2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Stabilization of discrete-time linear systems with saturating actuators using sliding modes: application to a twin-rotor system

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Abstract— **This paper investigates the stabilization problem for discrete-time linear controllable systems subject to actuator saturation. In the case of completely known plants, a control design technique based on a time varying sliding surface is proposed ensuring stabilization under the assumption of availability of all state measurements. Further, in the presence of matched disturbances with known constant bound, a discrete time sliding mode controller is proposed ensuring plant practical stabilization, and a conservative estimate of the attraction domain is given. Theoretical results have been validated by experimental data using a twin-rotor system.**

Keywords: Saturating Actuators, discrete time sliding mode control, twin rotor system.

I. Introduction

The phenomenon of amplitude saturation in actuators is due to inherent physical limitations of devices. Though often ignored, as happens in classical control theory, it cannot be avoided in practice. Unfortunately, failure in accounting for actuator saturation may lead to severe deterioration of closed loop system performance, even to instability. Just to mention a few examples, sliding mode control [1] [2] has been widely used in recent literature for the control of motors, precision devices and/or vehicles [3] [4], [5], but the problem of actuator saturation has been considered only rarely, at least from the theoretical viewpoint.

In the vast literature addressing the stabilization problem for discrete-time linear systems subject to actuator saturation, two lines of research have been mostly pursued. The first line focuses on the estimation of the asymptotic stability region, which often has a very conservative expression. To reduce this conservatism, estimates are given as solution of suitable LMI optimization problems in the continuous time [6], [7] and in the discrete time framework [8], [9], [10]. The other line of research focuses on the estimation, less conservative as possible, of the null controllable region, i.e. the set of state which can be driven towards the origin of the state space using saturating actuators. In this latter framework, the problem has been completely studied for plants known as Asymptotically Null Controllable with Bounded Controls (ANCBC), for which the null controllable region is the whole state space [11], [12], [13]. Moreover, some results are available for general discrete-time systems about feedback laws achieving semi-global stabilization on the null controllable region. Broadly speaking, such techniques consist either in dividing the null controllable region in polygons and finding suitable controls driving the vertices to the origin [14], or in designing a sequence of feedback laws such that the union of the corresponding invariant sets is an invariant set contained in the domain of attraction [15]. Both techniques, however, require a considerable computational burden also for plants with relatively low order.

Furthermore, the problem of disturbance rejection for linear systems subject to actuator saturation has been investigated only marginally in the discrete time framework. Note that for continuous time plant an interesting research line considers disturbances that are magnitude bounded. In such context, [16] proved that semi-global practical stabilization for a linear system subject to actuator saturation and input additive disturbances can be achieved as long as the open loop system is not exponentially unstable. For the same class of systems, Lin [17] constructed non-linear feedback laws that achieve global practical stabilization. Recently, it has been proved in [18] that a 2-dimensional linear systems subject to actuator saturation and bounded input additive disturbances can be globally practically stabilized by linear state feedback, while a sliding mode approach has been very recently presented [19], [20] for continuous-time Single Input plants.

In this paper, a novel control law is proposed, along with experimental tests on a twin-rotor system. The controller is aimed at ensuring that the motors driving the plant work far from the saturation limit. In fact, it is well known that it is not good practice to let a driving power device work near or in saturation, for obvious reasons concerning safety and energy supply of the device itself. To this purpose, a time-varying sliding surface, different from that in [19], [20], has been designed, and the corresponding controller has been shown to be able to provide finite time plant stabilization for completely known systems. An extension to the case when matched bounded uncertainties affect the plant has been also considered, and a discrete-time sliding mode controller has been proposed ensuring ultimate boundedness of state trajectories. Finally, experimental tests have been performed on a twin-rotor system.

It is worth noticing that the control technique proposed here contains a number of novelties with respect to our previous works. Firstly, it addresses discrete-time plants, while previously published results [19], [20] considered the continuous time framework. Moreover, the sliding surface and the corresponding control law here discussed is not

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978-1-61284-799-3/11/$26.00 ©2011 IEEE

simply the discrete-time counterpart of the surface studied in [19], [20], but has been differently designed. Finally, one of the main focuses of this paper is the experimental validation of the controller, never performed in our previous papers.

II. Problem statement

Consider the following discrete-time, time invariant SISO plant $S \stackrel{\text{def}}{=} \{\hat{\mathbf{A}}, \hat{\mathbf{B}}\}$ described by:

$$\hat{\mathbf{x}}(k+1) = \hat{\mathbf{A}}\hat{\mathbf{x}}(k) + \hat{\mathbf{B}}u(k) \tag{1}$$

where: $\hat{\mathbf{x}}(k) = [\hat{x}_1(k) \cdots \hat{x}_n(k)]^T \in \mathbb{R}^n$ is the state vector (assumed available for measurement), $u(k) \in \mathbb{R}$ is the control input, and $\hat{\mathbf{A}} \in \mathbb{R}^{n\times n}$, $\hat{\mathbf{B}} \in \mathbb{R}^n$ are the state and input distribution matrices, respectively.

