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Kinematics for rolling a Lorentzian sphere

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Abstract— **We derive the equations of motion for the n-dimensional Lorentzian sphere (one-sheet hyperboloid) rolling, without slipping and twisting, over the affine tangent space at a point. Both manifolds are endowed with semi-Riemannian metrics, induced by the Lorentzian metric on the embedding manifold which is the generalized Minkowski space. The kinematic equations turn out to be a nonlinear control system evolving on a connected subgroup of the Poincaré group. The controls correspond to the choice of the curves along which the Lorentzian sphere rolls. Controllability of this rolling system will be proved by showing that the corresponding distribution is bracket-generating.**

I. INTRODUCTION

Motions of systems with nonholonomic constraints can be found in the work of great mathematicians as Newton, Euler, Bernoulli and Lagrange. More recently, nonholonomic systems have attracted much attention in control literature due to their numerous applications in physics and engineering problems. For instance, in a robotic system if the controllable degrees of freedom are less than the total degrees of freedom, the system is nonholonomic. Nowadays, the interest in this area is increasing and one can find references to potential applications of nonholonomic systems, for instance, in neurobiology and economics. For a recent survey on nonholonomic systems we refer to [17].

Nonholonomic constraints are usually analyzed from the point of view of sub-Riemannian geometry. This is the case when the constraints define a non-integrable subbundle of the tangent bundle of a Riemannian manifold (see, for instance, [1], [12], [11] for work interconnecting sub-Riemannian geometry and control theory). But, if the manifold is only equipped with a semi-Riemannian metric (nondegenerate but not positive definite), we will be in the presence of problems in sub-semi-Riemannian geometry ([2], [4], [8], [9]).

A pair of n-dimensional Riemannian (or semi-Riemannian manifolds) rolling on each other without slipping and twisting also form a nonholonomic system posing many theoretical challenges and interesting control problems. To better understand the geometry of this motion, one needs tools from sub-semi-Riemannian geometry.

Our paper is devoted to studying a particular problem of sub-semi-Riemannian type and arises from the kinematic problem concerning rolling a sphere over its affine tangent space at a point, when they are both equipped with a Lorentzian metric (a metric with index 1). Here the semi-Riemannian manifold is the configuration space of the mechanical system, and the subbundle is a sub-semi-Riemannian distribution defined by the constraints on rolling: no slipping, no twisting. The 2-dimensional Lorentzian sphere may be represented by the surface known as the one-sheet hyperboloid. Physical rolling of this manifold on the affine tangent space at any point is impossible. Nevertheless, contrary to our intuition, the notion of rolling makes sense. Rollings are isometries in the embedding space, in particular preserving length of curves, satisfying several constraints. Knowing how to perform such "virtual" motions is important and, in particular, allows to solve hard problems on certain manifolds by reducing them to much simpler ones. These rolling notions have been applied successfully to generate interpolating curves on manifolds ([5]) and we intend to further research in this area for the semi-Riemannian case.

In this article, we start with the definition of rolling, an adaptation of the classical definition for rolling Euclidean manifolds, as given, for instance, in [15]. The kinematic equations for rolling the Lorentzian sphere are derived from the non-slip and non-twist conditions and solved completely when rolling along geodesics. In the last Section, these equations are rewritten as a left-invariant control system evolving on a connected subgroup of the Poincaré group, and a result on controllability of the kinematic equations is proven.

II. BASIC FACTS ON SEMI-RIEMANNIAN MANIFOLDS

A semi-Riemannian manifold is a smooth manifold $\overline{M}$ furnished with a metric tensor $\overline{g}$ (a symmetric nondegenerate $(0, 2)$ tensor field of constant index). The common value ν of the index $\overline{g}_x$ at each point x on a semi-Riemannian manifold $\overline{M}$ is called the index of $\overline{M}$ and $0 \leq \nu \leq \dim(\overline{M})$. If $\nu = 0$, each $\overline{g}_x$ is then a (positive definite) inner product on $T_x\overline{M}$ and $\overline{M}$ is a Riemannian manifold. If $\nu = 1$ and $\dim(\overline{M}) \geq 2$, $\overline{M}$ is a Lorentz manifold.

If $(\overline{M}, \overline{g})$ is a semi-Riemannian manifold and $v \in T_x\overline{M}$, then v is *spacelike* if $\overline{g}(v, v) > 0$ or $v = 0$; v is *timelike* if $\overline{g}(v, v) < 0$; v is *lightlike* if $\overline{g}(v, v) = 0$ and $v \neq 0$.

Let M be a submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ and $\imath : M \hookrightarrow \overline{M}$ the inclusion map. Then M is a semi-Riemannian submanifold of $\overline{M}$ if the pullback metric $g = \imath^*(\overline{g})$ is a metric tensor on M. If M is equipped with

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the induced metric g, then $\imath$ is an isometric embedding. (In subsequent sections, we use $\langle\cdot,\cdot\rangle$ as an alternative notation for g).

