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Observer-Based Pole Placement for Non-Lexicographically-fixed Linear Time-Varying Systems

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***Abstract*— In this paper, the observer based pole placement control for non-lexicographically-fixed linear time-varying MIMO systems is considered. Using the concept of the relative degrees of a MIMO system, Ackerman-like design algorithm for pole placement state feedback control is directly derived. This makes the calculation procedure to obtain the pole placement state feedback very simple, especially for MIMO systems. The separation principle is also shown in the case where both of the controllability indices and the observability indices are not lexicographically-fixed.**

I. INTRODUCTION

The design of the pole placement for linear time-varying systems is well established problem. The design procedure of the pole placement state feedback for linear time-varying system was proposed in[4][5] using the Flobenius standard form as linear time invariant case[1][2][3]. However, the time-varying transformation is complicated, so the design procedure is also complicated especially for MIMO systems. From this point of view, [7] proposed the Ackerman-like calculation algorithm of pole placement for linear time-varying systems.

Another problem is that even though the time-varying linear system is controllable and observable, its controllability indices and/or observability indices may not be fixed. Such a system is called non-lexicographically-fixed system. For this problem, M.Valasek et.al. proposed the pole placement control design method, by adding some dynamics so that the augmented system is controllable and lexicographically-fixed[6]. W.Chai et.al. proposed the design method of observer for the system which is observable, but, its observability indices are not lexicographically-fixed[8]. However, in [8], the observability canonical form is used, so the total system analysis becomes complicated and separation principle of the total system was not argued.

This paper will consider the following subject. 1. Using the concept of the relative degrees, we derive easily the Ackerman-like design algorithm for the pole placement state feedback for linear time-varying MIMO systems. 2. Using the duality of the adjoint system, the Ackerman-like design method of the observer is proposed. 3. The observer based pole placement control is considered for the system that is controllable and observable, but its controllability indices and observability indices are not lexicographically-fixed. The total closed loop is analyzed and the separation principle is also shown.

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II. POLE PLACEMENT OF MULTI-INPUT SYSTEMS

Consider the following linear time-varying (LTV) system with multi-input.

$$\dot{x}(t) \quad = \quad A(t)x(t) + B(t)u(t) \tag{1}$$

Here, $x(t) \in R^n$ and $u(t) \in R^m$ are the state variable and the input signal, and, $A(t) \in R^{n\times n}$ and $B(t) \in R^{n\times m}$ are the time varying coefficient matrices, which are smooth functions of t. The matrix $B(t)$ can be written as follows.

$$B(t) \quad = \quad \begin{bmatrix} b_1(t) & b_2(t) & \cdots & b_m(t) \end{bmatrix} \tag{2}$$

Let $B_j(t) \in R^{n\times m}$ be defined as follows.

$$\begin{aligned} B_0(t) &= B(t) \\ B_{j+1} &= A(t)B_j(t) - \dot{B}_j(t) \end{aligned} \tag{3}$$

Then, the following $U_C(t)$ is the controllability matrix.

$$\begin{aligned} U_C(t) &= \begin{bmatrix} B_0(t) & B_1(t) & \cdots & B_{n-1}(t) \end{bmatrix} \\ &= \Big[\ b_1^0(t) \ \cdots \ b_m^0(t) \ \Big|\ \cdots \\ &\qquad \cdots \ \Big|\ b_1^{n-1}(t) \ \cdots \ b_m^{n-1}(t) \ \Big] \end{aligned} \tag{4}$$

where $b_k^i(t)$ is the k-th column vector of $B_i(t)$. Hence, from (3), $b_k^i(t)$ satisfies the same equation, i.e.,

$$\begin{aligned} b_k^0(t) &= b_k(t) \\ b_k^{i+1}(t) &= A(t)b_k^i(t) - \dot{b}_k^i(t) \\ & \qquad k = 1, 2, \cdots, m \qquad i = 0, 1, 2, \cdots \end{aligned} \tag{5}$$

The system (1) is controllable if and only if

$$rank U_C(t) \quad = \quad n. \tag{6}$$

Suppose that the system (1) is controllable. Let μ_i $(i = 1, \cdots, m)$ be its controllability indices, then we have the following equations,

$$\begin{aligned} rank R(t) &= n \\ \sum_{i=1}^{m} \mu_i &= n \end{aligned} \tag{7}$$

where

$$\begin{aligned} R(t) &= \Big[\ b_1^0(t) \ \cdots \ b_1^{\mu_1 - 1}(t) \ \Big|\ \cdots \\ &\qquad \cdots \ \Big|\ b_m^0(t) \ \cdots \ b_m^{\mu_m - 1}(t) \ \Big]. \end{aligned} \tag{8}$$

In this section, the system (1) is assumed to be lexicographically-fixed, that is, it is assumed that μ_i does not change with t. It is also assumed that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ without loss of generality.