Assumption 2.1: The plant is controllable.

The plant is supposed to be preceded by a saturating device $u(k) = f(v(k))$ described by the following analytical expression:

$$u(k) = f(v(k)) = \begin{cases} M & if \;\; v(k) \geq M \\ v & if \;\; -M < v(k) < M \\ -M & if \;\; v(k) \leq -M \end{cases} \tag{2}$$

with threshold M known.

Under the controllability hypothesis, there exists a smooth change of coordinates: $\mathbf{x}(k) = \mathbf{T}_2\mathbf{T}_1\hat{\mathbf{x}}(k)$ such that by $\mathbf{T}_1$ the plant is transformed in the controllability form, and $\mathbf{T}_2$ is such that system (1) becomes:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k) \tag{3}$$

with: $\mathbf{A} = \mathbf{T}_2\mathbf{T}_1\hat{\mathbf{A}}\mathbf{T}_1^{-1}\mathbf{T}_2^{-1} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & A_{22} \end{bmatrix} =$

$$= \begin{bmatrix} 0 & \alpha_1 & 0 & \dots & 0 \\ 0 & 0 & \alpha_2 & \dots & 0 \\ \vdots & & \dots & & \alpha_{n-1} \\ a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}, \tag{4}$$

$$\mathbf{B} = \mathbf{T}_2\mathbf{T}_1\hat{\mathbf{B}} = \begin{bmatrix} \mathbf{0} \\ b \end{bmatrix} \quad |\alpha_1| < 1; \dots |\alpha_{n-1}| < 1 \tag{5}$$

Since $|\alpha_1| < 1; \dots |\alpha_{n-1}| < 1$, it is straightforward that, when all the eigenvalues of $\mathbf{A}$ are equal to zero, i.e. when $a_1 = a_2 = \cdots = a_n = 0$, matrix $\mathbf{A}$ has norm less than 1.

Recalling that only the input $v(k)$ is available for direct manipulation, the control problem addressed in this paper consists in finding a state feedback controller $v(k)$ guaranteeing the stabilization of the system (1), in the presence of a saturating non-linearity in the actuating device given by (2).

III. A finite time stabilizing controller with saturating inputs

The basic idea pursued in this section is to design a time-varying sliding surface such that the achievement of a quasi sliding motion on it can be ensured with saturating input. The associated sliding mode based controller is used to drive the plant state toward a suitable neighbourhood of the origin, where a standard state feedback controller can be used to achieve finite time stability.

First of all, a set of initial states can be easily found, starting from which the state vector can be directly steered to the origin using a standard linear state feedback controller. To this purpose, with reference to the transformed plant (3), consider the following control law:

$$u(k) = \mathbf{K}\mathbf{x}(k) \tag{6}$$

where $\mathbf{K}$ is such that the matrix $\mathbf{A}+\mathbf{B}\mathbf{K} \stackrel{\text{def}}{=} \mathbf{N}$ is nilpotent. In the following, the symbol $||\cdot||$ will denote $||\cdot||_2$.

Lemma 3.1: It is given the discrete-time system (3) preceded by the saturating device (2) under Assumption 2.1. The deadbeat controller (6) guarantees finite time stabilization with saturating inputs for any initial condition belonging to the set:

$$\mathcal{I} = \left\{ \mathbf{x}(0) : ||\mathbf{x}(0)|| \leq \frac{M}{||\mathbf{K}||} \stackrel{\text{def}}{=} \bar{M} \right\} \tag{7}$$

Such set is an invariant set.

Proof: Consider $\mathbf{x}(0)$ as the initial condition, and apply the deadbeat controller $u(k) = \mathbf{K}\mathbf{x}(k)$. The saturation constraint provides:

$$|\mathbf{K}\mathbf{x}(k)| \leq M \tag{8}$$

Moreover, the following chain of inequalities is straightforward:

$$\begin{aligned} |\mathbf{K}\mathbf{x}(k)| &\leq ||\mathbf{K}|| \cdot ||\mathbf{x}(k)|| \leq ||\mathbf{K}|| \cdot ||\mathbf{N}^k|| \cdot ||\mathbf{x}(0)|| \\ &\leq ||\mathbf{K}|| \cdot ||\mathbf{N}||^k \cdot ||\mathbf{x}(0)|| \leq ||\mathbf{K}|| \cdot ||\mathbf{x}(0)|| \end{aligned} \tag{9}$$