Let M be a semi-Riemannian submanifold of $\overline{M}$ (write $M\subset\overline{M}$), and $p\in M$. Each tangent space T_xM is, by definition, a nondegenerate subspace of $T_x\overline{M}$. Consequently, $T_x\overline{M}$ decomposes as a direct sum $T_x\overline{M}=T_xM\oplus(T_xM)^\perp$ and $(T_xM)^\perp$ is also nondegenerate. Vectors in $(T_xM)^\perp$ are said to be normal to M, while those in T_xM are, of course, tangent to M. Similarly, a vector field Z on $\overline{M}$ is normal (respectively tangent) to M provided each value Z_x, for $x\in M$ belongs to $(T_xM)^\perp$ (respectively T_xM). For more details on semi-Riemannian geometry we refer to [13].

III. DEFINITION OF ROLLING

For the sake of simplicity, whenever we use the word "rolling" we mean "rolling without slipping and twisting".

The classical definition of rolling, as given in Sharpe [15] for rolling manifolds embedded in the same Euclidean space, has been generalized to the situation when the embedding is on a Riemannian manifold ([10]) or on a semi-Riemannian manifold ([16]). Intrinsic rolling of two manifolds, a situation where no embedding is required, has also been defined in ([3]). Here we adopt the notion of rolling as it appears in [16]. The most classical of all rolling motions is that of the 2-dimensional (Euclidean) sphere S^2 rolling on the tangent plane at a point. This motion is easily visualized because it is physically possible. However, the notion of rolling doesn't exclude the situation when the "moving" manifold crashes into the "still" manifold. Such is the case for the one-sheet hyperboloid rolling on the tangent plane at a point. This may sound awkward, but the study of rolling maps is important even when rolling is physically impossible. This derives from the fact that rolling maps are isometries.

Assume that M and $\widehat{M}$ are isometrically embedded submanifolds of an N-dimensional manifold $\overline{M}$ endowed with a semi-Riemannian metric $\overline{g}$ of signature (p,q), with $p+q=N$. A rolling motion is described by the action of the group $\overline{G}$ of orientation preserving isometries of $\overline{M}$. The action of $\overline{G}$ on $\overline{M}$ is denoted by $\circ$ and orthogonality is taken with respect to the semi-Riemannian metric. To understand the definition bellow, we introduced some notations.

If $t\mapsto g(t)$ is a curve in $\overline{G}$, x a point in $\in\overline{M}$, and η a vector tangent to $\overline{M}$, so that there exists a smooth curve $t\mapsto y\in(-\varepsilon,\varepsilon)\to\overline{M}$ such that $\dot{y}(0)=\eta$, then

$$\dot{g}(t)\circ x:=\frac{\mathrm{d}}{\mathrm{d}\,\sigma}(g(\sigma)\circ x)\Big|_{\sigma=t},$$

$$\big(\dot{g}(t)\circ g(t)^{-1}\big)\circ x:=\frac{\mathrm{d}}{\mathrm{d}\,\sigma}((g(\sigma)\circ g(t)^{-1})\circ x)\Big|_{\sigma=t},$$

$$\big(\dot{g}(t)\circ g(t)^{-1}\big)\circ\eta:=\frac{\mathrm{d}}{\mathrm{d}\,\sigma}((\dot{g}(t)\circ g(t)^{-1})\circ y(\sigma))\Big|_{\sigma=0}.$$

***Definition** 3.1:* A smooth map

$$g\colon[0,\tau]\to\overline{G}$$

satisfying the following properties $1)-3)$, for each $t\in[0,\tau]$, is called a (smooth) *rolling of M on $\widehat{M}$ without slipping or twisting*.

1) There is a smooth curve on M, $x\colon[0,\tau]\to M$, such that for all $t\in[0,\tau]$
   – $g(t)\circ x(t)\in\widehat{M}$,
   – $T_{g(t)\circ x(t)}(g(t)\circ M)=T_{g(t)\circ x(t)}\widehat{M}$.
   x is called the rolling curve while $\widehat{x}$ defined by $\widehat{x}(t):=g(t)\circ x(t)$ is called the *development* of x on $\widehat{M}$.
2) – no-slip condition:
$$\big(\dot{g}(t)\circ g(t)^{-1}\big)\circ\widehat{x}(t)=0,$$
3) – no-twist conditions:

   (tangential part):
$$(\dot{g}(t)\circ g(t)^{-1})\circ T_{\widehat{x}(t)}\widehat{M}\subset T_{\widehat{x}(t)}\widehat{M}^\perp,$$
   (normal part):
$$(\dot{g}(t)\circ g(t)^{-1})\circ T_{\widehat{x}(t)}\widehat{M}^\perp\subset T_{\widehat{x}(t)}\widehat{M}.$$

In the classical case $\overline{M}$ is the Euclidean space $\mathbb{R}^{n+1}$ and $\overline{G}=\mathrm{SE}(n+1)$.