The problem is to find the state feedback

$$u(t) = K(t)x(t) \tag{9}$$

which makes the closed loop system equivalent to some time invariant linear system with arbitrarily stable poles.

Now, consider the problem of finding a new output signal $\tilde{y}(t)$ such that the relative degree from u to $\tilde{y}$ is n. Here, $\tilde{y}(t)$ has the following form.

$$\tilde{y}(t) = \tilde{C}(t)x(t) \tag{10}$$

Then, the problem is to find a matrix $\tilde{C}(t) \in R^{m\times n}$ that satisfies this condition. Eq. (10) can be rewritten as

$$\tilde{y}_i(t) = \tilde{c}_i(t)x(t), \qquad i = 1, \cdots, m \tag{11}$$

here,

$$\tilde{y}(t) = \begin{bmatrix} \tilde{y}_1(t) \\ \vdots \\ \tilde{y}_m(t) \end{bmatrix}, \quad \tilde{C}(t) = \begin{bmatrix} \tilde{c}_1(t) \\ \vdots \\ \tilde{c}_m(t) \end{bmatrix} \tag{12}$$

Let $\tilde{c}_i^j(t)$ be the i-th row of $\tilde{C}_j(t)$ defined by

$$\begin{aligned} \tilde{C}_{j+1}(t) &= \tilde{C}_j(t)A(t) + \dot{\tilde{C}}_j(t), \quad \tilde{C}_0(t) = \tilde{C}(t). \\ (\tilde{c}_i^{j+1}(t) &= \tilde{c}_i^j(t)A(t) + \dot{\tilde{c}}_i^j(t), \quad \tilde{c}_i^0(t) = \tilde{c}_i(t)) \end{aligned} \tag{13}$$

Lemma 1: The relative degree from u to $\tilde{y}$ is n, if and only if

$$\begin{aligned} \tilde{c}_j^0(t)b_1(t) &= \cdots = \tilde{c}_j^{\mu_j-1}(t)b_1(t) = 0 \\ \tilde{c}_j^0(t)b_2(t) &= \cdots = \tilde{c}_j^{\mu_j-1}(t)b_2(t) = 0 \\ &\vdots \\ \tilde{c}_j^0(t)b_j(t) &= \cdots = \tilde{c}_j^{\mu_j-2}(t)b_j(t) = 0, \\ &\qquad \tilde{c}_j^{\mu_j-1}(t)b_j(t) = 1 \\ &\vdots \\ \tilde{c}_j^0(t)b_{m-1}(t) &= \cdots = \tilde{c}_j^{\mu_j-1}(t)b_{m-1}(t) = 0 \\ \tilde{c}_j^0(t)b_m(t) &= \cdots = \tilde{c}_j^{\mu_j-1}(t)b_m(t) = 0 \\ &\qquad (j = 1, \cdots, m) \end{aligned} \tag{14}$$

(Proof) By differentiating $\tilde{y}$ μ_i times successively, we have

$$\begin{aligned} \tilde{y}_i(t) &= c_i^0(t)x(t) \\ \dot{\tilde{y}}_i(t) &= c_i^1(t)x(t) \\ \ddot{\tilde{y}}_i(t) &= c_i^2(t)x(t) \\ &\vdots \\ \tilde{y}_i^{(\mu_i)}(t) &= c_i^{(\mu_i)}(t)x(t) + c_i^{(\mu_i-1)}(t)b_i(t)u_i(t) \\ &= c_i^{(\mu_i)}(t)x(t) + u_i(t) \\ &\quad +\gamma_{i(i+1)}(t)u_{i+1}(t)\cdots + \gamma_{im}(t)u_m(t) \\ &\qquad (i = 1, \cdots, m) \end{aligned} \tag{15}$$

which implies that the relative degree from u to $\tilde{y}$ is n. Here,

$$\gamma_{ij}(t) = c_i^{(\mu_i-1)}(t)b_j(t)$$

Lemma 2: If $\tilde{C}(t)$ satisfies the condition of Lemma 1, we have

$$\begin{aligned} \tilde{c}_j^k(t)b_i(t) &= \tilde{c}_j(t)b_i^k(t) \\ (j, i &= 1, \cdots, m), \quad (k = 0, \cdots, \mu_j - 1) \end{aligned} \tag{16}$$

