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Compositional properties of passivity

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Abstract— **The classical passivity theorem states that the negative feedback interconnection of passive systems is again passive. The converse statement, - passivity of the interconnected system implies passivity of the subsystems -, turns out to be equally valid. This result implies that among all feasible storage functions of a passive interconnected system there is always one that is the sum of storage functions of the subsystems. Sufficient conditions guaranteeing that all storage functions are of this type are derived. Closely related is the question when and how the stability of the closed-loop interconnected system implies passivity of the subsystems. We recall a folklore theorem which was proved for SISO linear systems, and derive some preliminary results towards a more general result, using the theory of simulation relations.**

I. INTRODUCTION

The notion of passivity has been of crucial importance in many areas of systems and control, as well as in network analysis and design. The fundamental passivity theorem, rooted in physical systems and network theory, states that the negative feedback interconnection of passive systems results in an interconnected system that is again passive. Furthermore, a feasible storage function of the interconnected system is the sum of storage functions of the subsystems. Thus in a very fundamental sense, passivity is a *compositional property*.

In this paper we aim to take a fresh look at the compositional properties of passivity. We start with a basic result that seems to have been overlooked before: if an interconnected system is passive with regard to external inputs and outputs corresponding to all interconnection constraints, then the subsystems are passive as well. This converse result allows us to further study the feasible storage functions of a passive interconnected system. It is well-known that usually there is a whole class of feasible storage functions, possessing a *minimal* and generally a *maximal* element. The converse result implies that among all feasible storage functions there is always an *additive* storage function, that is, a function that is the sum of storage functions of the subsystems. Furthermore, it is well-known that *lossless* systems generally have a *unique* storage function. The converse result allows us to prove that if at least one subsystem of the interconnected system is lossless then, under additional accessibility conditions, all storage functions of the interconnected system are additive.

In the last section we explore a closely related, but different problem. For linear SISO systems it has been proved that if the *closed* (no external inputs anymore) negative feedback interconnection of a system with *any* passive system is stable then the system itself is necessarily passive. This is an interesting statement which seems to be valid for a general class of systems. We provide a preliminary result in this direction which is motivated by recent work on compositional reasoning using simulation relations.

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II. INTERCONNECTION OF PASSIVE SYSTEMS AND PASSIVE INTERCONNECTED SYSTEMS

Throughout this paper we will consider input-affine square nonlinear systems Σ with an equilibrium x^*

$$\Sigma: \quad \begin{array}{rcl} \dot{x} &=& f(x)+g(x)u = f(x)+\sum_{j=1}^{m} u^j g^j(x) \\ 0 &=& f(x^*) \\ y &=& h(x), \\ \multicolumn{3}{c}{x\in\mathscr{X}, \quad u\in\mathscr{U}, \quad y\in\mathscr{Y},} \end{array} \tag{1}$$

where $\mathscr{X}$ is an n-dimensional manifold, and $\mathscr{U}$ and $\mathscr{Y}$ are linear input and output spaces, both of dimension m.

We throughout assume smoothness of the vector fields $f, g^1, g^2, \cdots, g^m$ and the mapping h.

Definition 1: [9] A state space system Σ is *passive* if there exists a function $V:\mathscr{X}\to\mathbb{R}^+$, called the *storage function*, such that for all $x_0\in\mathscr{X}$, all $t_1\geq t_0$, and all input functions $u:[t_0,t_1]\to\mathscr{U}$

$$V(x(t_1)) \leq V(x(t_0)) + \int_{t_0}^{t_1} u^T(t)y(t)dt \tag{2}$$

where $x(t_0)=x_0$, and $x(t_1)$ denotes the state at time t_1 resulting from initial condition $x(t_0)=x_0$ and the input function $u:[t_0,t_1]\to\mathscr{U}$. If (2) holds with equality, then Σ is *lossless*.

If the storage function V is differentiable, the differential version of the dissipation inequality (2) is given by [1], [9]

$$\dot{V}(x) = V_x(x)\dot{x} \leq u^T y \tag{3}$$

for all $(x,\dot{x},u,y)$ satisfying (1). Here $V_x(x)$ denotes the row vector of partial derivatives

$$V_x(x) = \begin{pmatrix} \frac{\partial V}{\partial x_1}(x) & \dots & \frac{\partial V}{\partial x_n}(x) \end{pmatrix}$$

The differential dissipation inequality is equivalent [3], [6] to the following conditions for passivity, cq. losslessness, which will be used in the rest of the paper.