IV. ROLLING THE LORENTZIAN SPHERE

In this section we consider the n-dimensional Lorentzian sphere S_1^n rolling over the affine tangent space at a point $x_0\in S_1^n$, both embedded in the Lorentzian manifold $\mathbb{R}_1^{n+1}$, with Lorentzian scalar product defined as $\langle v,w\rangle_J=v^\top Jw$, where $J=\mathrm{diag}(I_n,-I_1)$. The hypersurface S_1^n is also known as the hyperboloid of one sheet and defined as:

$$S_1^n=\big\{x\in\mathbb{R}_1^{n+1}\colon\langle x,x\rangle_J=1\big\}.$$

Rolling motions of the hyperbolic sphere (the hyperboloid of two sheets) defined as $H^n=\big\{x\in\mathbb{R}_1^{n+1}\colon\langle x,x\rangle_J=-1\big\}$, have been studied in [7] and [16]. However, although $\langle\cdot,\cdot\rangle_J$ is indefinite, it turns out that its restriction to tangent spaces to H^n is positive definite. Thus, contrary to the present situation, $\langle\cdot,\cdot\rangle_J$ defines a Riemannian metric on the hyperbolic sphere.

Before deriving the equations of motion for the rolling, one needs to introduce some notation and adapt the definition of rolling to the present situation.

The group of isometries of the embedding space $\mathbb{R}_1^{n+1}$ is the Poincaré group (see, for instance, [13]), and the group of orientation preserving isometries $\overline{G}$ is the semidirect product $\overline{G}=\mathrm{SO}^+(n,1)\ltimes\mathbb{R}_1^{n+1}$, of the restricted Lorentz group $\mathrm{SO}^+(n,1)$, consisting of the connected component containing the identity of the special pseudo-orthogonal group $\mathrm{SO}(n,1)$, and the Abelian group of translations $\mathbb{R}_1^{n+1}$. The special pseudo-orthogonal group has a matrix representation by real $(n+1)\times(n+1)$-matrices:

$$\mathrm{SO}(n,1)=\{X:\ X^\top JX=J \text{ and } \det X=1\},$$

with Lie algebra

$$\mathrm{so}(n,1)=\{\Omega:\ \Omega^\top J=-J\Omega\}.$$

Some facts can be easily derived using the definitions of this Lie group and corresponding Lie algebra, namely

$$e^{\Omega t}\in \mathrm{SO}^+(n,1),\quad \forall\Omega\in\mathrm{so}(n,1),\quad \forall t\in\mathbb{R};$$

$$X(t)\in\mathrm{SO}^+(n,1)\Rightarrow \dot X(t)=\Omega(t)X(t),$$

for some $\Omega(t)\in\mathrm{so}(n,1)$.

We next give a characterization of the tangent and normal spaces to the Lorentzian sphere at a point x_0, that will be very useful later on.

***Proposition** 4.1:*

$$T_{x_0}S_1^n=\left\{v\in\mathbb{R}_1^{n+1}\colon v=\Omega x_0,\ \Omega\in\mathrm{so}(n,1)\right\};$$

$$\left(T_{x_0}S_1^n\right)^\perp=\mathrm{span}\{x_0\}.$$

Proof: Let $V=\left\{v\in\mathbb{R}_1^{n+1}\colon v=\Omega x_0,\ \Omega\in\mathrm{so}(n,1)\right\}$. It is clear that $V\subset T_{x_0}S_1^n$, since the curve $\gamma(t)=e^{\Omega t}x_0\in S_1^n$ satisfies $\gamma(0)=x_0$ and $\dot\gamma(0)=\Omega x_0$. We now show that all tangent vectors at x_0 are of the form Ωx_0. For that, let $\gamma(t)$ be an arbitrary smooth curve in S_1^n satisfying $\gamma(0)=x_0$. Since all curves in S_1^n result from the action of $\mathrm{SO}(n,1)$, we may write $\gamma(t)=X(t)x_0$, where $X(t)\in\mathrm{SO}(n,1)$, $X(0)=I$. So, $\dot\gamma(t)=\dot X(t)x_0=\Omega(t)X(t)x_0=\Omega(t)\gamma(t)$, for some $\Omega(t)\in\mathrm{so}(n,1)$. In particular, at $t=0$, $\dot\gamma(0)=\Omega(0)x_0$. So, $T_{x_0}S_1^n\subset V$. Consequently, $T_{x_0}S_1^n=V$.