since $||\mathbf{N}|| < 1$. In fact, $\mathbf{N} = \mathbf{A} + \mathbf{B}\mathbf{K}$ is nilpotent and therefore all the elements of its last row are equal to zero, due to transformation matrix $\mathbf{T}_1$: as a consequence $||\mathbf{N}|| < 1$, due to transformation matrix $\mathbf{T}_2$, i.e. to the choice of coefficients $\alpha_1 < 1 \dots \alpha_n < 1$. Expression (9) implies that: $||\mathbf{x}(0)|| \leq \frac{M}{||\mathbf{K}||} \Rightarrow ||\mathbf{x}(k)|| \leq \frac{M}{||\mathbf{K}||}$, i.e. the set $\mathcal{I}$ is invariant. In other words, if the initial state belongs to the set $\mathcal{I}$ and fulfills the saturation constraints, the entire dynamics satisfy the same constraint. Moreover, the deadbeat controller ensures stabilization in finite time. ■

As already mentioned, a time varying sliding surface will be introduced. As well known [1] [21], a vector $\mathbf{C} = [\mathbf{C}_1 \;\; C_2] \in \mathbb{R}^n$ can be chosen such that, when a sliding motion is achieved on the following sliding surface:

$$\hat{s}(k) = \mathbf{C}\mathbf{x}(k) = \mathbf{C}_1\mathbf{x}_1(k) + C_2x_2(k) = 0 \tag{10}$$

the corresponding reduced order system has assigned stable eigenvalues, and, as a consequence, system (3) is stable, too. It will be assumed here to choose $\mathbf{C}_1$ and C_2 such that the matrix $\mathbf{N}_1 = \mathbf{A}_{11} - \mathbf{A}_{12}\frac{\mathbf{C}_1}{C_2}$ has stable eigenvalues, and that, without loss of generality, $C_2 > 0$. Starting from the classical sliding surface (10), always with reference to the transformed plant (3), the following time varying sliding surface can be introduced:

$$s(k) = \mathbf{C}\mathbf{x}(k) - \lambda^k\mathbf{C}\mathbf{A}\mathbf{x}(k-1) = 0 \tag{11}$$

where $0 < \lambda < 1$ is a design parameter. It is straightforward that when $s(k) = 0$, the system is asymptotically stable, since for $k \to \infty$ surface (11) tends to surface (10). The equivalent control ensuring the achievement of a sliding motion on (11) can be obtained imposing the condition $s(k+1)=0$, i.e.:

$$u_{eq}(k) = -(\mathbf{CB})^{-1}\mathbf{CAx}(k)(1-\lambda^{k+1}) \tag{12}$$

After the application of the controller (12), the closed loop system for $k \geq 1$ is described by:

$$\begin{aligned}\mathbf{x}(k+1) &= \left[\mathbf{A} - (1-\lambda^{k+1})\mathbf{B}(\mathbf{CB})^{-1}\mathbf{CA}\right]\mathbf{x}(k) \\ &= \mathbf{F}(\lambda, k+1)\mathbf{x}(k)\end{aligned} \tag{13}$$

Due to the choice of $\mathbf{C}$ and λ in (11), the closed loop system (13) is asymptotically stable. The following result can be proved:

Theorem 3.1: It is given the discrete-time system (3) preceded by the saturating device (2) under Assumption 2.1. The controller (12) guarantees that any initial condition belonging to the set:

$$\mathcal{J} = \left\{ \mathbf{x}(0) \in \left(\prod_{j=0}^{k-1} \mathbf{Q}(\lambda, k-j) \right) \mathcal{I} \right\} \tag{14}$$

being $\mathbf{Q}(\lambda, k-j) = \mathbf{F}(\lambda, k-j)^{-1}$, is driven to the set $\mathcal{I}$ in k steps without violating the saturation constraints. Therefore, finite time stabilization with saturating inputs is guaranteed for any initial condition belonging to the set $\mathcal{J}$ by coupling (6) and (12).

Proof: The proof consists in showing that a procedure exists for selecting the parameter λ such that the control law (12) drives the state into the invariant set $\mathcal{I}$ in a finite number of steps k. Let's consider $\overline{\mathbf{x}} \in \partial\mathcal{I}$, hence $||\overline{\mathbf{x}}|| = \overline{M}$.

