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A Linear Multi-Agent Systems Approach to Diffusively Coupled Piecewise Affine Systems: Delay Robustness

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Abstract— **In this paper, we investigate ultimate boundedness of large-scale arrays consisting of piecewise affine (PWA) subsystems linearly interconnected through channels with delays. Under an assumption on subsystem dynamics, it is shown that ultimate boundedness can be reduced to the stability of a linear delay differential system. This enables us to use *linear* multi-agent system theory. As a result, we obtain sufficient conditions for ultimate boundedness taking the robustness of the interconnection topology into account. The usefulness of the results is examined through its application to the FitzHugh-Nagumo model.**

I. Introduction

In this paper, we investigate large-scale arrays consisting of linearly coupled subsystems. Such systems have attracted great attention due to their relevance to formation control, electric power networks, gene regulatory networks, and neuronal networks, to list a few. This has prompted intensive research on analysis/synthesis of such systems. In this line of work, *scalability* is an important requirement, that is, it is desirable that the obtained criteria are applicable to systems with extremely large dimension. A lot of research effort has been devoted to the reduction of the dependency of the complexity on the system scale. In particular, in the linear subsystems case, many scalable results are available for complicated connection settings such as delays and quantization; see, e.g., [15] and the references therein.

Also, in the nonlinear subsystem case, various kinds of theoretical analyses have been obtained both in the dynamical systems [3], [26], [4] and in the controls communities [1], [7]. These attempts lead to mathematical tools such as monotonicity [1], application of the Poincaré-Bendixon theorem [12], [11], [25], Hopf bifurcation [13], [21] and so on. Aside from the approaches listed above, *semi-passivity* is shown to be a powerful tool [19], [20], [22]. In particular, [23] successfully derived scalable criteria for synchronization despite the presence of the delays [16].

Concerning the representation of nonlinearity, the piecewise affine (PWA) approach has been adopted in many areas, e.g., mathematical physiology [10]. There are several numerical methods for the analysis of PWA systems, e.g., [6], [2], [21]. Unfortunately, it is still difficult to apply them to the systems considered here, since the state dimension and the number of modes quickly increases in the coupled dynamics. On the contrary, in some cases, e.g., [9], scalable criteria for Y-oscillatory behavior [24] were developed. As detailed in the next section, this result can be viewed under the linear multi-agent systems (MAS) framework. In [9], only a simple eigenvalue decomposition technique was borrowed from the linear MAS literature. The main purpose of this paper is to investigate the possibility of directly utilizing more advanced results. More specifically, we attempt to derive a sufficient condition for the robust ultimate boundedness based on [15]. See Table I for system classification.

TABLE I
System Classification

	Subsystem Dynamics	Delays	Strength
[15]	SISO Linear	Yes	Diffusive
Eq. (3), [9]	MIMO PWA	No	Arbitrary
Eq. (11), Theorem 1	MIMO PWA	Yes	Arbitrary
Eq. (16), Theorem 2	SISO PWA	Yes	Diffusive

The remainder of this paper is organized as follows: First, in Section II, we describe the class of linearly coupled PWA systems to be investigated, and review the result for the delay-free case in [9]. Then, in Section III, we provide a sufficient condition for the ultimate boundedness taking coupling topology (e.g., delays, connection graphs, strength etc.) robustness into account. In Section IV, the obtained theoretical results are applied to the piecewisely approximated FitzHugh-Nagumo model. Finally, we conclude the paper in Section V.

NOTATION AND CONVENTIONS:

The set of real and complex numbers are $\mathbb{R}$ and $\mathbb{C}$, and their right half subsets $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}$ and $\mathbb{C}_{\sigma+} := \{s \in \mathbb{C} : \mathrm{Re}\ s \geq \sigma\}$ for $\sigma \in \mathbb{R}$. As usual, p-Lebesgue and p-Hilbert spaces on these sets are denoted by L^p and H^p, respectively. For a matrix A, $\mathrm{eig}(A)$ denotes the set of all eigenvalues. A square matrix (resp. a polynomial) is said to be Hurwitz if it has no eigenvalue (resp. root) in $\mathbb{C}_{0+}$. The $(n \times n)$-identity matrix is I_n. The norm $\|\cdot\|$ denotes the Euclidean norm for vector $x \in \mathbb{R}^n$, i.e., $\|x\| := \sqrt{x^{\mathrm{T}}x}$, and the maximal singular value for other matrices. The (block-)diagonal matrix and the Kronecker product are represented by diag and $\otimes$, respectively. For a set $\mathcal{S}$, int$\mathcal{S}$, $\partial\mathcal{S}$ and $\bar{\mathcal{S}}$ denote the interior, boundary and closure of $\mathcal{S}$.