Now, for the normal space which is 1-dimensional, it is enough to prove that $x_0\in\left(T_{x_0}S_1^n\right)^\perp$. Since $\langle x_0,\Omega x_0\rangle_J=x_0^\top J\Omega x_0=-x_0^\top\Omega^\top Jx_0$ and, on the other hand $\langle x_0,\Omega x_0\rangle_J=\langle\Omega x_0,x_0\rangle_J=x_0^\top\Omega^\top Jx_0$, the result follows.

■

We will see later that integration of the kinematic equations is not trivial except when the rolling curves are geodesics. But before getting to that point, we revise some facts about geodesics on the Lorentzian sphere. Since the restrictions of the Lorentzian metric $\langle\cdot,\cdot\rangle_J$ to S_1^n and to its tangent spaces are indefinite and nondegenerate, the geodesics are of different causal types.

Geodesics of Minkowski space $\mathbb{R}_1^{n+1}$ are stationary points for the functional $\int_0^1\langle\dot x(t),\dot x(t)\rangle_J\,dt$. They are solutions of the corresponding Euler-Lagrange equation $\ddot x=0$ and so are straight lines. Geodesics on S_1^n are also stationary points for the same functional, with the additional constraint that $\langle x(t),x(t)\rangle_J=1$. The equation for geodesics on the Lorentzian sphere is easily derived to obtain

$$\ddot x-\langle\ddot x,x\rangle_J x=0. \tag{1}$$

Thus, geodesics on S_1^n are curves along which the extrinsic acceleration $\ddot x(t)$ belongs to $\left(T_{x_0}S_1^n\right)^\perp$. Since $\langle x,x\rangle_J=1$ implies $\langle\dot x,x\rangle_J=0$ and therefore $\langle\ddot x,x\rangle_J=-\langle\dot x,\dot x\rangle_J$, the geodesic equation (1) can be written alternatively as

$$\ddot x+\langle\dot x,\dot x\rangle_J x=0. \tag{2}$$

***Proposition** 4.2:* Let $x_0\in S_1^n$ and $v\in T_{x_0}S_1^n$. Then

(a) If the vector v is timelike, i.e. $\langle v,v\rangle_J=-1$,

$$t\mapsto\gamma(t)=x_0\cosh(t)+v\sinh(t) \tag{3}$$

is the unique timelike geodesic in S_1^n satisfying $\gamma(0)=x_0$, $\dot\gamma(0)=v$.

(b) If the vector v is spacelike, i.e. $\langle v,v\rangle_J=1$,

$$t\mapsto\gamma(t)=x_0\cos(t)+v\sin(t) \tag{4}$$

is the unique spacelike geodesic in S_1^n satisfying $\gamma(0)=x_0$, $\dot\gamma(0)=v$.

(c) If the vector v is lightlike, i.e. $\langle v,v\rangle_J=0$,

$$t\mapsto\gamma(t)=x_0+vt \tag{5}$$

is the unique lightlike geodesic in S_1^n satisfying $\gamma(0)=x_0$, $\dot\gamma(0)=v$.

Proof: The theory of semi-Riemannian geometry guarantees that geodesic starting at x_0 with initial velocity v is locally unique. It can easily be checked that the curves given by (3)–(5) satisfy the geodesic equation above. Finally, since $\langle\dot x,\dot x\rangle_J$ is an invariant for geodesics, the causal character is that of the initial velocity vector v.

■

The following result answers the question: can any two points x_0,x_1 in the Lorentzian sphere be joined by a geodesic? We will see that the answer is yes only if $\langle x_0,x_1\rangle_J>-1$ or if $\langle x_0,x_1\rangle_J=-1$ and $x_0=-x_1$.

***Proposition** 4.3:* Let x_0,x_1 be two distinct points in S_1^n. Then,

(a) If $\langle x_0,x_1\rangle_J>1$, say $\langle x_0,x_1\rangle_J=\cosh(\theta)$ for some $\theta\neq0$, the timelike geodesic given by

$$t\mapsto\gamma(t)=x_0\cosh(t)+\left(\frac{x_1-x_0\cosh(\theta)}{\sinh(\theta)}\right)\sinh(t)$$

satisfies $\gamma(0)=x_0$, $\gamma(\theta)=x_1$.

(b) If $\langle x_0,x_1\rangle_J=1$, the lightlike geodesic given by

$$t\mapsto\gamma(t)=x_0+t(x_1-x_0)$$

satisfies $\gamma(0)=x_0$, $\gamma(\theta)=x_1$.

(c) If $\langle x_0,x_1\rangle_J\in]-1,1[$, say $\langle x_0,x_1\rangle_J=\cos(\theta)$ for some $\theta\neq k\pi$, ($k\in\mathbb{Z}$), the spacelike geodesic given by

$$t\mapsto\gamma(t)=x_0\cos(t)+\left(\frac{x_1-x_0\cos(\theta)}{\sin(\theta)}\right)\sin(t)$$

satisfies $\gamma(0)=x_0$, $\gamma(\theta)=x_1$.