- *Step* 1 Consider the initial state $\hat{\mathbf{x}}_1$

  $$\hat{\mathbf{x}}_1 := \mathbf{Q}(\lambda, 1)\overline{\mathbf{x}} := [\mathbf{A} - (1-\lambda)\mathbf{B}(\mathbf{CB})^{-1}\mathbf{CA}]^{-1}\overline{\mathbf{x}}.$$

  Imposing that $|u_{eq}(0)| \leq M$, one gets

  $$||(1-\lambda)\mathbf{Q}(\lambda, 1)|| \leq \frac{||\mathbf{K}||}{||(\mathbf{CB})^{-1}\mathbf{CA}||} = \frac{M}{\overline{M}||(\mathbf{CB})^{-1}\mathbf{CA}||} \tag{15}$$

  Denoting by $\lambda_1 = \inf\{\lambda \in (0,1) : \text{(15) is fulfilled}\}$, one immediately gets that initial conditions belonging to the set $\mathbf{Q}(\lambda, 1)\mathcal{I}$ can be driven to the set $\mathcal{I}$ simply setting $1 > \lambda > \lambda_1$ in (12), in fact

  $$\begin{aligned}\mathbf{x}(1) &= \mathbf{A}\hat{\mathbf{x}}_1 + \mathbf{B}u_{eq}(0) = \mathbf{F}(\lambda, 1)\hat{\mathbf{x}}_1 = \\ &= \mathbf{F}(\lambda, 1)\mathbf{Q}(\lambda, 1)\overline{\mathbf{x}} = \overline{\mathbf{x}}.\end{aligned} \tag{16}$$

- *Step* 2 Define $\hat{\mathbf{x}}_2$ as $\hat{\mathbf{x}}_2 := \mathbf{Q}(\lambda, 1)\mathbf{Q}(\lambda, 2)\overline{\mathbf{x}}$. The saturation constraints is fulfilled if the parameter λ in (12) is chosen as $1 > \lambda > \lambda_2$, with λ_2 solution of
  1) $||(1-\lambda)\mathbf{Q}(\lambda, 1)\mathbf{Q}(\lambda, 2)|| < \mu$,
  2) $||(1-\lambda^2)\mathbf{Q}(\lambda, 2)|| < \mu$,

  having defined $\mu = \frac{||\mathbf{K}||}{||(\mathbf{CB})^{-1}\mathbf{CA}||}$. It follows that all initial conditions belonging to $\mathbf{Q}(\lambda, 1)\mathbf{Q}(\lambda, 2)\mathcal{I}$ can be driven to the set $\mathcal{I}$ in 2 steps.
- *Step* k As before define $\hat{\mathbf{x}}_k := \left(\prod_{j=0}^{k-1} \mathbf{Q}(\lambda, k-j)\right)\overline{\mathbf{x}}$ and imposing that

  $$\left\| (1-\lambda^r) \prod_{j=0}^{k-r} \mathbf{Q}(\lambda, k-j) \right\| < \mu \quad \forall r \leq k \tag{17}$$

  one gets that, denoting $\lambda_k := \inf\{\lambda \in (0,1) : \text{(17) is fulfilled}\}$, and choosing $1 > \lambda > \lambda_k$ in (12), any initial condition belonging to the set $\mathcal{J}$ can be driven to the set $\mathcal{I}$ in k steps. Therefore the statement follows.

■

IV. PRESENCE OF BOUNDED UNCERTAINTIES

The case when matched bounded disturbances or uncertainties affect the plant will be now considered. As well known, such class of disturbances is traditionally dealt with by sliding mode control, though may be restrictive for some plants. Reference is made here to the plant:

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{Ax}(k) + \mathbf{B}\left[u(k) + d(k)\right] = \\ &= \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & A_{22} \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \mathbf{0} \\ b \end{bmatrix} [u(k) + d(k)]\end{aligned} \tag{18}$$

under the following assumption:

Assumption 4.1: The uncertain term $d(k)$ is such that: $|d(k)| \leq \rho$, being ρ a known constant. Moreover, ρ is such that $\rho < \frac{M}{n||\mathbf{B}||\cdot||\mathbf{K}||}$.

It is straightforward to verify that Lemma 3.1 can be extended to cope with the uncertain plant (18) as follows:

Lemma 4.1: It is given the discrete-time system (18) preceded by the saturating device (2) under Assumptions 2.1, 4.1. The deadbeat controller (6) guarantees that for any initial condition belonging to the set:

$$\mathcal{I}_\rho = \left\{ \mathbf{x}(0) : ||\mathbf{x}(0)|| \leq \frac{M}{||\mathbf{K}||} - L \stackrel{\text{def}}{=} M^* \right\} \tag{19}$$

with $L = n\rho||\mathbf{B}||$, ultimate boundedness of the state trajectories is ensured according to

$$\lim_{k\to\infty} ||\mathbf{x}(k)|| \leq L \tag{20}$$

Following the lines of the previous section, the following result can be stated.