The proof is omitted, here. From Lemma 2, (14) can be written as

$$\tilde{C}(t)R(t) = W \tag{17}$$

where

$$W = diag(w_1, w_2, \cdots, w_m) \tag{18}$$

and

$$w_i = [0, \cdots, 0, 1] \in R^{1\times\mu_i} \quad (i = 1, \cdots, m) \tag{19}$$

Hence, such $\tilde{C}(t)$ can be calculated by

$$\tilde{C}(t) = WR^{-1}(t). \tag{20}$$

From this, the pole placement state feedback is obtained as follows. Let $\alpha_j^i (i = 1, \cdots, m,\ j = 0, \cdots, \mu_i)$ be the coefficients of an ideal polynomial

$$\alpha^i(s) = s^{\mu_i} + \alpha_{\mu_i-1}^i s^{\mu_i-1} + \cdots + \alpha_0^i \tag{21}$$

where s is a differential operator. By multiplying in sequence from the first equation of (15) by $\alpha_0^i, \cdots, \alpha_{\mu_i-1}^i, 1$, and summing them up, we have

$$\alpha^i(s)y_i(t) = D_i(t)x(t) + \Lambda_i(t)u(t) \tag{22}$$

where $D_i(t) \in R^{1\times n}$ and $\Lambda_i(k) \in R^{1\times m}$ are as follows

$$\begin{aligned} D_i(t) &= [\alpha_0^i, \alpha_1^i, \cdots, \alpha_{\mu_i-1}^i, 1] \begin{bmatrix} \tilde{c}_i^0(t) \\ \tilde{c}_i^1(t) \\ \vdots \\ \tilde{c}_i^{\mu_i}(t) \end{bmatrix} \\ \Lambda_i(t) &= [0, \cdots, 0, 1, \gamma_{i(i+1)}(t), \cdots, \gamma_{im}(t)] \end{aligned} \tag{23}$$

Thus, by the state feedback

$$\begin{aligned} u(t) &= -\Lambda^{-1}(t)D(t)x(t) \\ D(t) &= \begin{bmatrix} D_1(t) \\ D_2(t) \\ \vdots \\ D_m(t) \end{bmatrix}, \quad \Lambda(t) = \begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \\ \vdots \\ \Lambda_m(t) \end{bmatrix} \end{aligned} \tag{24}$$

the closed loop system becomes as follows.

$$\begin{bmatrix} \alpha^1(s) & & \\ & \ddots & \\ & & \alpha^m(s) \end{bmatrix} \tilde{y}(t) = 0 \tag{25}$$

This implies that (24) is the pole placement state feedback.

From (1) and (24), the closed loop state equation becomes

$$\dot{x}(t) = (A(t) - B(t)D(t))x(t) \tag{26}$$

Let $T(t)$ be the time varying matrix defined by

$$T(t) = \left[\begin{array}{c} \tilde{c}_1^0(t) \\ \vdots \\ \tilde{c}_1^{\mu_1-1}(t) \\ \hline \vdots \\ \hline \tilde{c}_m^0(t) \\ \vdots \\ \tilde{c}_m^{\mu_m-1}(t) \end{array}\right] \tag{27}$$

and define the new state variable w by

$$w(t) = T(t)x(t), \qquad w = \left[\begin{array}{c} \tilde{y}_1(t) \\ \vdots \\ \tilde{y}_1^{(\mu_1-1)}(t) \\ \hline \vdots \\ \hline \tilde{y}_m(t) \\ \vdots \\ \tilde{y}_m^{(\mu_m-1)}(t) \end{array}\right] \tag{28}$$