Proposition 2: Let Σ be a nonlinear system of the form (1) and let $V(x)$ be a differentiable storage function of Σ. Then Σ is passive (lossless) if and only if

$$\begin{array}{rcl} V_x(x)f(x) &\leq& 0\,(=0) \\ V_x(x)g(x) &=& h^T(x) \end{array} \tag{4}$$

It is well-known that in general the storage function for a passive system is intrinsically *non-unique*. In fact [9], the set of storage functions is convex, has a minimum (the *available storage*), and, if the system is *reachable* from some state, has a maximum (the *required supply*). If the system is *lossless* and *reachable* from some state then the storage function *is* unique (up to a constant).

Given two nonlinear systems $\Sigma_i, i = 1,2$, with equal dimension of their input and output spaces we define their *negative feedback interconnection*

$$u_1 = -y_2 + e_1 \quad , \quad u_2 = y_1 + e_2$$

where e_1, e_2 are new external inputs.

The resulting interconnected system, with inputs e_1, e_2 and outputs $z_1 = y_1, z_2 = y_2$, is given as

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} f_1(x_1) - g_1(x_1)h_2(x_2) \\ f_2(x_2) + g_2(x_2)h_1(x_1) \end{bmatrix} + \begin{bmatrix} g_1(x_1)e_1 \\ g_2(x_2)e_2 \end{bmatrix} \\ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} h_1(x_1) \\ h_2(x_2) \end{bmatrix} \end{aligned} \tag{5}$$

and will be denoted as $\Sigma_1 \| \Sigma_2$.

A fundamental result in passivity theory is that the property of passivity is preserved under negative feedback interconnection, while the sum of any storage functions for each subsystem serves as a feasible storage function for the interconnected system (*additivity* of 'energy'). For later reference we summarize this in the following theorem, and provide for completeness its proof.

Theorem 3: For any two passive (lossless) nonlinear systems $\Sigma_i, i = 1,2$, with storage functions $V_i, i = 1,2$, the interconnected system $\Sigma_1 \| \Sigma_2$ with inputs e_1, e_2 and outputs $z_1 = y_1, z_2 = y_2$ is passive (lossless) with storage function $V_1(x_1) + V_2(x_2)$.

Proof: Since the two systems $\Sigma_i, i = 1,2$, are passive, their storage functions $V_i(x_i)$ satisfy

$$\dot{V}_i(x_i) \leq u_i^T y_i, \quad i = 1,2 \tag{6}$$

Hence the system $\Sigma_1 \| \Sigma_2$ satisfies

$$\begin{aligned} \dot{V}_1(x_1) + \dot{V}_2(x_2) &\leq (-y_2 + e_1)^T y_1 + (y_1 + e_2)^T y_2 \\ &= e_1^T z_1 + e_2^T z_2 \end{aligned} \tag{7}$$

which proves that $\Sigma_1 \| \Sigma_2$ is passive, with storage function $V_1(x_1) + V_2(x_2)$. The argument immediately extends to the lossless case. ∎

The *converse* statement of the fundamental passivity theorem 3, i.e., passivity of the interconnected system $\Sigma_1 \| \Sigma_2$ implying passivity of the two subsystems $\Sigma_i, i = 1,2$, seems not to have been investigated in the literature, but turns out to be equally valid.

Theorem 4: Consider two nonlinear systems $\Sigma_i, i = 1,2$, such that $\Sigma_1 \| \Sigma_2$ is passive (lossless). Then also the component systems $\Sigma_i, i = 1,2$, are passive (lossless).

Proof: We will only prove the passive case, the same arguments hold for the lossless case. The interconnected system $\Sigma_1 \| \Sigma_2$ being passive is equivalent to the existence of a storage function $V : \mathscr{X}_1 \times \mathscr{X}_2 \to \mathbb{R}^+$ for $\Sigma_1 \| \Sigma_2$, that is

$$\begin{aligned} V_{x_1}(x_1,x_2)\,(f_1(x_1) - g_1(x_1)h_2(x_2)) + & \\ V_{x_2}(x_1,x_2)\,(f_2(x_2) + g_2(x_2)h_1(x_1)) &\leq 0 \end{aligned} \tag{8}$$

$$V_{x_1}(x_1,x_2)g_1(x_1) = h_1^T(x_1) \tag{9}$$

$$V_{x_2}(x_1,x_2)g_2(x_2) = h_2^T(x_2) \tag{10}$$

This results in

$$\begin{aligned} & V_{x_1}(x_1,x_2)f_1(x_1) - \underbrace{V_{x_1}(x_1,x_2)g_1(x_1)}_{=h_1^T(x_1)} h_2(x_2) + \\ & + V_{x_2}(x_1,x_2)f_2(x_2) + \underbrace{V_{x_2}(x_1,x_2)g_2(x_2)}_{=h_2^T(x_2)} h_1(x_1) = \\ & = V_{x_1}(x_1,x_2)f_1(x_1) + V_{x_2}(x_1,x_2)f_2(x_2) \leq 0 \end{aligned} \tag{11}$$