(d) If $\langle x_0,x_1\rangle_J=-1$ and $x_1=-x_0$, any spacelike geodesic given by

$$t\mapsto\gamma(t)=x_0\cos(t)+v\sin(t)$$

satisfies $\gamma(0)=x_0$, $\gamma(\pi)=x_1$.
(e) If $\langle x_0,x_1\rangle_J \le -1$ and $x_1 \neq -x_0$, then x_0 can not be joined to x_1 by a geodesic. However, they can be joined by a broken geodesic.

Proof: The proof of the first four statements follows easily from some computations using the results contained in Proposition 4.2. In particular, if we assume that there exists a geodesic joining x_0 (at $t=0$) to x_1 (at $t=\theta$) (there are three possibilities only), the following observations are useful. If this geodesic is timelike, it follows from (3) that $\langle x_0,x_1\rangle_J = \langle x_0,\gamma(\theta)\rangle_J = \cosh(\theta) > 1$. Similarly, if the geodesic is spacelike, it follows from (4) that $\langle x_0,x_1\rangle_J = \langle x_0,\gamma(\theta)\rangle_J = \cos(\theta) \in [-1,1]$. Hence, if $\cos(\theta)=-1$ then $x_1=-x_0$ and if $\cos(\theta)=1$ then $x_1=x_0$ (impossible). Finally, if the geodesic is lightlike, (5) implies that $\langle x_0,x_1\rangle_J = \langle x_0,\gamma(\theta)\rangle_J = 1$.

It is clear from here that, under the conditions in (e), the points x_0 and x_1 can not be joined by a geodesic. Moreover, when $\langle x_0,x_1\rangle_J \le -1$ and $x_1 \neq -x_0$, $\langle x_0,-x_1\rangle_J \ge 1$ and, according to (b) and (c), x_0 and $-x_1$ can be joined by a lightlike geodesic. But, by (d), $-x_1$ and x_1 can be joined by a spacelike geodesic. So, x_0 and x_1 may be joined by a curve with is the concatenation of a lightlike geodesic (joining x_0 to $-x_1$) and a spacelike geodesic (joining $-x_1$ to x_1). ■

We now turn our attention towards rolling S_1^n over the affine tangent space at $x_0 \in S_1^n$, defined as

$$T_{x_0}^{\mathrm{aff}} S_1^n = \left\{x \in \mathbb{R}^{n+1} \colon x = x_0 + \Omega x_0,\ \ \Omega \in \mathrm{so}(n,1)\right\}.$$

***Remark** 4.1:* Contrary to the rolling of the Euclidean sphere or the hyperbolic sphere, where there is only one point of contact between the manifold and the affine tangent space at any point, here the two rolling manifolds always intersect along light-like geodesics. More precisely,

$$T_{x_0}^{\mathrm{aff}} S_1^n \cap S_1^n = \{x_0 + \Omega x_0 :\ \langle \Omega x_0, \Omega x_0\rangle_J = 0\}.$$

To see this, assume that $x_0 \in S_1^n$ and $x = x_0 + \Omega x_0 \in T_{x_0}^{\mathrm{aff}} S_1^n \cap S_1^n$. So, we must have $\langle x,x\rangle_J = 1$. But

$$\langle x,x\rangle_J = \langle x_0,x_0\rangle_J + 2\langle x_0,\Omega x_0\rangle_J + \langle \Omega x_0,\Omega x_0\rangle_J.$$

According to Proposition 4.1, $x_0 \in S_1^n \cap (T_{x_0}S_1^n)^\perp$ and $\Omega x_0 \in T_{x_0}S_1^n$, so the first term above is equal to 1 and the second term is equal to 0. So, we must have $\langle \Omega x_0,\Omega x_0\rangle_J = 0$, and clearly Ωx_0 runs over the set of lightlike geodesics (5) when Ω runs over $\mathrm{so}(n,1)$.

This can be visualized for the 2-dimensional one-sheet hyperboloid (see Fig.1). As already mentioned earlier, physical rolling is impossible, but we will proceed for the kinematic equations of "virtual" rolling.

A. KINEMATIC EQUATIONS FOR ROLLING S_1^n

We recall that a rolling is a curve in $\overline{G} = \mathrm{SO}^+(n,1) \ltimes \mathbb{R}_1^{n+1}$. For convenience we represent elements in this Lie group as pairs (R,s), with $R \in \mathrm{SO}^+(n,1)$, $s \in \mathbb{R}_1^{n+1}$. $(I,0)$ is the identity and the group operations are defined by

$$(R_1,s_1)(R_2,s_2) = (R_1R_2, R_1s_2 + s_1),$$

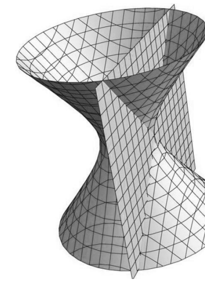

Fig. 1. S_1^2 and its tangent plane at a point.

$$(R,s)^{-1} = (R^{-1}, -R^{-1}s).$$

The following theorem is the analogue for the Lorentzian sphere of results for the Euclidean sphere (as, for instance, in [5]) and for the hyperbolic sphere (as, for instance, in [16]).