Theorem 4.1: It is given the discrete-time system (18) preceded by the saturating device (2) under Assumptions 2.1, 4.1. For perturbing terms $d(k)$ bounded by a constant satisfying:

$$\rho < \min\left\{ \frac{M}{2n||\mathbf{K}|| \cdot ||\mathbf{B}||}, \frac{M}{n\rho||\mathbf{A}||} \right\}. \tag{21}$$

the control law $u(k) = u^{eq}(k) + v(k)$, with $u^{eq}(k)$ given by (12) and $v(k)$ of the form

$$v(k) = \begin{cases} 0 & \text{if } k = 0 \\ -\bar{\mathbf{D}}\mathbf{F}(\lambda, k+1)(\mathbf{x}(k) - \prod_{j=1}^{k} \mathbf{F}(\lambda, j)\mathbf{x}(0)) & \\ & \text{if } k \geq 1 \end{cases} \tag{22}$$

with $\bar{\mathbf{D}} = [0\ 0\ \cdots\ 0\ \frac{1}{b}]$, guarantees that any initial condition belonging to the set:

$$\mathcal{J}_\rho = \left\{ \mathbf{x}(0) \in \left(\prod_{j=0}^{k-1} \mathbf{Q}(\lambda, k-j) \right) \mathcal{I}_\rho \right\} \tag{23}$$

is driven to the set $\mathcal{I}_\rho$ in k steps without violating the saturation constraints. Therefore, ultimate boundedness of state trajectories is guaranteed for any initial condition belonging to the set $\mathcal{J}_\rho$ by coupling (6), (12) and (22).

Proof: The proof consists in showing that a procedure exists for selecting the parameter λ such that the control laws (12) and (22) drive the state into the invariant set $\mathcal{I}_\rho$ in a finite number of steps k. Noticing that $(\mathbf{I}-\mathbf{B}\bar{\mathbf{D}})\mathbf{F}(\lambda,1) = (\mathbf{I}-\mathbf{B}\bar{\mathbf{D}})\mathbf{A} = \mathbf{N}$, since $\mathbf{N}^n = 0$, the following expressions can be easily derived:

$$
\begin{aligned}
&\mathbf{x}(1) = \mathbf{F}(\lambda,1)\mathbf{x}(0) + \mathbf{B}d(0)\\
&\mathbf{x}(2) = \mathbf{F}(\lambda,2)\left(\mathbf{F}(\lambda,1)\mathbf{x}(0)) + \mathbf{B}d(0)\right) + \mathbf{B}v(1) + \mathbf{B}d(1)\\
&\ = \mathbf{F}(\lambda,2)\mathbf{F}(\lambda,1)\mathbf{x}(0) + \mathbf{N}\mathbf{B}d(0) + \mathbf{B}d(1)\\
&\mathbf{x}(3) = \mathbf{F}(\lambda,3)\mathbf{F}(\lambda,2)\mathbf{F}(\lambda,1)\mathbf{x}(0) + \mathbf{F}(\lambda,3)(\mathbf{B}d(1)\\
&\ + \mathbf{N}\mathbf{B}d(0)) + \mathbf{B}d(2) + \mathbf{B}v(2)\\
&\ = \mathbf{F}(\lambda,3)\mathbf{F}(\lambda,2)\mathbf{F}(\lambda)\mathbf{x}(0) + \mathbf{B}d(2) + \mathbf{N}\mathbf{B}d(1) + \mathbf{N}^2\mathbf{B}d(0)\\
&\quad \dots\\
&\mathbf{x}(k) = \prod_{j=1}^{k}\mathbf{F}(\lambda,j)\mathbf{x}(0) + \sum_{i=0}^{n-1}\mathbf{N}^i\mathbf{B}d(k-i)
\end{aligned}
$$

Moreover, the control input (22) fulfills the following inequality

$$
\begin{aligned}
|v(k)| &= \left|\bar{\mathbf{D}}\mathbf{F}(\lambda,k+1)\sum_{i=0}^{n-1}\mathbf{N}^i\mathbf{B}d(k-i-1)\right|\\
&\le n\rho||\bar{\mathbf{D}}||\cdot||\mathbf{B}||\cdot||\mathbf{A}|| = n\rho||\mathbf{A}||.
\end{aligned}
$$

and accounting for the saturation constraint requires that $\rho < \frac{M}{n||A||}$ therefore the condition (21) is found coupling the previous inequality and Assumption 4.1. Following the same approach of the proof of Theorem 3.1, it is now enough to impose $\forall r \le k$

$$
\left\|(1-\lambda^r)\prod_{j=0}^{k-r}\mathbf{Q}(\lambda,k-j)\right\| < \frac{M-n\rho||\mathbf{A}||}{||(\mathbf{C}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}||(M^*-n\rho||\mathbf{B}||)} \tag{24}
$$