A (delay-)differential dynamical system with trajectory denoted by $x(t)$ is *ultimately bounded* if $\limsup_{t\to\infty} \|x(t)\| < R$ where R is independent of initial conditions.

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II. Linearly Coupled PWA Systems without Delay

Piecewise affine (PWA) models provide intuitive understanding of the dynamical behaviors, and also some analytical expressions of trajectory, which are often used for detailed analysis and/or synthesis. One typical way of obtaining PWA models is by approximating the nonlinear function in the dynamics; see Section IV. In this case, in practice, we can increase the number of modes to obtain a sufficiently accurate PWA vector field, depending on the smoothness of the original nonlinearity and required accuracy. Note that it would sometimes be valid to approximate physical systems directly by PWA systems, in particular when the dynamics are discontinuous. See [5] and the references therein for a biological example and related methods.

In this paper, we investigate not only an individual PWA subsystem, but also their large-scale coupled array. Let J be the number of subsystems. Throughout this paper, the index $j \in \mathbb{J} := \{1, 2, \cdots, J\}$ is used to index the j-th subsystem. Next, let $\{\mathcal{S}_i\}_{i\in\mathbb{I}}$ be a family of closed subset of $\mathbb{R}^{\bar{n}}$ indexed by mode labels $\mathbb{I} := \{1, 2, \cdots, L\}$. This plays a role of state partitions since we assume $\bigcup_{i\in\mathbb{I}} \mathcal{S}_i = \mathbb{R}^{\bar{n}}$ and $\mathcal{S}_i$ have disjoint interiors. For simplicity, we suppose all PWA subsystems are equipped with the common mode labels $\mathbb{I}$ and partitions $\{\mathcal{S}_i\}$. Then, the subsystem dynamics are given by

$$\dot{x}_j = A^{(j)}_{i_j} x_j + b^{(j)}_{i_j} + Bu_j, \text{ if } x_j \in \mathcal{S}_{i_j} \tag{1}$$

$$y_j = Cx_j \tag{2}$$

where $x_j \in \mathbb{R}^{\bar{n}}$ and $i_j \in \mathbb{I}$ denote the state variable and the mode of the j-th subsystem, and $A^{(j)}_i \in \mathbb{R}^{\bar{n}\times\bar{n}}, B, C^{\mathrm{T}} \in \mathbb{R}^{\bar{n}\times\bar{p}}$, $b^{(j)}_i \in \mathbb{R}^{\bar{n}\times 1}$. Note that $A^{(j)}_{i_j}$, $b^{(j)}_{i_j}, \mathcal{S}_{i_j}$ represent $A^{(j)}_i$, $b^{(j)}_i, \mathcal{S}_i$ with $i = i_j$. When $B = 0$, every subsystem is an autonomous L-mode PWA system with the switching signal i_j. In other words, the third right hand side term with nonzero B specifies the linear interaction between these PWA subsystems.

Let us begin with

$$u_j = \sum_{k=1}^{J} \gamma_{jk} y_k. \tag{3}$$

Note that the coupled system given by (1), (2) and (3) is again a PWA system equipped with L^J-modes and state

$$x(t) := \left[x_1(t)^{\mathrm{T}}, x_2(t)^{\mathrm{T}}, \ldots, x_J(t)^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^n$$

with $n := J\bar{n}$. To see this, define

$$\mathbf{i} := (i_1, i_2, \cdots, i_J) \in \mathbb{I}^J.$$

We say $x \in \mathcal{S}_{\mathbf{i}}$ if $x_j \in \mathcal{S}_{i_j}$ for all $j \in \mathbb{J}$. The coupling matrix Γ is given by