Then, from (25), (26) is transformed into

$$\begin{aligned} \dot{w} &= \{T(t)(A(t) - B(t)D(t))T^{-1}(t) - T(t)\dot{T}^{-1}(t)\}w \\ &= \begin{bmatrix} A_1^* & & 0 \\ & \ddots & \\ 0 & & A_m^* \end{bmatrix} w \end{aligned} \tag{29}$$

where

$$A_i^* = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & 1 \\ -\alpha_0^i & \cdots & \cdots & -\alpha_{\mu_i-1}^i \end{bmatrix} \quad (i = 1, \cdots, m) \tag{30}$$

which is the realization of (25). This implies that the closed loop system is equivalent to the time invariant linear system which has the desired closed loop poles. $(\det(sI - A^*) = \alpha^1(s) \cdot \alpha^2(s) \cdots \alpha^m(s))$

The non-singularity of $T(t)$ is guaranteed by the following Theorem.

Theorem 1: If the system (1) is controllable, then, the matrix for the change of variable, $T(t)$, given by (27) is nonsingular for all t. $\nabla\nabla$

This theorem can be proved by straightforward calculation as for the time invariant case.

It is well known that the exponential stability is preserved between two equivalent linear time-varying systems if the transformation matrix is Lyapunov transformation. Note that $T(t)$ is Lyapunov transformation if it is nonsingular and both of $T(t)$ and $T^{-1}(t)$ are continuous and bounded for all t.

Then, to guarantee the stability of the closed loop system, we need the following Theorem.

Theorem 2: In the above pole placement control, the closed loop system is exponentially stable if the transformation matrix $T(t)$ in (27) is Lyapunov transformation.

The pole placement design procedure is as follows.

- STEP 1 Check the controllability of the system (1) and find the controllability indices μ_i.
- STEP 2 Calculate $\tilde{C}(t)$ using (17).
- STEP 3 Determine the desired characteristic polynomials in (21).
- STEP 4 (24) is the pole placement state feedback, i.e., the disired state feedback gain matrix is $K(t) = -\Lambda^{-1}(t)D(t)$.

III. State Observer

In this section, we consider the design of the state observer for the following linear time-varying MIMO system.

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x \end{aligned} \tag{31}$$

Here, $y \in R^m$ is the output signal of this system and $C(t) \in R^{m\times n}$ is a time varying matrix. Other variables and matrices are the same as those in (1). The system (31) is supposed to be observable. The problem is to design the full order state observer of (31). Consider the following system as a candidate of the observer.

$$\begin{aligned} \dot{\hat{x}} &= A(t)\hat{x} + B(t)u + H(t)(C(t)\hat{x} - y) \\ &= A(t)\hat{x} + B(t)u + H(t)C(t)(\hat{x} - x) \end{aligned} \tag{32}$$

where $H(t) \in R^{n\times m}$ is an observer gain matrix. From this, $e = \hat{x} - x$ satisfies the following error equation.

$$\dot{e} = (A(t) + H(t)C(t))e \tag{33}$$

Hence, as well known, (32) is a state observer of (31) if $H(t)$ satisfies the following condition.

$$A(t) + H(t)C(t) \ : \ \text{arbitrarily stable matrix} \tag{34}$$

Then, the problem is to find $H(t)$ such that $A(t)+H(t)C(t)$ is equivalent to some constant matrix that has desired constant exponentially stable poles.

For this purpose, consider the pole placement control problem of the following adjoint system of (31).

$$\dot{x} = -A^T(t)x + C^T(t)u \tag{35}$$

From the property of the duality of the time varying adjoint system, if the pair $(A(t), C(t))$ is observable, the pair $(-A^T(t), C^T(t))$ is controllable. This implies that if the system (31) is observable, there is an **anti-stablized** pole placement state feedback for the system (35).

Using the result of the previous section, the calculation procedure to obtain the observer for (31) is summarized as follows.

- STEP 1 Check the observability of the system (31) and find the observablility indices ν_i $(i = 1, \cdots, m)$.
- STEP 2 Since ν_i's are controllability indices of the adjoint system (35), calculate the matrix $\tilde{C}(t)$ for (35) using (17) by replacing $A(t)$, $B(t)$ and μ_i by $-A^T(t)$, $C^T(t)$ and ν_i respectively.
- STEP 3 Determine the desired **anti-stable** characteristic polynomial for the observer by (21), by replacing

μ_i by ν_i. That is, if the desired stable poles for the observer are $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, the desired unstable poles in this case are $\{-\lambda_1, -\lambda_2, \cdots, -\lambda_n\}$.