***Theorem** 4.1:* Let x_0 be an arbitrary point in S_1^n and $t \mapsto u(t) \in \mathbb{R}_1^{n+1}$ a piecewise smooth function satisfying $\langle u(t),x_0\rangle_J = 0$. If $R \in \mathrm{SO}_+(n,1)$, $s \in \mathbb{R}_1^{n+1}$ is the solution of

$$\begin{aligned} \dot R(t) &= R(t)\left(u(t)x_0^\top - x_0u^\top(t)\right) J, \\ \dot s(t) &= u(t), \end{aligned} \tag{6}$$

satisfying the initial condition $R(0)=I$, $s(0)=0$, then $t \mapsto g(t) = (R^{-1}(t), s(t)) \in \mathrm{SO}_0(n,1) \ltimes \mathbb{R}_1^{n+1}$ is a rolling map (in the sense of definition 3.1) of S_1^n over its affine tangent space at x_0, with rolling curve $t \mapsto x(t) = R(t)x_0$. Consequently, (6) are the Kinematic equations for the rolling motion.

Proof: First of all, we show that the statement makes sense.
- The rolling curve $t \mapsto x(t) = R(t)x_0 \in S_1^n$ since, for any $R \in \mathrm{SO}_0(n,1)$,

$$\langle Rx_0,Rx_0\rangle_J = x_0^\top R^\top J R x_0 = x_0^\top J x_0 = \langle x_0,x_0\rangle_J = 1.$$

- All curves in S_1^n starting at x_0 are of that form, because $\mathrm{SO}_+(n,1)$ acts transitively on S_1^n.
- The first equation in (6) makes sense, due to the fact that the matrix $u(t)x_0^\top - x_0u^\top(t)$ is skew-symmetric, and so, when multiplied by J belongs to $\mathrm{so}(n,1)$.
- Finally, we explain why we need the restriction $\langle u(t),x_0\rangle_J = 0$. According to the definition 3.1, the development curve $t \mapsto \widehat{x}(t)$ is defined by

$$\widehat{x}(t) = g(t)\circ x(t) = R^{-1}(t)x(t) + s(t) = x_0 + s(t) \in T_{x_0}^{\mathrm{aff}} S_1^n.$$

Consequently, $s(t)$, $\dot s(t)$ and $u(t)$ all belong to $T_{x_0}S_1^n$. So, we must have $\langle u(t),x_0\rangle_J = 0$.

We are now in condition to show that the no-slip and no-twist conditions are satisfied.

The no-slip condition is verified as follows:

$$\begin{aligned} R^{-1}(t)\dot R(t)(\widehat{x}(t) - s(t)) - \dot s(t) &= \left(u(t)x_0^\top - x_0u^\top(t)\right)Jx_0 \\ -u(t) &= u(t)x_0^\top Jx_0 - x_0u^\top(t)Jx_0 - u(t) = 0. \end{aligned}$$

The no-twist conditions are:

$$R^{-1}(t)\dot{R}(t)(kx_0) \in T_{x_0}S_1^n, \forall k \in \mathbb{R};$$

$$R^{-1}(t)\dot{R}(t)(\Omega x_0) \in (T_{x_0}S_1^n)^\perp, \forall \Omega \in \mathrm{so}(n,1).$$

The first inclusion is straightforward. For the second we need to observe that $\langle x_0, \Omega x_0\rangle_J = \langle x_0, J\Omega x_0\rangle = 0$, since $J\Omega$ is skew-symmetric. So,

$$\begin{aligned} R^{-1}(t)\dot{R}(t)(\Omega x_0) &= \left(-u(t)x_0^\top + x_0 u^\top(t)\right) J\Omega x_0 \\ &= -u(t)x_0^\top J\Omega x_0 + x_0 u^\top(t) J\Omega x_0 \\ &= -u(t)\ \langle x_0, \Omega x_0\rangle_J + x_0\ \langle u(t), \Omega x_0\rangle_J \\ &= x_0 \langle u(t), \Omega x_0\rangle_J, \end{aligned}.$$

Since the expression $x_0 \langle u(t), \Omega x_0\rangle_J$ is the product of a scalar function by x_0, the proof is complete. ■

In order to characterize the solutions of the kinematic equations, let us concentrate on the coefficient matrix of the first equation in (6).

