,f(t)}{dt}$ and $x_1(t)\to D_n(f)$.

Secondly, if $x_2(t)\to f(t)$, its input $v_2(t)$ converges necessarily to $D_n(f)$, because $v_2(t)=x_1(t)$ is the implicit fractional derivative of $f(t)$ (refer to II. B).

C. Convergence of the algorithm

The objective is to analyze the tracking and disturbance rejection capabilities of the considered system. In this section, initial states $z_1(\omega,0)$ and $z_2(\omega,0)$ are considered as disturbances in the context of tracking.

Using Laplace transform, we can write:

$$X_2(s)=\frac{1}{s^n}X_1(s)+\int_2 \tag{50}$$

$$\text{with } \int_2=\int_0^\infty \mu_n(\omega)\frac{z_2(\omega,0)}{s+\omega}d\omega \tag{51}$$

$$\text{and } X_1(s)=\frac{1}{s^{1-n}}(E(s)+K\,(F(s)-X_2(s))+\int_1 \tag{52}$$

$$\text{with } \int_1=\int_0^\infty \mu_{1-n}(\omega)\frac{z_1(\omega,0)}{s+\omega}d\omega \tag{53}$$

$$\text{and } E(s)=L\left[\frac{d\,f(t)}{d\,t}\right] \tag{54}$$

It is straightforward to write:

$$X_2(s)=\frac{1}{s+K}E(s)+\frac{K}{s+K}F(s)+\frac{s^{1-n}\int_1+s\int_2}{s+K} \tag{55}$$

and:

$$X_1(s)=\frac{s^n}{s+K}E(s)+\frac{K}{s+K}s^n\,F(s)+\frac{s\int_1-K\,s^n\int_2}{s+K} \tag{56}$$

The transfer function $\dfrac{K}{s+K}$ is a first order system, with unity gain and a time constant equal to $1/K$, i.e. very low if $K>>1$.

We can conclude that $x_2(t)$ tracks $f(t)$ with a fast transient.

$s^n\,F(s)$ is the Laplace transform of $D_n(f)$, so for the same reasons, $x_1(t)$ tracks $D_n(f)$ with a fast transient.

On the other hand, all the other terms correspond to rejection of disturbances: it is important to notice that the dynamics of this rejection is imposed by the transfer function $\dfrac{1}{s+K}$.

We can conclude that this algorithm accelerates the convergence of $x_1(t)$ and $x_2(t)$ respectively to $D_n(f)$ and $f(t)$. Moreover, the influence of initial conditions vanishes quickly, compared to the long memory effect introduced by the terms $\int_1$ and $\int_2$.

VII. NUMERICAL SIMULATIONS

A. Experiment conditions

Remind that $f(t)=\sin(t)$ and $D_n(f)=\sin(t+n\frac{\pi}{2})$.

The period of $f(t)$ is $T=6.28\,s$.

All experiments have been performed with $n=0.5$.

According to III. C the fractional integrator has been frequency discretized into $J+1=21$ cells, ranging from $\omega_1=10^{-4}\,rd/s$ to $\omega_{20}=10^{2}\,rd/s$, with $\omega_0=0\,rd/s$. For time simulation, we have used the sampling period $T_e=2.5\,ms$.

B. Convergence of the algorithm

We present on figure n°3 the graphs of $x_1(t)$, $x_2(t)$, $f(t)$ and $D_n(f)$ for $K=0.5$ and on figure n°4 for $K=50$.

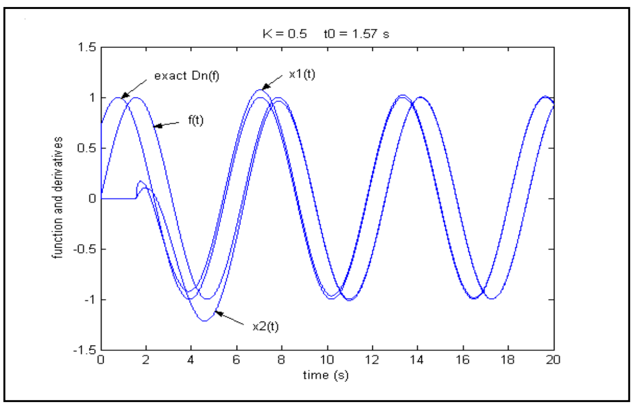

Figure n°3: Convergence for $K=0.5$

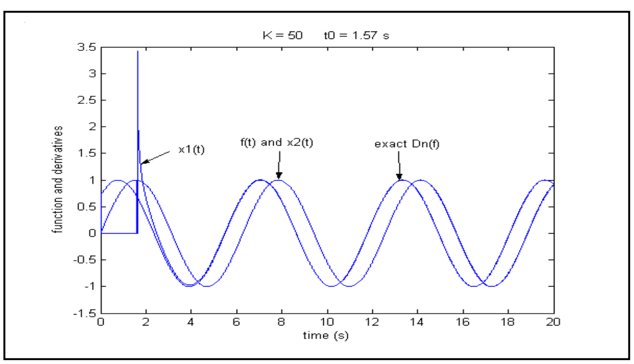

Figure n°4: Convergence for $K=50$

We can notice that convergence is slow for the low values of K, while convergence rate is increased for $K>>1$, with an impulsive behavior for $x_1(t)$, according to the theoretical analysis of section VI. C.