STEP 4 Using (24), the **anti-stablized** pole placement state feedback

$$u = -H^T(t)x$$

for the system (35) is obtained, by replacing μ_i by ν_i.

This implies that, from the previous section, using the appropriate transformation matrix, $P(t) \in R^{n\times n}$, we have the following equation.

$$P^{-1}(t)(-A^T(t) - C^T(t)H^T(t))P(t) \\ -P^{-1}(t)\dot{P}(t) = -F^{*T} \quad (36)$$

Here, the eigenvalues of $-F^*$ are $\{-\lambda_1, -\lambda_2, \cdots, -\lambda_n\}$.

From this, we have the following.

$$\begin{aligned} & P^T(t)\{A(t) + H(t)C(t)\}(P^T(t))^{-1} \\ & \quad -P^T(t)\frac{d}{dt}(P^T(t))^{-1} \\ = \; & P^T(t)\{A(t) + H(t)C(t)\}(P^T(t))^{-1} \\ & \quad +\dot{P}^T(t)(P^T(t))^{-1} \\ = \; & \big(P^{-1}(t)\{A^T(t) + C^T(t)H^T(t)\}P(t) \\ & \quad +P^{-1}(t)\dot{P}(t)\big)^T \\ = \; & F^* \end{aligned} \quad (37)$$

The eigenvalues of F^* are the desired observer poles, $(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Hence, $H(t)$ is an observer gain matrix.

Note that the system (32) is the observer for the system (31) if the transformation matrix $P(t)$ is Lyapunov transformation.

IV. Non-Lexicographically-Fixed Systems

A. Pole Placement for NLF systems

In this section, we assume that the system (1) is controllable. However, its controllability indices μ_i's are not fixed. Such a system is called "Non-Lexicographically-Fixed" (NLF) system. For such systems, the truncated controllability matrix $R(t)$ in (8) is not always non-singular, which makes the design problem difficult. M.Valasek and N.Olgac proposed the pole placement control design method, by adding some dynamics so that the augmented system is controllable and lexicographically-fixed[6] In this paper, this design procedure is not repeated. Please see [6] for details. Using this technique, basic design procedure is as follows.

Since, the system (1) is assumed to be controllable, the rank of $U_C(t)$ is always n. In addition, it is supposed that the maximum values of all controllablity indices are known, i.e., the following $\bar{\mu}_i$'s are known.

$$\bar{\mu}_i = \max_t \mu_i(t) \qquad (i = 1, \cdots, m) \quad (38)$$

Let n_e be defined as

$$n_e = \sum_{i=1}^{m} \bar{\mu}_i - n \quad (39)$$

and consider the following augmented system.

$$\frac{d}{dt}\begin{bmatrix} x \\ x_e \end{bmatrix} = \begin{bmatrix} A(t) & 0 \\ A_2(t) & A_1(t) \end{bmatrix}\begin{bmatrix} x \\ x_e \end{bmatrix} + \begin{bmatrix} B(t) \\ B_e(t) \end{bmatrix} u \quad (40)$$

where $x_e \in R^{n_e}$, $A_2(t) \in R^{n_e\times n}$, $A_1(t) \in R^{n_e\times n_e}$ and $B_e(t) \in R^{n_e\times m}$. The calculation method for $A_1(t)$, $A_2(t)$ and $B_e(t)$ is shown in [6] so that the augmented system (39) is controllable and its controllability indices are $\bar{\mu}_i$ ($i = 1, \cdots, m$) and they are lexicographically-fixed.

Then, by using the simple design procedure in Section II, the following pole placement augmented state feedback can be obtained.

$$u = \begin{bmatrix} K(t) & K_e(t) \end{bmatrix}\begin{bmatrix} x \\ x_e \end{bmatrix} \quad (41)$$

Define $A_{aug}(t)$ by

$$\begin{aligned} A_{aug}(t) \;=\; & \begin{bmatrix} A(t) & 0 \\ A_2(t) & A_1(t) \end{bmatrix} \\ & + \begin{bmatrix} B(t) \\ B_e(t) \end{bmatrix}\begin{bmatrix} K(t) & K_e(t) \end{bmatrix} \end{aligned} \quad (42)$$

then, there exists the state transformation matrix $T_{aug}(t) \in R^{(n+n_e)\times(n+n_e)}$, and it satisfies the following equation.

$$T_{aug}(t)A_{aug}(t)T_{aug}^{-1}(t) - T_{aug}(t)\dot{T}_{aug}^{-1}(t) = A_{aug}^* \quad (43)$$

where $A_{aug}^* \in R^{(n+n_e)\times(n+n_e)}$ is a constant matrix that has the desired augmented closed loop eigenvalues.