Now define the non-negative functions

$$V_1(x_1) := V(x_1,x_2^*) \quad , \quad V_2(x_2) := V(x_1^*,x_2) \tag{12}$$

as candidate storage functions for the component systems $\Sigma_i, i = 1,2$. For $x_2 = x_2^*$, (11) then becomes

$$\begin{aligned} & V_{x_1}(x_1,x_2^*)f_1(x_1) + V_{x_2}(x_1,x_2^*)f_2(x_2^*) = \\ & = V_{x_1}(x_1,x_2^*)f_1(x_1) = V_{1x_1}(x_1)f_1(x_1) \leq 0\,, \end{aligned} \tag{13}$$

since $f_2(x^*) = 0$, while (9) becomes

$$V_{x_1}(x_1,x_2^*)g_1(x_1) = V_{1x_1}(x_1)g_1(x_1) = h_1^T(x_1) \tag{14}$$

Hence, $V_1(x_1) = V(x_1,x_2^*)$ satisfies conditions (4) and thus is a storage function for Σ_1. The same reasoning leads to Σ_2 being passive with storage function $V_2(x_2) := V(x_1^*,x_2)$. ∎

An important implication of Theorems 3 and 4 is therefore the following

Corollary 5: If the interconnection $\Sigma_1 \| \Sigma_2$ of two nonlinear systems Σ_1, Σ_2 is passive then there exists an *additive* storage function

$$V_1(x_1) + V_2(x_2) \tag{15}$$

where $V_i(x_i)$ are storage functions of the components $\Sigma_i, i = 1,2$.

Indeed, since Theorem 4 states that the component systems $\Sigma_i, i = 1,2$, are passive with storage functions $V_1(x_1), V_2(x_2)$ it follows from Theorem 3 that $V_1(x_1) + V_2(x_2)$ is a storage function for the interconnected system $\Sigma_1 \| \Sigma_2$.

Note that in general

$$V_1(x_1) + V_2(x_2) \neq V(x_1,x_2),$$

where $V(x_1,x_2)$ is the storage function of the interconnected system $\Sigma_1 \| \Sigma_2$ that we started with. Of course, this is in accordance with the fact that storage functions for a passive system are in general not unique.

Finally, Theorems 3 and 4 can be combined into

Corollary 6: $\Sigma_1 \| \Sigma_2$ is passive (lossless) if and only if Σ_1 and Σ_2 are passive (lossless).

Theorems 3 and 4 and their corollaries can be generalized to interconnections of *multiple* systems in the following way.

Theorem 7: Consider nonlinear systems $\Sigma_i, i = 1,\cdots,k$, interconnected to each other by interconnection constraints of the form[1]

$$u_i = F_i(y_1,\cdots,y_k) + e_i, \quad i = 1,\cdots,k, \tag{16}$$

where the functions F_i satisfy

$$\sum_{i=1}^{k} F_i(y_1,\cdots,y_k)y_i = 0, \quad \text{for all } y_1,\cdots,y_k, \tag{17}$$

Denote the resulting interconnected system with inputs $e_1,\cdots,e_k$ and outputs $z_1 = y_1,\cdots,z_k = y_k$ by Σ_{int}. Then

1) Suppose that the systems $\Sigma_i, i = 1,\cdots,k$, are passive (lossless) with storage functions $V_i, i = 1,\cdots,k$. Then the interconnected system Σ_{int} is passive (lossless), with storage function $V_{\text{int}} = V_1 + \cdots + V_k$.
2) Suppose that the interconnected system Σ_{int} is passive (lossless). Then also the component systems $\Sigma_i, i = 1,\cdots,k$, are passive (lossless). In particular, let $V_i(x_i), i = 1,\cdots,k$, be storage functions for the component systems then $V_1(x_1) + \cdots V_k(x_k)$ is a storage function for the interconnected system Σ_{int}.