$$\Gamma := \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1J} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{J1} & \gamma_{J2} & \cdots & \gamma_{JJ} \end{bmatrix}, \tag{4}$$

where no specific structure is assumed. Then, the interconnected system can be rewritten as

$$\dot{x} = (\mathbb{A}_{\mathbf{i}} + \Gamma \otimes BC)x + b_{\mathbf{i}}, \text{if } x \in \mathcal{S}_{\mathbf{i}} \tag{5}$$

with

$$\mathbb{A}_{\mathbf{i}} := \mathrm{diag}\{A^{(j)}_{i_j}\}_{j=1}^{J}, \tag{6}$$

$$b_{\mathbf{i}} := \left[b^{(1)\mathrm{T}}_{i_1}, b^{(2)\mathrm{T}}_{i_2}, \ldots, b^{(k)\mathrm{T}}_{i_J}\right]^{\mathrm{T}}. \tag{7}$$

Thus, $\mathbf{i} \in \mathbb{I}^J$ and its j-th entry $i_j \in \mathbb{I}$ always represent the mode of the coupled systems and of its j-th subsystem. We also assume the continuity of the vector field to avoid the nonessential well-posedness issue [8].

In [9], the stability of such structured large-scale PWA systems was investigated under the following assumption:

Assumption 1: 1) There exists a unitary matrix $T := [T_1^{\mathrm{T}},\ T_2^{\mathrm{T}}]^{\mathrm{T}}$ such that, for any $j \in \mathbb{J}$ and $i \in \mathbb{I}$ associated with unbounded $\mathcal{S}_i$, we can take $p \in \tilde{\mathbb{I}}$ satisfying $(A^{(j)}_i - A^{(j)}_p)T_2^{\mathrm{T}} = 0$ where

$$\tilde{\mathbb{I}} := \{i \in \mathbb{I} : T_1\mathcal{S}_i \text{ is unbounded}\}. \tag{8}$$

2) $A^{(j)}_i \equiv \hat{A}_j$ for all $i \in \tilde{\mathbb{I}}$ and $j \in \mathbb{J}$. □

Intuitively, this assumption is a relaxation of the requirement that for every subsystem, all modes associated with unbouded region have a common A-matrix, i.e.,

$$A^{(j)}_i \equiv \hat{A}_j, \text{ if } \mathcal{S}_i \text{ is unbounded.}$$

See also the piecewise affine approximated FitzHugh-Nagumo equations in Section IV, which do not satisfy this requirement, but Assumption 1.

Proposition 1 ([9]): Under Assumption 1, if

$$\hat{\mathbb{A}} + \Gamma \otimes BC \tag{9}$$

$$\hat{\mathbb{A}} := \mathrm{diag}\{\hat{A}_j\}_{j=1}^{J} \tag{10}$$

is Hurwitz, then the coupled system given by (1), (2) and (3) is ultimately bounded. □

When $\hat{A}_j$ does not depend on $j \in \mathbb{J}$, we readily obtain the following:

Corollary 1: Under Assumption 1, further suppose $\hat{A}_j \equiv \hat{A}$ for all $j \in \mathbb{J}$. Then, if

$$\hat{A} + \lambda BC$$

is Hurwitz for all $\lambda \in \mathrm{eig}(\Gamma)$, then the coupled system given by (1), (2) and (3) is ultimately bounded. □

The size of $A + \lambda BC$ is independent of the number of subsystems. Thus, this criterion is easily checkable if the eigenvalues of Γ are available. In Proposition 1, we firstly reduced the ultimate boundedness of the coupled PWA systems to the stability of the (still large-scale) linear system whose A-matrix is (10). Secondly, in Corollary 1, we applied an eigenvalue decomposition technique that is standard tool in recent research on linear MAS. This shows a clear contrast to the passivity-based approach mentioned in Section I. This fact motivates us to utilize more advanced results from linear MAS theory in this context.