B. Observer for NLF systems

Consider the system (31) which is supposed to be observable, but, its observability indices are not lexicographically-fixed. It is assumed that the following $\bar{\nu}_i$'s are known.

$$\bar{\nu}_i = \max_t \nu_i(t) \qquad (i = 1, \cdots, m) \quad (44)$$

Let n_f be defined as

$$\hat{n}_e = \sum_{i=1}^{m} \bar{\nu}_i - n \quad (45)$$

and consider the following candidate of the augmented observer for the system (31).

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \hat{x} \\ \epsilon \end{bmatrix} \;=\; & \begin{bmatrix} A(t) & \hat{A}_2(t) \\ 0 & \hat{A}_1(t) \end{bmatrix}\begin{bmatrix} \hat{x} \\ \epsilon \end{bmatrix} + \begin{bmatrix} B(t) \\ 0 \end{bmatrix} u \\ & + \begin{bmatrix} H(t) \\ H_e(t) \end{bmatrix}\begin{bmatrix} C(t) & C_e(t) \end{bmatrix}\begin{bmatrix} \hat{x} - x \\ \epsilon \end{bmatrix} \end{aligned} \quad (46)$$

where $\epsilon \in R^{\hat{n}_e}$, $\hat{A}_2(t) \in R^{n\times\hat{n}_e}$, $\hat{A}_1(t) \in R^{\hat{n}_e\times\hat{n}_e}$, $H_e(t) \in R^{\hat{n}_e\times m}$, and $C_e(t) \in R^{m\times\hat{n}_e}$

Define $e = \hat{x} - x$, then, from (31) and (46) we have

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} e \\ \epsilon \end{bmatrix} \;=\; & \begin{bmatrix} A(t) & \hat{A}_2(t) \\ 0 & \hat{A}_1(t) \end{bmatrix}\begin{bmatrix} e \\ \epsilon \end{bmatrix} \\ & + \begin{bmatrix} H(t) \\ H_e(t) \end{bmatrix}\begin{bmatrix} C(t) & C_e(t) \end{bmatrix}\begin{bmatrix} e \\ \epsilon \end{bmatrix} \end{aligned} \quad (47)$$

Hence, (46) is an augmented observer for the system (31), if $F_{aug}(t)$ defined by

$$\begin{aligned} F_{aug}(t) &= \begin{bmatrix} A(t) & \hat{A}_2(t) \\ 0 & \hat{A}_1(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} H(t) \\ H_e(t) \end{bmatrix} \begin{bmatrix} C(t) & C_e(t) \end{bmatrix} \end{aligned} \tag{48}$$

is equivalent to some constant matrix, F^* which has desired stable constant eigenvalues. For this purpose, we consider the following augmented adjoint system.

$$\dot{\zeta} = -\begin{bmatrix} A^T(t) & 0 \\ \hat{A}_2^T(t) & \hat{A}_1^T(t) \end{bmatrix} \zeta + \begin{bmatrix} C^T(t) \\ C_e^T(t) \end{bmatrix} v \tag{49}$$

Then, the observer problem is to find the anti-stabilized pole placement state feedback

$$v = -\begin{bmatrix} H^T(t) & H_e^T(t) \end{bmatrix} \begin{bmatrix} e \\ \epsilon \end{bmatrix} \tag{50}$$

for (49), so that $-F_{aug}^T(t)$ is equivalent to some constant matrix $-F_{aug}^{*T}$. The eigenvalues of $-F_{aug}^{*T}$ are chosen as $\{-\gamma_1, \cdots, -\gamma_{(n+\hat{n}_e)}\}$, where $\{\gamma_1, \cdots, \gamma_{(n+\hat{n}_e)}\}$ are desired augmented observer poles.