Denoting by $\lambda_k = \inf\{\lambda \in (0,1) :$ condition (24) is fulfilled$\}$, one can conclude that choosing $1 > \lambda > \lambda_k$ in (12) any initial condition belonging to the set $\mathcal{J}_\rho$ can be driven to the set $\mathcal{I}_\rho$ in k steps using a control law satisfying $|u_{eq}(k)| \le M - n\rho||\mathbf{A}|| < M$.

In fact, if $\bar{\mathbf{x}} \in \mathcal{I}_\rho$, then for the initial condition $\mathbf{x}(0) := \left(\prod_{j=0}^{k-1}\mathbf{Q}(\lambda,k-j)\right)\bar{\mathbf{x}}$ it holds

$$
\begin{aligned}
\mathbf{x}(k) &= \prod_{j=1}^{k}\mathbf{F}(\lambda,j)\mathbf{x}(0) + \sum_{i=0}^{n-1}\mathbf{N}^i\mathbf{B}d(k-i)\\
&= \prod_{j=1}^{k}\mathbf{F}(\lambda,j)\left(\prod_{j=0}^{k-1}\mathbf{Q}(\lambda,k-j)\right)\bar{\mathbf{x}} + \sum_{i=0}^{n-1}\mathbf{N}^i\mathbf{B}d(k-i)\\
&= \bar{\mathbf{x}} + \sum_{i=0}^{n-1}\mathbf{N}^i\mathbf{B}d(k-i)
\end{aligned}
$$

hence $||\mathbf{x}(k)|| \le ||\bar{\mathbf{x}}|| + n\rho||B|| < M^* - n\rho||\mathbf{B}|| + n\rho||\mathbf{B}|| = M^*$. ■

V. Experimental Results

Previous theoretical results have been experimentally validated on the twin rotor shown in Fig.1. The plant has been built in our laboratory for educational purposes, and is constituted of two metal arms: the first is locked to the ground, while the second is linked to the first one, and can move with two degrees of freedom. The movements are generated by two brushless D.C. motors (produced by AIRPAX©), placed on the two ends of the free arm. Moreover, two potentiometers are in charge of measuring the angular displacements of the free arm. The controller code is written in MATLAB/SIMULINK©, running on a Personal Computer (PC). The PC is equipped with a Plug-and-Play general purpose board, namely **NI-PCI6024e**, produced by NATIONAL INSTRUMENTS©, which is connected to MATLAB/SIMULINK© by means of Real Time Workshop and Real Time Windows Target MATLAB© packages. The **NI-PCI6024e** allows data exchange between PC and the plant, but it is not directly connected to the potentiometers and to the motors. An interface board, made in our Lab, is in charge to filter and to adapt signals coming from (and to) the sensors, before (and after) passing them to the **NI-PCI6024e** board i.e. to the PC. The power board, made in our Lab, too, mounts a PWM modulator and drives the motors with suitable voltages, corresponding to the control actions produced by the control law implemented in SIMULINK©. The maximum range that the power board can supply is $\pm 12V$. Nevertheless, the safer voltage saturation limit of $\pm 7V$ was chosen, because of the problems we encountered during the testing phase of the overall control system. Indeed sudden changes in the control variables caused damages to the boards (microchips and capacitors, for example), and, more seldom, a risk occurred to burn the motors. Finally, note that the angular velocities, required by the state feedback control law, were obtained by filtering and differentiating the signals coming from the potentiometer. The chosen sampling time was $T_c = 0.05\ s$. The mathematical model of the plant can be derived using well known theoretical physics results. Making reference to Fig.2, and introducing the state vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} \theta & \phi & \dot{\theta} & \dot{\phi} \end{bmatrix}^T$, where θ and ϕ are the pitch and yaw angles, respectively, the twin rotor is described by the following nonlinear equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{h}(\mathbf{x})\mathbf{u}$, with:

$$
\mathbf{f}(\mathbf{x}) = \begin{bmatrix} x_3 \\ x_4 \\ \dfrac{-mgl_c\cos x_1 - J_L x_4^2\cos x_1 \sin x_1 - \alpha_\theta x_3}{J_L} \\ \dfrac{2J_L x_3 x_4 \cos x_1 \sin x_1 - \alpha_\phi x_4}{(\cos^2 x_1 J_L + J_A)} \end{bmatrix} \tag{25}
$$