III. Diffusively coupled PWA systems with delays

A. Reduction to the stability of a linear system

Consider the delayed coupling

$$u_j = \sum_{k=1}^{J} \gamma_{jk} y_k(t - T_{jk}) \tag{11}$$

where delay lengths T_{ji} are non-negative constants. Proposition 1 can be generalized as follows:

Theorem 1: Under Assumption 1, define $\Gamma_T(s)$ by

$$\Gamma_T := \begin{bmatrix} \gamma_{11}\mathrm{e}^{-T_{11}s} & \gamma_{12}\mathrm{e}^{-T_{12}s} & \cdots & \gamma_{1J}\mathrm{e}^{-T_{1J}s} \\ \gamma_{21}\mathrm{e}^{-T_{21}s} & \gamma_{22}\mathrm{e}^{-T_{22}s} & \cdots & \gamma_{2J}\mathrm{e}^{-T_{1J}s} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{J1}\mathrm{e}^{-T_{J1}s} & \gamma_{J2}\mathrm{e}^{-T_{J2}s} & \cdots & \gamma_{JJ}\mathrm{e}^{-T_{JJ}s} \end{bmatrix}. \tag{12}$$

Then, the linearly coupled PWA systems with delays defined by (1), (2) and (11) is ultimately bounded if

$$G(s) := (sI_n - \hat{\mathbb{A}} - \Gamma_T(s) \otimes BC)^{-1}$$

has no poles in $\mathbb{C}_{0+}$.

Proof: Denote

$$u(t) := \left[u_1(t)^{\mathrm{T}}, u_2(t)^{\mathrm{T}}, \ldots, u_J(t)^{\mathrm{T}}\right]^{\mathrm{T}}. \tag{13}$$

Then, the overall system can be rewritten as

$$\dot{x}(t) = \hat{\mathbb{A}}x(t) + u(t) + d(t), \tag{14}$$

$$d(t) = (\mathbb{A}_{\mathbf{i}} - \hat{\mathbb{A}})x(t) + b_{\mathbf{i}},\ x(t) \in \mathcal{S}_{\mathbf{i}} \tag{15}$$

with $\mathbb{A}_{\mathbf{i}}$ in (6) and $\hat{\mathbb{A}}$ in (10). Let us regard these equations as the linear delay-differential system whose transfer function is $G(s)$ having $d(t)$ as its input.

Note that the j-th entry of d is given by

$$d_j = (A_{i_j}^{(j)} - \hat{A}_j)x_j + b_{i_j}.$$

Furthermore, for each j, at least one of the following holds,

a) $i_j \in \tilde{\mathbb{I}}$, consequently $(A_{i_j}^{(j)} - \hat{A}_j)x_j = 0$,
b) $\mathcal{S}_{i_j}$ is bounded,
c) $i_j \notin \tilde{\mathbb{I}}$ and $\mathcal{S}_{i_j}$ is unbounded.

In case c),

$$\begin{aligned} &\|(A_{i_j}^{(j)} - \hat{A}_j)x_j\| \\ =&\|(A_{i_j}^{(j)} - \hat{A}_j)(T_1^{\mathrm{T}}T_1 + T_2^{\mathrm{T}}T_2)x_j\| \\ =&\|(A_{i_j}^{(j)} - \hat{A}_j)T_1^{\mathrm{T}}(T_1x_j)\| \\ \le&\|T_1x_j\| \cdot \|(A_{i_j}^{(j)} - \hat{A}_j)T_1^{\mathrm{T}}\|, \end{aligned}$$

where we used the unitarity of T for the first equality and Assumption 1 for the second equality, respectively. This directly implies the existence of an initial-state-independent $r > 0$ such that $\|d(t)\| < r$ for all t.

Note that there exists no pole chain of G that asymptotically approaches the imaginary axis. This fact combined with the assumption on G means that G is in $H^\infty(\mathbb{C}_{\sigma+})$ for some negative σ. Hence, this system is associated with an exponentially decaying semigroup, and consequently the effect of initial conditions on $x(t)$ exponentially converges to 0. On the other hand, we have $G \in H^2(\mathbb{C}_{0+})$ since $\|\Gamma_T(j\omega)\|$ is bounded and consequently

$$\int_{-\infty}^{\infty} \mathrm{tr}\ \overline{G(j\omega)}^{\mathrm{T}} G(j\omega)d\omega < \infty.$$