Since the system (31) is assumed to be observable, the pair $(-A^T(t), C^T(t))$ is controllable. Then, from the previous section, we can find $\hat{A}_1^T(t)$, $\hat{A}_2^T(t)$ and $C_e^T(t)$ such that the augmented adjoint system (49) is controllable and its controllability indices are $\{\nu_1, \cdots, \nu_m\}$ and they are lexicographically-fixed. Thus, we can calculate the state feedback

$$v = \begin{bmatrix} H^T(t) & H_e^T(t) \end{bmatrix} \zeta \tag{51}$$

such that the closed loop system matrix, $-F_{aug}^T(t)$, is equivalent to $-F_{aug}^{*T}$. This implies that there exists a state transformation matrix $P_{aug}(t) \in R^{(n+\hat{n}_e)\times(n+\hat{n}_e)}$ satisfying the following relation.

$$\begin{aligned} &P_{aug}^T(t)(-F_{aug}^T(t))P_{aug}^{-1}(t) \\ &\quad -P_{aug}^T(t)\frac{d}{dt}(P_{aug}^T(t))^{-1} = -F_{aug}^* \end{aligned} \tag{52}$$

Then, we have the following.

$$\begin{aligned} &P_{aug}^T(t)F_{aug}^T(t)(P_{aug}^T(t))^{-1} \\ &\qquad -P_{aug}^T(t)\frac{d}{dt}(P_{aug}^T(t))^{-1} \\ = \; &P_{aug}^T(t)F_{aug}^T(t)(P_{aug}^T(t))^{-1} \\ &\qquad +\dot{P}_{aug}^T(t)(P_{aug}^T(t))^{-1} \\ = \; &-\big(P_{aug}^{-1}(t)(-F_{aug}^T(t))P_{aug}(t) \\ &\qquad -P_{aug}^{-1}(t)\dot{P}_{aug}(t)\big)^T \\ = \; &F_{aug}^* \end{aligned} \tag{53}$$

C. Observer Based Control of a NLF System

In this section, the observer based pole placement control for a system which has both of NLF controllability indices and NLF observability indices.

In this case, x should be replaced by $\hat{x}$ in the controller. This implies that to obtain x_e, the equation (40) should be modified as follows using $\hat{x}_e$ instead of x_e.

$$\dot{\hat{x}}_e = A_2(t)\hat{x} + A_1(t)\hat{x}_e + B_e(t)u \tag{54}$$

Thus, the pole placement state feedback is

$$u = \begin{bmatrix} K(t) & K_e(t) \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{x}_e \end{bmatrix} \tag{55}$$

The augmented system to be controlled is described by

$$\dot{x} = A(t)x + B(t)u \tag{56}$$

$$\dot{\hat{x}}_e = A_2(t)\hat{x} + A_1(t)\hat{x}_e + B_e(t)u \tag{57}$$

and the augmented observer is described by (46). By substituting the state feedback (55) into (46),(56) and (57), the total closed loop system is as follows.

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ \hat{x}_e \\ \hat{x} \\ \epsilon \end{bmatrix} = &\left[\begin{matrix} A & BK_e \\ 0 & A_1 + B_eK_e \\ -HC & BK_e \\ -H_eC & 0 \end{matrix}\right. \\ &\left.\begin{matrix} BK & 0 \\ A_2 + B_eK & 0 \\ A + BK + HC & \hat{A}_2 + HC_e \\ H_eC & \hat{A}_1 + H_eC_e \end{matrix}\right] \begin{bmatrix} x \\ \hat{x}_e \\ \hat{x} \\ \epsilon \end{bmatrix} \end{aligned} \tag{58}$$

(Here, (t) is omitted because of the narrow space.) By the following change of variable,

$$\begin{bmatrix} x \\ \hat{x}_e \\ e \\ \epsilon \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} x \\ \hat{x}_e \\ \hat{x} \\ \epsilon \end{bmatrix} \tag{59}$$

the total system (58) is described as follows.