Proof: The first statement follows, like Theorem 3, from classical passivity theory [9], [6]. For the second statement we consider any storage function $V(x_1,\cdots,x_k)$ for the interconnected system Σ_{int}, satisfying

$$\begin{aligned} &V_{x_1}(x_1,\cdots,x_k)[f_1(x_1) + g_1(x_1)u_1] + \cdots + \\ &+V_{x_k}(x_1,\cdots,x_k)[f_k(x_k) + g_k(x_k)u_k] \leq e_1^T y_1 + \cdots e_k^T y_k \end{aligned}$$

where $u_1,\cdots,u_k,y_1,\cdots,y_k,e_1,\cdots,e_k$ are related by (16, 17). It follows that

$$V_{x_i}(x_1,\cdots,x_k)g_i(x_i) = h_i^T(x_i), \quad i = 1,\cdots,k \tag{18}$$

together with (leaving out all arguments x_i)

$$V_{x_1}(f_1 + g_1F_1) + \cdots + V_{x_k}(f_k + g_kF_k) \leq 0 \tag{19}$$

Substitution of (18) in (19) thus yields

$$V_{x_1}f_1 + \cdots V_{x_k}f_k + \sum_{i=1}^{k} F_i(y_1,\cdots,y_k)y_i \leq 0,$$

which in view of (17) yields

$$V_{x_1}(x_1,\cdots,x_k)f_1(x_1) + \cdots V_{x_k}(x_1,\cdots,x_k)f_k(x_k) \leq 0 \tag{20}$$

Then define the non-negative functions

$$\begin{aligned} &V_1(x_1) := V(x_1,x_2^*,\cdots,x_k^*), \quad V_2(x_2) := V(x_1^*,x_2,\cdots,x_k^*), \\ &\cdots V_k(x_k) := V(x_1^*,x_2^*,\cdots,x_{k-1}^*,x_k) \end{aligned}$$

By using $f_i(x_i^*) = 0$ we obtain

$$(V_i)_{x_i}(x_i)f_i(x_i) \leq 0, \quad i = 1,\cdots,k,$$

showing, together with (18), that the subsystems Σ_i are passive with storage functions $V_i, i = 1,\cdots,k$. Furthermore, it follows that $V_1(x_1) + \cdots + V_k(x_k)$ is a storage function for the interconnected system.

The lossless case uses the same arguments, leading to $(V_i)_{x_i}(x_i)f_i(x_i) = 0, \quad i = 1,\cdots,k.$ ■

Remark 8: The first statement of the theorem continues to hold with regard to passivity of the interconnected system for interconnections (16) satisfying the inequality

$$\sum_{i=1}^{k} F_i(y_1,\cdots,y_k)y_i \leq 0, \quad \text{for all } y_1,\cdots,y_k$$

Dually, the second statement of the theorem regarding passivity of the subsystems continues to hold for interconnections (16) satisfying the reverse inequality

$$\sum_{i=1}^{k} F_i(y_1,\cdots,y_k)y_i \geq 0, \quad \text{for all } y_1,\cdots,y_k$$

Remark 9: Theorem 7 can be easily generalized to *strict output passivity*. Recall, see e.g. [6], that a system Σ is strictly output passive if there exists $\varepsilon > 0$ such that

$$\begin{aligned} V_x(x)f(x) &\leq -\varepsilon h^T(x)h(x) \\ V_x(x)g(x) &= h^T(x) \end{aligned}$$

It follows that the interconnection (17,17) of strictly output passive systems is again strictly output passive (with $\varepsilon = \min\{\varepsilon_1,\cdots.\varepsilon_k\}$), while the strict output passivity of the interconnected system Σ_{int} implies strict output passivity (all for the same ε) of the subsystems provided that $h_j(x_j^*) = 0, j = 1,\cdots,k$.

III. Structure of the set of storage functions for passive interconnected systems

Now let us look more closely at the issue of *additivity* and (partial) *uniqueness* of the storage function of an interconnected system $\Sigma_1\|\Sigma_2$, which is passive or lossless. For brevity we will only do this for the case of the interconnection of two systems; the results can be directly extended to interconnections (16, 17) of multiple systems.

As recalled before, in case $\Sigma_1\|\Sigma_2$ is lossless and reachable from some ground state, then it follows from passivity theory [9] that its storage function is unique. As a direct consequence we obtain

Proposition 10: Let $\Sigma_1\|\Sigma_2$ be lossless and reachable from some state $\bar{x}$. Then its unique storage function is an additive function

$$V(x_1,x_2) = V_1(x_1) + V_2(x_2)$$

We will now show that similar results can be obtained under the much weaker assumption that only one of the two system components is lossless, and both components satisfy *accessibility* assumptions.