C. Initializations

Thanks to these convergence results, we have used $K=50$ and measured the state $z_1(\omega,t_0^{'})$ of the $I_{1-n}(s)$ integrator at the instant $t_0^{'}=T+t_0$ (where $t_0^{'}$ is equivalent to t_0 because of the periodicity of $f(t)$).

Then, this state has been used to initialize the Caputo derivative at $t_0^{'}$ (without the feedback initialization system). Two initializations have been performed (refer to figure n°5) for $t_0^{'}=6.28\,s$ and $t_0^{'}=7.85\,s$, corresponding to $t_0=0\,s$ and $t_0=1.57\,s$ of figures n°1 and n°2. It is obvious that $z_1(\omega,t_0^{'})$ provides a very good initialization for $x_1(t)$ because there is perfect coincidence with $D_n(f)$ for $t\geq t_0^{'}$ as exhibited by figure n°5.

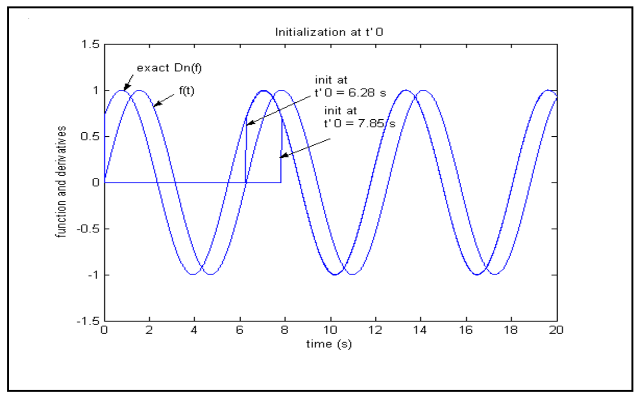

Figure n°5: Initialization of the Caputo derivative

In order to analyze the frequency distribution of the components of the initial state $z_1(\omega,t_0^{'})$, we have represented figure n°6 the amplitudes of the different modes for thee values of $t_0^{'}$ ($i=1$ corresponds to $\omega=0$ while $i=21$ corresponds to the highest frequency).

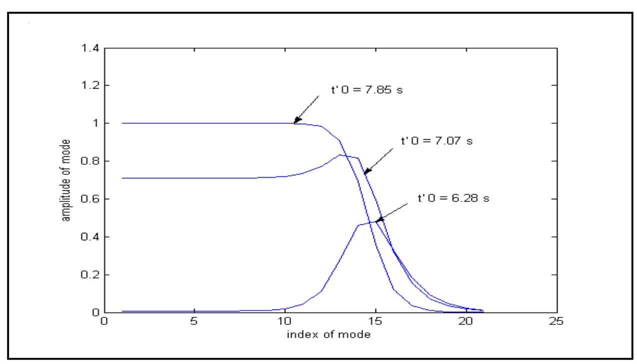

Figure n°6: Evolution of the modal distribution with $t_0^{'}$

We can notice that this modal distribution depends highly on the initialization instant.

For $t_0' = 6.28\,s$ (or $t_0 = 0\,s$), the low modes have a non significant amplitude, thus the transient is relatively quick (compare with $t_0 = 0\,s$ of figure n°2). On the other hand, for $t_0' = 7.85\,s$ (or $t_0 = 1.57\,s$) the low modes have a dominant amplitude and thus there is a long range memory effect (compare with $t_0 = 1.57\,s$ of figure n°2).

Finally, in order to analyze more quantitatively this long range phenomenon, we have represented figure n°7 the free response of four of these modes ($\omega_7 = 0.0106\,rd/s$, $\omega_{10} = 0.0841\,rd/s$, $\omega_{13} = 0.6683 rd/s$, $\omega_{16} = 5.3088\,rd/s$) for $t_0' = 7.07\,s$.

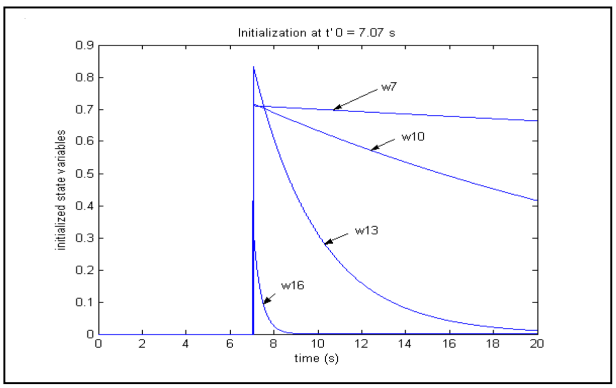

Figure n°7: Four components of the free response of the fractional integrator

First, we can verify that the initial amplitudes correspond to the values of figure n°6. Secondly, we can verify that the low frequency modes decrease very slowly, thus contributing highly to the long range memory effect, as noticed on figure n°2.