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x \\ \hat{x}_e \\ e \\ \epsilon \end{bmatrix} &= \left[\begin{matrix} A + BK & BK_e \\ A_2 + B_eK & A_1 + B_eK_e \\ 0 & 0 \\ 0 & 0 \end{matrix}\right. \\ &\quad \left.\begin{matrix} BK & 0 \\ A_2 + B_eK & 0 \\ A + HC & \hat{A}_2 + HC_e \\ H_eC & \hat{A}_1 + H_eC_e \end{matrix}\right] \begin{bmatrix} x \\ \hat{x}_e \\ e \\ \epsilon \end{bmatrix} \\ &= \begin{bmatrix} A_{aug}(t) & E(t) \\ 0 & F_{aug}(t) \end{bmatrix} \begin{bmatrix} x \\ \hat{x}_e \\ e \\ \epsilon \end{bmatrix} \end{aligned} \tag{60}$$

where

$$E(t) = \begin{bmatrix} BK & 0 \\ A_2 + B_eK & 0 \end{bmatrix} \tag{61}$$

From this, the further transformation matrix $Q(t)$ defined by

$$Q(t) = \begin{bmatrix} T_{aug}(t) & 0 \\ 0 & P_{aug}^T(t) \end{bmatrix} \tag{62}$$

the system matrix of (60) is transformed as follows.

$$\begin{aligned} Q(t) \begin{bmatrix} A_{aug}(t) & E(t) \\ 0 & F_{aug}(t) \end{bmatrix} Q^{-1}(t) - Q(t)\dot{Q}^{-1}(t) \\ = \begin{bmatrix} A_{aug}^* & T_{aug}(t)E(t)(P_{aug}^T(t))^{-1} \\ 0 & F_{aug}^* \end{bmatrix} \end{aligned} \tag{63}$$

This implies the separation principle of the observer based pole placement control for NLF systems. Note that $Q(t)$ should be Lyapunov transformation for the exponential stability of the closed loop system.

V. Example

Cosider the following system.

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sin t & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \sin t & 0 \end{bmatrix} u \tag{64}$$

$$y = \begin{bmatrix} 0 & 1 & 1 \\ \sin t & 1 & 0 \end{bmatrix} x \tag{65}$$

This system is controllable and observable. However, both of its controllability indices and observability indices are non-lexicographically-fixed.

For this system, the augmented system is

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \sin t & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \sin t & 0 \end{bmatrix} u \\ \dot{\hat{x}}_e &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} \cos t & -\sin t \end{bmatrix} u \end{aligned} \tag{66}$$

and the augmented observer is

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\epsilon} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & \sin t \\ 0 & 0 & 0 & \cos^2 t \\ 0 & \sin t & 0 & \cos t - \cos^2 t - \cos^3 t \\ 0 & 0 & 0 & \sin t \cos t \end{bmatrix} \\ &\times \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \epsilon \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \sin t & 0 \\ 0 & 0 \end{bmatrix} u \\ &+ \begin{bmatrix} H(t) \\ H_e(t) \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & \sin t \\ \sin t & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 - x_1 \\ \hat{x}_2 - x_2 \\ \hat{x}_3 - x_3 \\ \epsilon \end{bmatrix} \end{aligned} \tag{67}$$

One simulation result of observer based pole placement control for this system is shown in Fig.1 and Fig.2.

VI. Conclusions

In this paper, using the concept of the relative degrees and the adjoint system, the Ackerman-like design algorithm for the pole placement state feedback and observer for linear time-varying MIMO systems was considered. Then, the observer based pole placement control was considered for the non-lexicographically-fixed system. The separation principle of the total closed loop system was also shown.

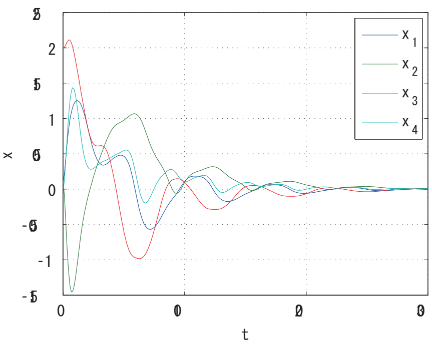

Fig. 1. Response of the Controlled Augmented System ($x_4 = \hat{x}_e$)

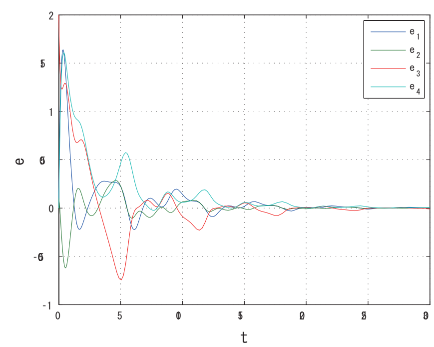

Fig. 2. State Estimation Error ($e_4 = \epsilon$)

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