Therefore, the impulse response $g(t) \in L^2(\mathbb{R}_+)$ and

$$\begin{aligned} &\left\|\int_0^t g(t-\tau)d(\tau)d\tau\right\|^2 \\ \le\ & \int_0^t \|g(t-\tau)d(\tau)\|^2 d\tau \\ \le\ & r^2 \cdot \int_0^t \|g(t-\tau)^{\mathrm{T}}g(t-\tau)\|d\tau \\ \le\ & r^2 \cdot \int_0^\infty \|g(\tau)^{\mathrm{T}}g(\tau)\|d\tau < \infty. \end{aligned}$$

This completes the proof. ■

Here, we reduced the ultimate boundedness of the PWA delay-differential system to the stability of a linear delay system. The stability of delay systems is difficult to check in general due to its infinite-dimensional nature. Furthermore, most of numerical tools such as Lyapunov-Krasovskii functional construction via semidefinite programming do not suitably work for such large-scale systems. However, for certain class of delay systems closely related to consensus problems, scalable stability criteria are already developed. In the next section, we apply the results from [15] for the reduced stability analysis required in Theorem 1, that is, parallel step from Proposition 1 to Corollary 1.

B. Connection topology robustness

Hereafter, we assume that $B, C^{\mathrm{T}} \in \mathbb{R}^{\bar{n}\times 1}$. Thus, all subsystems are single-input single-output. In this section, we focus on the following specific coupling topology:

$$u_j = -\sum_{k=1}^{J} \eta_{jk}(y_k(t - T_{jk}) - y_j(t - T_{jk})),\ T_{ji} = T_{ij} \tag{16}$$

investigated (2b) in [15], where $\eta_{jk} \ge 0$ and $\eta_{jj} = 0$ for all $j, k \in \mathbb{J}$. Note that the following discussion straightforwardly applies to the *different self-delay case* in [15] as well.

If $\hat{A}_j$ are identical, then Theorem 1 in [15] is applicable to check the nonexistence of unstable poles.

Denote

$$\frac{n(s)}{d(s)} := H(s) := C(sI - \hat{A})^{-1}B. \tag{17}$$

Assumption 2: The polynomial $d(s) + \rho n(s)$ is Hurwitz for all $\rho \in (0, 2]$. □

Theorem 2: Under Assumptions 1 and 2, assume further $\hat{A}_j \equiv \hat{A}$ for all $j \in \mathbb{J}$. Then, if

$$1 + H^{-1}(j\omega) \notin \Omega_{2s}(\omega\bar{\tau}) \tag{18}$$

for all $\omega \in \mathbb{R}\backslash\{0\}$, where $\Omega_{2s}(\omega\bar{\tau})$ is defined by

$$\Omega_{2s}(\omega\bar{\tau}) := \mathrm{Co}\{1 - 2\mathrm{e}^{-j\phi} : \phi \in [0, \omega\bar{\tau}]\}, \tag{19}$$

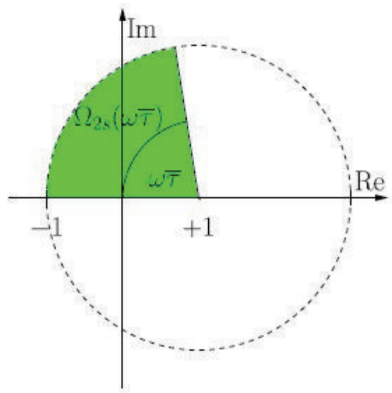

Fig. 1. $\Omega_{2s}(\omega\bar{\tau})$.

then the delay diffusively coupled PWA systems with delays defined by (1), (2) and (16) is ultimately bounded with arbitrary connected undirected topologies, arbitrary heterogeneous bounded delays $T_{ij} \le \bar{\tau}$ and arbitrary coupling satisfying

$$\eta_j := \sum_{k \neq j} \eta_{jk} \in (0,1], \ j \in \mathbb{J}.$$

Proof: If $\eta_j = 1$ for all $j \in \mathbb{J}$, the result is a direct consequence of Theorem 1 and the results in [15]. When $\eta_j \in (0,1)$, we only have to replace $d_j = \sum_{j=1}^{N} a_{ji}$ by $d_j > \sum_{j=1}^{N} a_{ji}$ in all statements in [15]. ■

The set $\Omega_{2s}(\omega, \bar{\tau})$ is illustrated in Fig. 1. As in the example in the next section, the criterion (18) is easily checkable graphically. It should be emphasized that this criterion is independent of the number of subsystems J.