VIII. Conclusion

In this paper, we have demonstrated that the initial conditions of fractional derivatives correspond to the initial state vector of the associated fractional integrator $I_{1-n}(s)$. The validation of this concept has been possible thanks to a feedback tracking system able to provide an efficient estimate of the Caputo derivative and indirectly of the initialization state vector. Numerical simulations have shown the efficiency of the proposed initialization technique and have provided interpretations of the long range memory phenomenon which is the main feature of fractional systems.

In future works, some points will deserve more investigation. The relation between the initialization function of Lorenzo and Heartley and the integrator state vector will have to be analyzed. The initial vector has been estimated with a numerical algorithm, but is it possible to formulate an analytical solution ? Indeed, the proposed initialization technique will have to be adapted to the case of the Riemann-Liouville fractional derivative.

References

[1] Caputo M., « Elasticita e Dissipazione », Zanichelli, Bologna, 1969

[2] Heartley T.T., C.L. Lorenzo, « Dynamics and control of initialized fractional order system », Non linear Dynamics 29 (2002) pp. 201-233.

[3] Heleschewitz D. and D. Matignon, "Diffusive realizations of fractional integro-differential operators: structural analysis under approximation". Conference IFAC, System, Structure and Control. Vol 2, pp. 243-248, Nantes, France, July 1998.

[4] Kailath T., " Linear Systems". Prentice Hall Inc., Englewood Cliffs, 1980

[5] Lorenzo C.F., T.T. Heartley, "Initialization in fractional order systems", in Proceedings of the European Control Conference (ECC'01), Porto, Portugal, 2001, pp. 1471-1476.

[6] Lorenzo C.F. and T.T. Heartley, 'Initialization of fractional order operators and fractional differential equations'. Special issue on Fractional Calculus of ASME 'Journal of computational and non linear dynamics' Vol 3, n°2, 021101, April 2008

[7] Matignon D., "Représentations en variables d'état de modèles de guides d'ondes avec dérivation fractionnaire". Thèse de Doctorat. Université de Paris XI, ORSAY, 1994

[8] Miller K.S., B. Ross « An introduction to the fractional calculus and fractional differential equations » John Wiley and Sons New-York 1993

[9] Montseny G., « Diffusive representation of pseudo differential time operators », Proceedings ESSAIM, Vol 5, 1998, pp. 159-175.

[10] Oldham K.B., J. Spanier "The fractional calculus" Academic Press New-York 1974

[11] Ortigueira M.D., "On the initial conditions in continuous-time fractional linear systems", Signal Processing 83 (2003) pp. 2301-2309

[12] Oustaloup A., "La dérivation non-entière : théorie, synthèse et applications" . Hermès Editeur, Paris, 1995

[13] Podlubny I., "Fractional differential equations". Academic Press, San Diego, 1999.

[14] Sabatier J. et al, " On a representation of fractional order systems : interests for the initial condition problem " . 3rd IFAC workshop, FDA'08, Ankara, Turkey, 5-7 november 2008.

[15] Trigeassou J.C. et al., "Modelling and identification of a non integer order system". ECC'99 European Control Conference, KARLSRUHE, Germany, 1999.

[16] Trigeassou J.C., N. Maamri , "State-space modelling of fractional differential equations and the initial condition problem", IEEE SSD'09, Djerba, Tunisia, 2009.

[17] Trigeassou J.C., N. Maamri, "The initial conditions of Riemman-Liouville and Caputo derivatives: an integrator interpretation". FDA'2010 Conference, October 2010, Badajoz, Spain.

[18] Trigeassou J.C, N. Maamri, " Initial conditions and initialization of linear fractional differential equations " in Signal Processing , Vol 91 Issue 3 March 2011 pp 427-436

[19] Trigeassou J.C., N. Maamri, A. Oustaloup, "Initialization of Riemann-Liouville and Caputo fractional derivatives", Proceedings of IDETC/CIE FDTA'2011 Conference, August 2011, Washington DC, USA

[20] Tricaud Christophe, Yang Quan Chen, "Solution of fractional order optimal control problems using SVD-based rational approximations", Proceedings of the ACC'2009 Conference, St Louis, Missouri, USA

[21] Tricaud Christophe, Yang Quan Chen, "An approximated method for numerically solving fractional order optimal control problems of general form", in Computers and Mathematics with Applications 59 (2010) 1644-1655