***Proposition** 4.4:* Let $A(t) := \left(u(t)x_0^\top - x_0 u^\top(t)\right)J$, and $w(t) := \langle u(t), u(t)\rangle_J$. Then, the following properties hold for any $j \in \mathbb{N}$.

$$\begin{aligned} A^{4j-3}(t) &= w(t)^{j-1} A(t), \\ A^{4j-2}(t) &= w(t)^{j-1} A^2(t), \\ A^{4j-1}(t) &= -w(t)^{j-1} A(t), \\ A^{4j}(t) &= -w(t)^{j-1} A^2(t). \end{aligned}$$

Proof: The proof is straightforward if taken in consideration that $\langle x_0, x_0\rangle_J = 1$ and $\langle u(t), x_0\rangle_J = 0$. ■

***Corollary** 4.1:* If $u(t) = u$ is a constant vector satisfying $\langle u(t), x_0\rangle_J = 0$, the solution of the kinematic equations (6), with the initial conditions $R(0) = I$, $s(0) = 0$, is

$$R(t) = \exp(At), \quad s(t) = ut.$$

Moreover, the rolling curve $x(t) = \exp(At)x_0$ and its development $\widehat{x}(t) = x_0 + s(t)$ are geodesics on S_1^n and $T_{x_0}^{\mathrm{aff}}$ respectively, having the same causality as the vector u.

Proof: The first part is obvious. For the second part, without loss of generality we my normalize the vector u and consider the three possible situations:

- $\langle u, u\rangle_J = 1$. Using Proposition 4.4, we can write

$$\exp(At) = I + \sin(t)A + (1 + \cos(t))A^2.$$

But, in this case, $Ax_0 = u$, $A^2x_0 = -x_0$. Therefore, the rolling curve is given by

$$x(t) = \exp(At)x_0 = x_0\cos(t) + u\sin(t)$$

while its development is

$$\widehat{x}(t) = x_0 + s(t) = x_0 + ut.$$

These geodesics satisfy $\dot{x}(0) = \dot{\widehat{x}}(0) = u$, so they are spacelike.

- $\langle u, u\rangle_J = -1$. Now,

$$\exp(At) = I + \sinh(t)A - (1 + \cosh(t))A^2.$$

In this case, $Ax_0 = u$, $A^2x_0 = x_0$. Therefore, the rolling curve is given by

$$x(t) = \exp(At)x_0 = x_0\cosh(t) + u\sinh(t)$$

and its development is

$$\widehat{x}(t) = x_0 + s(t) = x_0 + ut.$$

These geodesics satisfy $\dot{x}(0) = \dot{\widehat{x}}(0) = u$, so they are timelike.

- $\langle u, u\rangle_J = 0$. Then $A^{2j-1} = A^{2j} = 0$ for $j \geqslant 2$, so

$$\exp(At) = I + tA + t^2/2A^2.$$

In this case $Ax_0 = u$, $A^2x_0 = 0$, so

$$x(t) = \exp(At)x_0 = x_0 + ut, \quad \widehat{x}(t) = x_0 + s(t) = x_0 + ut.$$

In this case the geodesics coincide, $\dot{x}(0) = \dot{\widehat{x}}(0) = u$, and they are lightlike. ■

***Remark** 4.2:* The calculations done here support the well know facts that the group $\mathrm{SO}^+(n,1)$ acts transitively on S_1^n and the geodesics in the Lorentzian sphere result from the action of this group.

V. CONTROLLABILITY

From the discussion in the last section, it is clear that the choice of a rolling curve is equivalent to choosing a particular function $u(t)$. So, in terms of coordinates $x^1, \ldots, x^{n+1}$, in $\mathbb{R}_1^{n+1}$, we can treat $u = [\ u^1 \quad u^2 \quad \ldots \quad u^{n+1}\ ]^\top \subset \mathbb{R}^{n+1}$ as a control vector and the kinematic equation (6) becomes a control system with states (R, s). Thus, it makes sense to study the controllability of this system.

Without loss of generality, we may assume that

$$x_0 = [\ 1 \quad 0 \quad \ldots \quad 0\ ]^\top \in S_1^n.$$

So, $T_{x_0}^{\mathrm{aff}}S_1^n$ and $T_{x_0}S_1^n$ are, respectively, the hyperplanes $\{x^1 = 1\}$ and $\{x^1 = 0\}$, $u = [\ 0 \quad u^2 \quad \ldots \quad u^{n+1}\ ]^\top$, $s = [\ 0 \quad s^2 \quad \ldots \quad s^{n+1}\ ]^\top$, and the matrix A in (6) reduces to

$$A = \left[\begin{array}{c|cccc} 0 & -u^2 & \cdots & -u^n & u^{n+1} \\ \hline u^2 & & & & \\ \vdots & & & & \\ u^n & & \mathbf{0} & & \\ u^{n+1} & & & & \end{array}\right].$$