IV. Application to FitzHugh-Nagumo model

In this section, we apply the obtained results to the analysis of spatio-temporal neuronal model.

A. Piecewisely approximated FitzHugh-Nagumo model

Let us consider the FitzHugh-Nagumo equation ([10]) given by

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} g(X_1) - X_2 \\ \epsilon(X_1 - bX_2) - \epsilon\alpha \end{bmatrix}, \tag{20}$$

$$g(X_1) := -10X_1(X_1 - 1)(X_1 - 0.5) \tag{21}$$

where $\epsilon, b > 0$ and α are real constants. We approximate the nonlinear term in (21) by the PWA function depicted in Fig. 2:

$$g(X_1) \approx \tilde{g}(X_1) := \begin{cases} -5X_1 + 5, & \text{if } 0.8333 \le X_1, \\ -5X_1, & \text{if } X_1 \le 0.1667, \\ 2.5X_1 - 1.25, & \text{otherwise.} \end{cases}$$

We set $b = \epsilon = 0.1$ and show the time response of $X_1(t)$ of the approximated system for $\alpha = 0.5$ and $\alpha = 0.2$ in Figs. 3 and 4. Though the initial state is the same ($X(0) = \begin{bmatrix} 0.6 & 0.1 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0.2 & -0.1 \end{bmatrix}^{\mathrm{T}}$), the behavior is completely different. In mathematical physiology, these properties are refered to as the *self-oscillation* and *excitability*

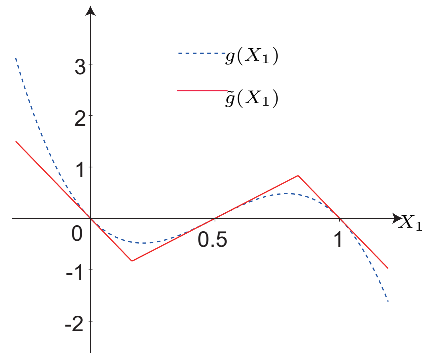

Fig. 2. Approximation $\tilde{g}$ of g.

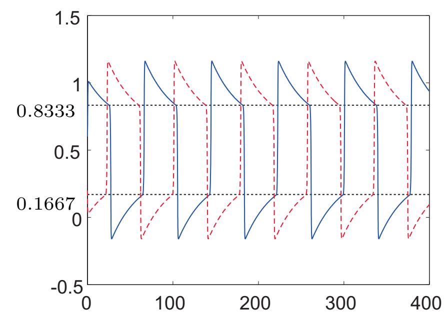

Fig. 3. Self-oscillatory property with $\alpha = 0.5$ for two initial conditions.

(convergence after a possible temporal perturbation), and used to model different behavior of neural networks. See [10], [8] for the actual roles of these properties in living organisms.

Note that $T_1 := \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $T_2 := \begin{bmatrix} 0 & 1 \end{bmatrix}$ satisfy all assumptions in Theorem 2:

$$\mathbb{I} := \{0,1,2\}, \ \tilde{\mathbb{I}} := \{1,2\}.$$
$$\mathcal{S}_0 := \{X \in \mathbb{R}^2 : 0.1667 \le T_1 X \le 0.8333\},$$
$$\mathcal{S}_1 := \{X \in \mathbb{R}^2 : T_1 X \ge 0.8333\},$$
$$\mathcal{S}_2 := \{X \in \mathbb{R}^2 : T_1 X \le 0.1667\}.$$
$$A_0^{(j)} := \begin{bmatrix} 2.5 & -1 \\ \epsilon & -\epsilon b \end{bmatrix}, \ A_1^{(j)} = A_2^{(j)} := \hat{A} := \begin{bmatrix} -5 & -1 \\ \epsilon & -\epsilon b \end{bmatrix},$$
$$b_0^{(j)} := \begin{bmatrix} -1.25 \\ \epsilon\alpha_j \end{bmatrix}, \ b_1^{(j)} := \begin{bmatrix} 5 \\ \epsilon\alpha_j \end{bmatrix}, \ b_2^{(j)} := \begin{bmatrix} 0 \\ \epsilon\alpha_j \end{bmatrix}.$$

It should be emphasized that all partitions $\mathcal{S}_i, \ i \in \mathbb{I}$ are unbounded.