$$
\mathbf{h}(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{l_1 p_1}{J_L} & \dfrac{p_2}{J_L} \\ \dfrac{p_3 cos(x_1)}{(\cos^2 x_1 J_L + J_A)} & \dfrac{l_2 p_4 cos x_1}{(\cos^2 x_1 J_L + J_A)} \end{bmatrix} \tag{26}
$$

where $\mathbf{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ is the input vector, i.e. the voltages of the two motors driving the twin-rotor system; J_L and J_A are the inertia moments of the free arm and of locked one, respectively; l_c is the centre of gravity of the free arm; l_1 and l_2 are the distances between the ends and the centre of the free arm; g is the gravity acceleration; α_ϕ and α_θ are the damper coefficients for angles ϕ and θ, respectively; p_1, p_2, p_3, p_4 are suitable coefficients, correlating the voltages and the moments supplied by the motors. The non-linear model has been linearised with respect to the equilibrium point $\begin{bmatrix}\mathbf{x}_e^T & \mathbf{u}_e^T\end{bmatrix}^T = \begin{bmatrix} \mathbf{0}_{4x1}^T & 1.24 & -0.21 \end{bmatrix}^T$, obtaining the following continuous time linear time invariant state space representation: $\frac{d}{dt}(\Delta\mathbf{x}) = \mathbf{A}_\Delta \Delta\mathbf{x} + \mathbf{B}_\Delta \Delta\mathbf{u}$, being $\Delta\mathbf{u} = \mathbf{u} - \mathbf{u}_e$, $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_e$, and with:

$$\mathbf{A}_\Delta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.46 & 0 \\ 0 & 0 & 0 & -0.46 \end{bmatrix} \quad \mathbf{B}_\Delta = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.276 & 0.046 \\ 0.046 & 0.276 \end{bmatrix}$$

Considering θ and ϕ as the system output, i.e. $\mathbf{y} = \begin{bmatrix} \phi & \theta \end{bmatrix}^T$, the corresponding input-output transfer matrix is given by:

$$\mathbf{F}(s) = \begin{bmatrix} \frac{0.276}{s^2+0.46s} & \frac{0.046}{s^2+0.46s} \\ \frac{0.046}{s^2+0.46s} & \frac{0.276}{s^2+0.46s} \end{bmatrix}$$

Since $\mathbf{F}(s)$ is a diagonally dominant matrix, the coupling terms between θ and ϕ dynamics have been neglected in the linearised plant. The twin rotor has been considered as made of two independent SISO plants, characterized by the same transfer function $F_{11}(s) = F_{22}(s) = \frac{0.276}{s^2+0.46s}$. In order to apply the proposed control law, the above transfer functions have been discretized with a sampling time $T_c = 0.05\ s$, obtaining two equal subsystems of the form (1), with: $\hat{\mathbf{A}} = \begin{bmatrix} 1 & 0.0494 \\ 0 & 0.9773 \end{bmatrix}$, $\hat{\mathbf{B}} = \begin{bmatrix} 0.0003 \\ 0.0136 \end{bmatrix}$. Successively, $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ have been transformed as in (3), (4), (5), obtaining: $\mathbf{A} = \begin{bmatrix} 0 & 0.8 \\ -1.22 & 1.98 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1.25 \end{bmatrix}$. Finally, two controllers have been built according to the approach described in Theorem 3.1, with $\mathbf{K} = \begin{bmatrix} 0.98 & -1.58 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & 1.2 \end{bmatrix}$ and $\lambda = 0.999988$. The saturation threshold of actuators is $M = 7\ V$, as explained at the beginning of this section. With reference to the theoretical development presented in Sections III, the sets of initial conditions from which the state can be steered to the set $\mathcal{I}$ is reported in Fig.3. Accordingly, the initial conditions have been chosen as $-31\ deg$ for the pitch angle θ and $24\ deg$ for the yaw angle ϕ, with null initial velocities, in order to drive the state to the set $\mathcal{I}$ in just one sampling time. Two experiments have been performed. In the first one, the control law based on (11), and given by (12), (6), has been implemented for each subsystem. In the second one, a standard equivalent control law based on (10) has been implemented for each subsystem, i.e.

$$u_{eq}(k) = -(\mathbf{CB})^{-1}\mathbf{CAx}(k) \quad (27)$$

Results of the first experiment have been reported in Fig.4, showing the experimental pitch angle (the yaw angle has not been reported due to space limitations), and Fig.5 displaying the control input u_1. The corresponding variables for the second experiment are reported in Figs.6-7. It can be noticed that when using control law (12), (6), the initial value of the control variables is always 0, regardless of the initial state, while using control law (27) the initial control effort depends on the initial state (the farther the initial state is from the origin, the larger will be the initial control effort). This fact can be seen comparing Fig.5 with Fig.7. However, after few time instants, control variables produced by (27) assume values comparable with signal produced by control law (12), (6). Anyway, the smaller initial control activity produced by (12), (6) is paid by the presence of some overshoots in the case of the behavior of the pitch angle, arising when controlled by the same controller (12), (6).