Definition 11: Consider a nonlinear system Σ of the form (1) with $g^1,\cdots,g^m$ the m columns of g. Then the *accessibility algebra* $\mathscr{C}$ is the smallest subalgebra of the Lie algebra of vector fields on $\mathscr{X}$ that contains f and all input vector fields $g^1,\ldots,g^m$. Define $\mathscr{C}_0$ as the smallest subalgebra containing $g^1,\ldots,g^m$ and satisfying $[f,X] \in \mathscr{C}_0$ for all $X \in \mathscr{C}_0$.
Σ is *locally strongly accessible* if the sets

$$\begin{aligned} R^V(x_0,T) \quad = \quad &\{(x \in \mathscr{X} \mid \exists u : [0,T] \to \mathscr{U} \text{ s. t. } x(t) \in V, \\ &0 \leq t \leq T, x(0) = x_0, x(T) = x\} \end{aligned}$$

[1]In many physical situations this will have the interpretation of a *power-conserving interconnection* [6].

for all $x_0 \in \mathscr{X}$ contains a non-empty open set of $\mathscr{X}$ for all neighborhoods V of x_0 and any sufficiently small $T > 0$. Define the reachable set

$$R_T^V(x_0) = \cup_{\tau \le T} R^V(x_0, \tau) \tag{21}$$

Then Σ is *reachable* from x_0 if $R_T^V(x_0) = \mathscr{X}$ for some $T > 0$. As shown in [5], every element of the subalgebra $\mathscr{C}_0$ is a linear combination of repeated Lie brackets $[X_k, [X_{k-1}, [\ldots, [X_1, g] \ldots]]], k = 0, 1, \ldots,$ where we will throughout use the shorthand notation $[X, g]$ for any of the Lie brackets $[X, g^j], j = 1, \cdots, m$. We recall from [5]

Proposition 12: Let Σ be a nonlinear system of the form (1). If Σ is locally strongly accessible then $\dim(\text{span}\{X(x_0) \mid X \in \mathscr{C}_0\}) = n = \dim \mathscr{X}$ for x_0 in an open and dense subset of $\mathscr{X}$.

We are now able to state the first result concerning the negative feedback interconnection of a passive and a lossless component.

Proposition 13: Consider two nonlinear systems $\Sigma_i, i = 1, 2$, of the form (1). Let Σ_1 be passive and Σ_2 be lossless. Assume that Σ_1 is locally strongly accessible. Then all storage functions $V(x_1, x_2)$ of the interconnection $\Sigma_1 \| \Sigma_2$ are of the form $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ where $V_1(x_1)$ is a storage function of Σ_1 and $V_2(x_2)$ is the unique storage function of Σ_2.

Proof: Since Σ_1 is passive and Σ_2 lossless, the interconnection $\Sigma_1 \| \Sigma_2$ by Theorem 3 is also passive. Consider any storage function $V(x_1, x_2)$ for $\Sigma_1 \| \Sigma_2$. Rewriting the dissipation inequality, this implies

$$\begin{aligned} & L_{f_1} V + L_{g_1} V (e_1 - y_2) + L_{f_2} V + L_{g_2} V (e_2 + y_1) = \\ & \quad e_1^T y_1 + e_2^T y_2 - W \end{aligned}$$

for all e_1, e_2, with $W = W(x_1)$ a nonnegative function of x_1 only. Equivalently

$$\begin{aligned} L_{f_1} V(x_1, x_2) + L_{f_2} V(x_1, x_2) + W(x_1) &= 0 \\ L_{g_1} V(x_1, x_2) &= h_1^T(x_1) \\ L_{g_2} V(x_1, x_2) &= h_2^T(x_2) \end{aligned}$$

We claim that $L_X V$ is a function of x_1 only for all $X \in \mathscr{C}_0^1$. Clearly, $L_{g_1} V = h_1^T$ is a function of x_1. Moreover, $L_{[f_1, g_1]} V = L_{f_1} L_{g_1} V - L_{g_1} L_{f_1} V = L_{f_1} h_1^T + L_{g_1} L_{f_2} V + L_{g_1} W$ is a function of x_1 only since $L_{g_1} L_{f_2} V = L_{f_2} L_{g_1} V = L_{f_2} h_1^T = 0$. In fact, $L_{X_i} L_{X_j} V = L_{X_j} L_{X_i} V, (i, j) \in \{(1, 2), (2, 1)\}$ due to $[f_i, g_j] = 0$. Assume now that $L_{[X_k, [X_{k-1}, [\ldots, [X_1, g] \ldots]]]} V$ is a function of x_1 only, denoted by $R(x_1)$. To complete the induction step, consider first the case $X_{K+1} = g_1$. Then

$$\begin{aligned} & L_{[g_1, [X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]]} V = \\ & + L_{g_1} L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} V - L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} L_{g_1} V = \\ & = L_{g_1} R(x_1) - L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} L_{g_1} V = \\ & = L_{g_1} R(x_1) - L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} h_1^T \end{aligned}$$

is a function of x_1 only since all $X_i, i = 1, 2, \ldots,$ depend on x_1 only. If $X_{k+1} = f_1$, then