In this model, it is standard to consider the diffusion through the first variable, that is, $B^{\mathrm{T}} = C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. In the next section, the array consisting of these PWA systems interconnected through (16) is denoted as the heterogeneous PWA-FN network with delays.

B. Stability analysis of delay-diffusive network

By applying Theorem 1, we can analyze the robust stability against arbitrary heterogeneous

- bounded delay length T_{ji},
- bounded coupling strength η_{jk}, and

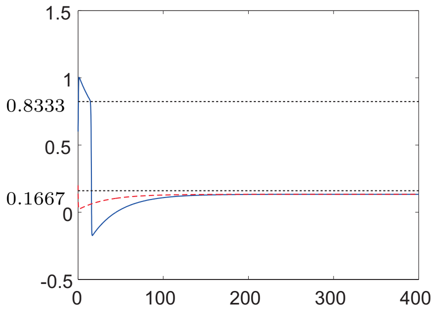

Fig. 4. Excitability property with $\alpha = 0.2$ for two initial conditions.

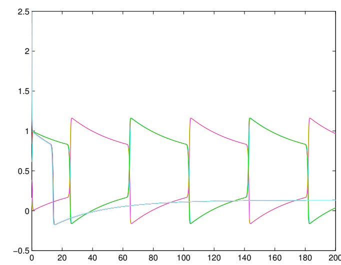

Fig. 5. Time response of all autonomous (uncoupled) subsystems.

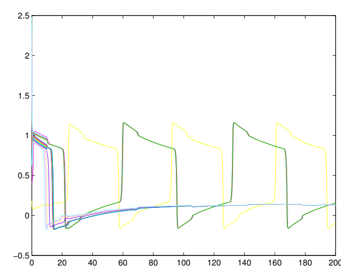

Fig. 6. Time response of all subsystems in coupled network with $2\eta\|H\|_\infty < 1$.

- *oscillation/excitability property* α_j.

Corollary 2: Given $\{\eta_{jk}\}_{j,k}$, define the coupling strength

$$\eta := \max_j \eta_j.$$

and

$$\tilde{H}(s) := \frac{\tilde{n}(s)}{\tilde{d}(s)} := \frac{s + \epsilon b}{s^2 + (\epsilon + 5)s + \epsilon(5b + 1)}.$$

1) If $2\eta\|\tilde{H}\|_\infty < 1$, the heterogeneous PWA-FN network with delays is ultimately bounded for arbitrary delay length T_{jk} and arbitrary α_j.
2) Suppose $2\eta\|\tilde{H}\|_\infty \geq 1$. Define

$$\varpi := \{\omega > 0 : 2\eta|\tilde{H}(j\omega)| \geq 1\}, \qquad (22)$$

$$\bar{\tau} < \inf_{\omega \in \varpi} \frac{\pi + \angle \tilde{H}(j\omega)}{\omega}, \qquad (23)$$

where $\angle$ gives the argument (taking value in $(-\pi, \pi]$) of the complex number. Then, the heterogeneous PWA-FN network with delays is ultimately bounded for any delay length T_{jk} less than $\bar{\tau}$ and arbitrary α_j.

Proof: Since all coefficients of $\tilde{d}(s)$ and $\tilde{n}(s)$ are positive, $\tilde{d}(s) + \rho\tilde{n}(s)$ is Hurwitz for any positive ρ. Then, we readily obtain the desired results by applying Theorem 2 after redefining $C := \eta C$ and $\eta_{jk} := \eta_{jk}/\eta$, i.e., $H = \eta\tilde{H}$. Note that it suffices to show that at least one of

a) $|H^{-1}(j\omega)| > 2$,
b) $\bar{\tau}\omega + \angle H^{-1}(j\omega) \leq \pi$

holds for all $\omega \in \mathbb{R}$. Moreover, a) automatically holds for $\omega \notin \varpi$ because