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Fig.1 - The twin rotor

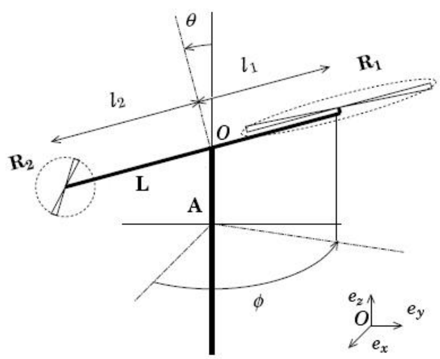

Fig.2 - The twin rotor model

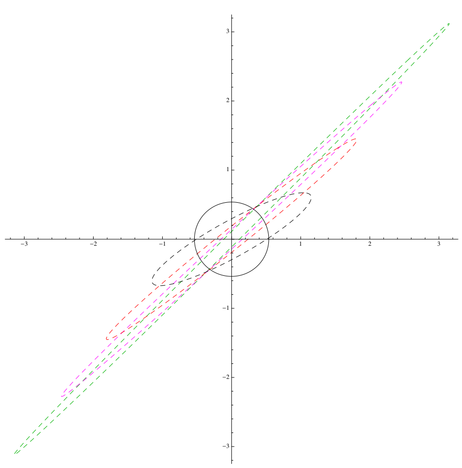

Fig.3 - Sets of initials conditions from which the state can be steered to the set $\mathcal{I}$ (the circle) in 1 (black), 2 (red), 3 (purple), 4 (green) steps

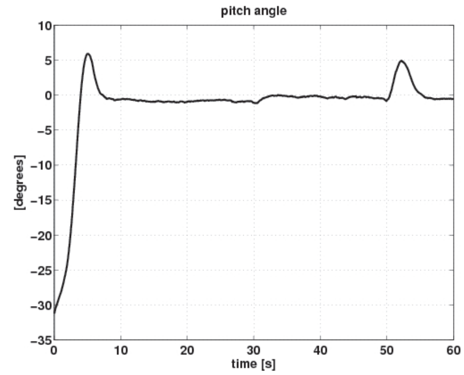

Fig.4 - Pitch angle θ (control law (12), (6))

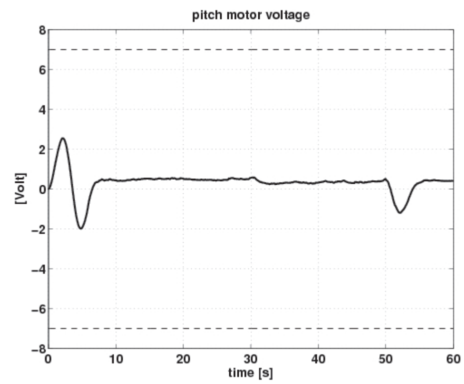

Fig.5 - Contr. input u_1 (control law (12), (6))

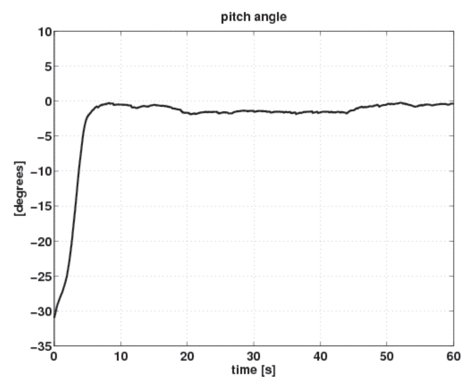

Fig.6 - Pitch angle θ (control law (27))

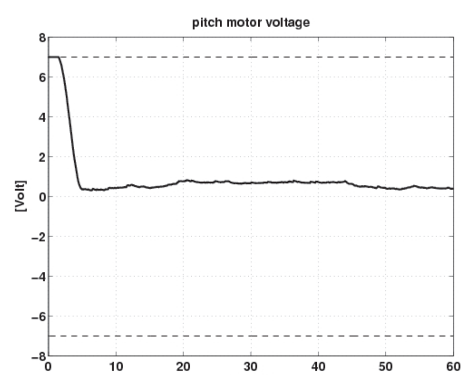

Fig.7 - Contr. input u_1 (control law (27))