$$\begin{aligned} & L_{[f_1, [X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]]} V = \\ & = L_{f_1} L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} V - L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} L_{f_1} V = \\ & = L_{f_1} R(x_1) - L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} L_{f_1} V = \\ & = L_{f_1} R(x_1) + L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} (L_{f_2} V + W(x_1)) = \\ & = L_{f_1} R(x_1) + L_{f_2} L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} V + \\ & \quad + L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} W(x_1) = \\ & = L_{f_1}(x_1) + L_{f_2} R_{x_1} + L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} W(x_1) \end{aligned}$$

is also a function of x_1 only. Thus, $L_{\mathscr{C}_0^1} V$ is indeed a function of x_1 only, i.e.

$$\frac{\partial}{\partial x_2} \left\{ L_{g_1} V, L_{[f_1, g_1]} V, \ldots, L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} V \right\} = 0 \tag{22}$$

Since Σ_1 is locally strongly accessible, $\dim(\text{span}\{X_1(x_1) \mid X_1 \in \mathscr{C}_0^1\}) = n_1$ for x_1 in an open and dense subset of $\mathscr{X}_1$. By continuity (22) implies that $\frac{\partial^2}{\partial x_2 \partial x_1} V(x_1, x_2) = 0$ for all x_1, x_2, and thus the storage function $V(x_1, x_2)$ of $\Sigma_1 \| \Sigma_2$ is of the form $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$ up to a constant. As a consequence of Theorem 4, $V_i(x_i)$ are storage functions of $\Sigma_i, i = 1, 2$. Since Σ_2 is lossless, its storage function $V_2(x_2)$ is unique up to a constant. ■

In case $\Sigma_i, i = 1, 2$, are both lossless we have the following stronger result.

Proposition 14: Consider two nonlinear systems $\Sigma_i, i = 1, 2$, of the form (1). Let $\Sigma_i, i = 1, 2$, be lossless and at least one of them locally strongly accessible. Then every storage function $V(x_1, x_2)$ of the interconnection $\Sigma_1 \| \Sigma_2$ is of the form (15).

Proof: Observe first that by Theorems 3 and 4, $\Sigma_1 \| \Sigma_2$ being lossless with storage function $V(x_1, x_2)$ is equivalent to both $\Sigma_i, i = 1, 2$, being lossless with storage functions $V_i(x_i), i = 1, 2$. Furthermore, $\Sigma_1 \| \Sigma_2$ being lossless implies by (8) – (10) that

$$\begin{aligned} & L_{f_1} V(x_1, x_2) + L_{f_2} V(x_1, x_2) = 0, \\ & \qquad L_{g_1} V(x_1, x_2) = h_1^T(x_1), \quad L_{g_2} V(x_1, x_2) = h_2^T(x_2) \end{aligned} \tag{23}$$

Now let us assume that Σ_1 is locally strongly accessible. We want to show that

$$L_{g_1} V, L_{[f_1, g_i]} V, \ldots, L_{[X_1^1, [X_2^1, \ldots, [X_k^1, X_{K+1}^i], \ldots]]} V, \quad i = 1, 2, \tag{24}$$

are functions of x_1 only for all $X_j^1, j \in k$ from the set $\{f_1, g_i\}$, $k \ge 1$. Clearly, $L_{g_1} V = h_1^T$ is a function of x_1 only. The proof that also $L_{[X_1^1, [X_2^1, \ldots, [X_k^1, X_{K+1}^1], \ldots]]} V$ is a function of x_1 only relies on the same arguments as used in the proof of Proposition 13. Hence, differentiation of (24) with respect to x_2 yields

$$\frac{\partial}{\partial x_2} \left\{ L_{g_1} V, L_{[f_1, g_i]} V, \ldots, L_{[X_k, [X_{k-1}, [\ldots, [X_1, g_1] \ldots]]]} V \right\} = 0, \tag{25}$$

Since Σ_1 is locally strongly accessible, $\mathscr{C}_0^1(x)$ has full rank for x in an open and dense subset of $\mathscr{X}_i$. Hence, 25 implies by continuity of $V(x_1, x_2)$ that $\frac{\partial^2 V}{\partial x_2 \partial x_1} = 0$. Hence, any storage function $V(x_1,x_2)$ of $\Sigma_1\|\Sigma_2$ is of the form $V(x_1,x_2) = V_1(x_1)+V_2(x_2)$ where $V_i(x_i)$ by Theorem 4 is a storage function of Σ_i. ■

Remark 15: Compare Proposition 14 with the following classical reasoning from passivity theory. If $\Sigma_i, i=1,2$, are both reachable from some point $x_i^\star, i=1,2$ then both $\Sigma_i, i=1,2$, have unique storage functions $V_i(x_i), i=1,2$ (up to a constant). But then the interconnected system $\Sigma_1\|\Sigma_2$ is reachable from $(x_1^\star,x_2^\star)$ using the inputs $e_1=u_1+h_2(x_2), e_2=u_2-h_1(x_1)$ and thus $\Sigma_1\|\Sigma_2$ has a unique storage function $V(x_1,x_2)$ as well. Theorem 3 then tells us that $V(x_1,x_2)$ is given as the sum of the unique storage functions $V_i(x_i)$.