$$|H^{-1}(j\omega)| = \left|\frac{1}{\eta\tilde{H}(j\omega)}\right| > 2.$$

In case 1), ϖ is empty. In case 2), we have

$$\bar{\tau}\omega \leq \pi + \angle\tilde{H}(j\omega) = \pi - \angle H^{-1}(j\omega),$$

and thus b) for all $\omega \in \varpi$. This completes the proof. ■

C. Numerical Example

We take $b = \epsilon = 0.1$, which leads to $\|\tilde{H}\|_\infty = 0.1961$. The number of subsystems is $N = 15$. We mixed oscillatory and excitable subsystems: $\alpha = 0.5$ for $i = 1, \cdots, 5$ (oscillatory) and $\alpha = 0.2$ for $i = 6, \cdots, 15$ (excitable). Fig. 5 shows time response of uncoupled dynamics.

Hereafter, we use the uniform delay lengths $T_{ji} \equiv \bar{\tau}$. Let us take a gain matrix Γ with $\eta = 2.4 < 2.5491 \approx 1/(2\|\tilde{H}\|_\infty)$. In this case, the coupled dynamics is delay-independently ultimately bounded. Fig. 6 shows time response for $\bar{\tau} = 10$.

Next, we take a gain matrix Γ with $\eta = 4$. Then, by the Bode gain plot in Fig. 7, $\varpi \approx (0.0206, 6.19)$. Fig. 8 shows $\frac{\pi + \angle\tilde{H}(j\omega)}{\omega}$ in this region. We can guarantee the ultimate boundedness according to Corollary 2 when $\bar{\tau} < 0.3644$. Fig. 9 shows $\bar{\tau} = 0.30$, and Fig. 10 for $\bar{\tau} = 0.40$. We see that only the former case is ultimately bounded, where the sufficient condition in Corollary 2 is satisfied.

V. Conclusion

In this paper, we investigated the ultimate boundedness of large-scale arrays consisting of delay diffusively coupled piecewise affine subsystems. At the cost of Assumption 1, we can reduce the problem to the stability of a linear system. Moreover, the latter linear system can easily be analyzed by invoking delay robustness analysis results from MAS theory.

In [23], assuming the semi-passivity of subsystems, similar, but not sharp bounds are obtained for the ultimate boundedness and further synchronization problems. In Section IIV in [23], the use of Master Stability Function [14], [18] is suggested. It would be interesting to compare these different approaches.

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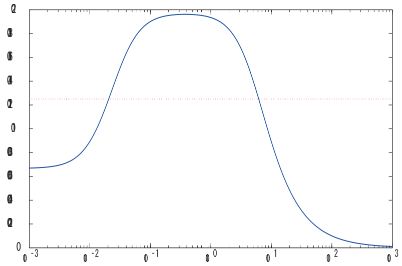

Fig. 7. Bode gain plot $|\tilde{H}(j\omega)|$.

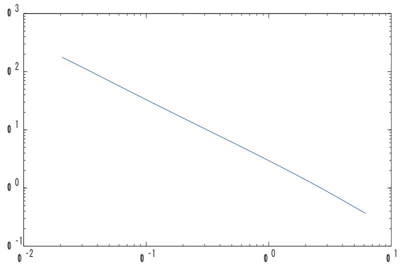

Fig. 8. $\frac{\pi+\angle \tilde{H}(j\omega)}{\omega}$ in ϖ.

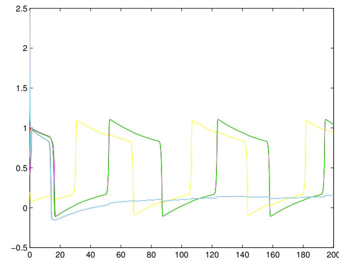

Fig. 9. Time response of all subsystems in coupled network with $\bar{\tau} < 0.3644$.

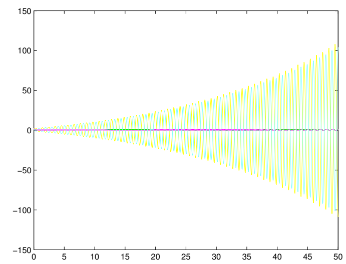

Fig. 10. Time response of all subsystems in coupled network with $\bar{\tau} > 0.3644$.

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