The proof of controllability of the kinematic equations follows the same arguments as those used in [18] for the Euclidean sphere. First we rewrite the control system (kinematic equation (6)) in a more convenient form in order to be able to apply available results for controllability on Lie groups. Following an idea from [14], for the kinematics of rolling the S^2 sphere, we identify the states (R, s) with a single matrix

$$X = \left[\begin{array}{c|c} R & 0 \\ \hline 0 & \begin{array}{c|ccc} 1 & s^2 & \cdots & s^{n+1} \\ \hline 0 & & & \\ \vdots & & \mathrm{I} & \\ 0 & & & \end{array} \end{array}\right],$$

so that (6) is equivalent to the following system evolving on the connected Lie group $G = \mathrm{SO}^+(n,1) \times \mathbb{R}^n$:

$$\dot{X}(t) = X(t)\, B(t), \tag{7}$$

where

$$B = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & \begin{array}{c|ccc} 0 & u^2 & \cdots & u^{n+1} \\ \hline 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{array} \end{array}\right].$$

To simplify notations, let $E_{i,j}$ denote the matrices whose (i,j) entries are equal to δ_{ij}, $A_{i,j} := E_{i,j} - E_{j,i}$ are skew-symmetric matrices, and $B_{i,j} := E_{i,j} + E_{j,i}$ are symmetric matrices. For $k = 2, \ldots, n$, define

$$A_k := -A_{1,k} + E_{n+k,n+k+1},$$
$$A_{n+1} := B_{1,n+1} + E_{n+2,2n+2}.$$

Then, the control system (7) may be written as

$$\dot{X}(t) = X(t) \left(\sum_{i=2}^{n+1} u_i(t) A_i \right).$$

This is a left invariant control system, without drift, evolving on a connected Lie group G. And according to well known results (see, for instance, the pioneer work [6] or the more recent [14]), the system is controllable if and only if the control vector fields are bracket generating. For the present situation, proving controllability of the kinematic equations on $G = \mathrm{SO}^+(n,1) \times \mathbb{R}^n$ amounts to show that the smallest Lie algebra containing $A_2, \ldots, A_{n+1}$ is the Lie algebra of G, $\mathcal{L} = \mathrm{so}(n,1) \oplus \mathbb{R}^n$.

***Lemma** 5.1:* The smallest Lie algebra containing $A_i, i = 2, \ldots, n+1$, is $\mathcal{L} = \mathrm{so}(n,1) \oplus \mathbb{R}^n$.

Proof: It is enough to show that every element in the canonical basis of $\mathcal{L}$,

$$\begin{aligned} \mathcal{B} = \quad & \{A_{i,j}, 1 \leqslant i < j \leqslant n\} \cup \{B_{i,n+1}, 1 \leqslant i \leqslant n\} \\ & \cup \{E_{n+2,n+j}, 3 \leqslant j \leqslant n+2\}, \end{aligned} \tag{8}$$

can be obtained as linear combinations of the matrices $A_i, i = 2, \ldots, n+1$, and their Lie brackets. Computing commutators, one gets:

$$A_{i,j} = -[A_i, A_j],\ 2 \leqslant i < j \leqslant n.$$

$$B_{i,n+1} = [A_i, A_{n+1}],\ 2 \leqslant i \leqslant n.$$

$$A_{1,i} = [A_{n+1}, B_{i,n+1}] = \big[A_{n+1}, [A_i, A_{n+1}]\big],\ 2 \leqslant i \leqslant n.$$

$$\begin{aligned} B_{1,n+1} &= [A_{1,2}, B_{2,n+1}] \\ &= \big[[A_{n+1}, [A_2, A_{n+1}]], [A_2, A_{n+1}]\big]. \end{aligned}$$

$$\begin{aligned} E_{n+2,n+j} &= A_{j-1} + A_{1,j-1} = A_{j-1} \\ &\quad + \big[A_{n+1}, [A_{n+1}, A_{j-1}]\big],\ 3 \leqslant j \leqslant n+1 \end{aligned}$$

$$\begin{aligned} E_{n+2,2n+2} &= A_{n+1} - B_{1,n+1} = A_{n+1} \\ &\quad + \big[[A_{n+1}, [A_2, A_{n+1}]], [A_2, A_{n+1}]\big]. \end{aligned}$$

This concludes the proof.

■

Notice that the bracket generating property here is of constant step 4.

Thus, we have proven the following main result about controllability.

***Theorem** 5.1:* The control system (7) (or, equivalently, the kinematic equation (6)), describing the rolling of S_1^n over its affine tangent space, is controllable on $G = \mathrm{SO}^+(n,1) \times \mathbb{R}^n$.

VI. CONCLUSIONS

We derived the kinematic equations for rolling, without slipping and without twisting, the Lorentzian sphere over its affine tangent space at a point, both embedded in the generalized Minkowski space. We also showed how the kinematic equations can be rewritten as a left-invariant control system evolving on a connected Lie group, and proved controllability of this system.

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