IV. Passivity resulting from stability of the interconnection with arbitrary passive system

Corollary 6 and Theorem 7 express the following *compositional* property of passivity: *an interconnected system is passive if and only if the component systems are all passive*.

Note, however, that the 'only if' part requires passivity of the interconnected system Σ_{int} with respect to all new inputs $e_1,\cdots,e_k$ and all outputs $z_1=y_1,\cdots,z_k=y_k$. Indeed, typically passivity of the component system Σ_j is only implied when the interconnected system is passive with respect to input e_j and output y_j.

Example 16: Consider an RC-circuit with current source

$$\begin{aligned}\dot{Q} &= -G\tfrac{Q}{C}+u_1\\ y_1 &= \tfrac{Q}{C}\end{aligned}$$

where Q is the charge at the condensator, $C>0$ is its capacitance, G is the conductance of the resistor, u_1 is the input current of the current source, and y_1 is its output voltage. Clearly the system is passive if and only if $G\geq 0$.

Analogously, consider an RL-circuit with voltage source

$$\begin{aligned}\dot{\varphi} &= -R\tfrac{\varphi}{L}+u_2\\ y_2 &= \tfrac{\varphi}{L}\end{aligned}$$

where φ is the flux of the inductor, $L>0$ is its inductance, R is the resistance of the resistor, u_2 is the input voltage, and y_2 the output current. Again, the system is passive if and only if $R\geq 0$.

The *closed* negative feedback interconnection $u_1=-y_2, u_2=y_1$ of the RC-circuit with the RL-circuit (corresponding to Kirchhoff's current and voltage laws) results in the autonomous system

$$\begin{bmatrix}\dot{Q}\\ \dot{\varphi}\end{bmatrix} = \begin{bmatrix}-G & -1\\ 1 & -R\end{bmatrix}\begin{bmatrix}\frac{Q}{C}\\ \frac{\varphi}{L}\end{bmatrix},$$

which is *stable* if and only if

$$G+R\geq 0,$$

and asymptotically stable if and only if $G+R>0$. Thus it is *not* necessary that both subsystems are passive in order to guarantee stability of the interconnected system: the lack of passivity of e.g. the RC-circuit (corresponding to the case $G<0$) can be compensated by a 'surplus' of passivity of the RL-circuit (i.e., R such that $G+R\geq 0$).

Note furthermore that if $G+R\geq 0$ but *not* both $G\geq 0$ and $R\geq 0$ (the case of one 'non-physical' resistor) then, although the system is stable, the physical energy $\frac{1}{2C}Q^2+\frac{1}{2L}\varphi^2$ is *not* a Lyapunov function anymore.

This motivates the interest to derive conditions ensuring that passivity of the component systems *is* implied by passivity of the interconnected system with regard to a smaller number of inputs and outputs. As a typical case of such considerations let us consider, as in the previous example, the following situation. Consider two systems Σ_1 and Σ_2 interconnected by the *closed* negative feedback interconnection

$$u_1=-y_2,\quad u_2=y_1 \tag{26}$$

(no external inputs e_1,e_2). Denote the autonomous interconnected system by $\Sigma_1\|_{\text{cl}}\Sigma_2$. Clearly, passivity of the interconnected system $\Sigma_1\|_{\text{cl}}\Sigma_2$ cannot even be defined. Nevertheless the following *folklore theorem* can be stated:

Suppose that $\Sigma_1\|_{\text{cl}}\Sigma_2$ is stable for all passive systems Σ_2, then Σ_1 is passive.

In fact, the above statement has been proved for single-input single-output linear systems in [2], making use of a Nyquist plot argument. The proof line is to suppose that Σ_2 is *not* passive, and then to construct a passive (even lossless) Σ_1 which is destabilizing the closed-loop system $\Sigma_1\|_{\text{cl}}\Sigma_2$, thus leading to a contradiction.

Example 17 (Example 16 continued): Consider again the RC-circuit from above with $G\in\mathbb{R}$, i.e., not necessarily non-negative. Clearly, if this circuit whenever interconnected with an RL-circuit results in a *stable* autonomous system for any $R\geq 0$ (or equivalently, the RL-circuit is passive), then necessarily $G\geq 0$, and thus the RC-circuit is passive. The same reasoning holds for the interconnection of an RL-circuit with R of arbitrary sign with a passive RC-circuit: stability for any $G\geq 0$ implies $R\geq 0$.

However, up to the knowledge of the authors of the present paper, no proof of this folklore theorem for more general systems is available.

In the rest of this paper we will approach the folklore theorem in the following modified sense. Replace the lossless system Σ_2 by its *abstraction*

$$\Xi_2:\dot{\xi}_2=u_2^Ty_2,\quad \xi_2\in\mathbb{R}^+ \tag{27}$$

(keeping only track of the energy balance of the arbitrary lossless system). Then consider the interconnection of Σ_1 with Ξ_2 via (26), leading to the interconnected system $\Sigma_1\|_{\text{cl}}\Xi_2$ given as

$$\begin{aligned}\dot{x}_1 &= f_1(x_1)+g_1(x_1)u_1\\ \dot{\xi}_2 &= -h_1^T(x_1)u_1,\quad \xi_2\geq 0\end{aligned} \tag{28}$$

(Note that this is a system description of a generalized type, since u_1 is not uniquely determined by (28). It means that we consider all x_1,ξ_2,u_1 satisfying (28).)

Proposition 18: Suppose that $\Sigma_1\|_{\text{cl}}\Xi_2$ is stable in the sense that there exists a non-negative function $V(x_1,\xi_2)$

satisfying

$$V_{x_1}(x_1,\xi_2)[f_1(x_1)+g_1(x_1)u_1]+V_{\xi_2}(x_1,\xi_2)\dot{\xi}_2 \leq 0 \quad (29)$$

for all $x_1, \xi_2, \dot{\xi}_2, u_1$ satisfying (28). Furthermore, assume that there exists a ξ_2^* such that

$$V_{\xi_2}(x_1,\xi_2^*)=\alpha>0, \quad (30)$$

with α a constant (independent of x_1). Then Σ_1 is passive.

Proof: Since (29) holds for all $x_1, \xi_2, \dot{\xi}_2, u_1$ satisfying (28), it follows that

$$V_{x_1}(x_1,\xi_2)[f_1(x_1)+g_1(x_1)u_1]-V_{\xi_2}(x_1,\xi_2)h_1^T(x_1)u_1 \leq 0$$

for all x_1, u_1. This is equivalent to

$$\begin{aligned} &V_{x_1}(x_1,\xi_2)f_1(x_1) \leq 0 \\ &V_{x_1}(x_1,\xi_2)g_1(x_1)-V_{\xi_2}(x_1,\xi_2)h_1^T(x_1)=0 \end{aligned} \quad (31)$$

Evaluating the second equation at any point (x_1, ξ_2^*) yields

$$V_{x_1}(x_1,\xi_2^*)g_1(x_1)=\alpha h_1^T(x_1)$$

Then it follows that $V(x_1) := \frac{1}{\alpha}V(x_1,\xi_2^*)$ is a storage function for Σ_1. ∎

A. Passivity as a nonlinear simulation relation

The introduction of the abstraction system (27) can be interpreted from a *simulation* point of view as follows.

Recall that a system Σ is passive if there exists a (differentiable) storage function $V : \mathscr{X} \to \mathbb{R}^+$ satisfying

$$V_x(x) \leq u^T y, \text{ for all } x,u,y \text{ satisfying (1)} \quad (32)$$

This can be also expressed by saying that Σ is *simulated* by the abstraction system

$$\Xi : \dot{\xi} \leq u^T y, \quad \xi \in \mathbb{R}^+, \quad (33)$$

where the simulation relation $S \subset \mathscr{X} \times \mathbb{R}^+$ is given by

$$S = \{(x,\xi) \in \mathscr{X} \times \mathbb{R}^+ \mid \xi = V(x)\} \quad (34)$$

Indeed, starting at every $(x,\xi) \in S$ it follows that for every common input u to Σ and Ξ there exists a scalar v such that

$$\begin{aligned} &(f(x)+g(x)u, v) \in T_{(x,\xi)}S \\ &v \leq u^T h(x) \end{aligned} \quad (35)$$

where $T_{(x,\xi)}S$ denotes the tangent space to the submanifold S at the point $(x,\xi) \in S$. This implies that for every initial state $(x,\xi) \in S$ and for every input function $u(\cdot)$ there corresponds to the solution trajectory $x(\cdot)$ of Σ a solution trajectory $\xi(\cdot)$ of the generalized system Ξ such that for all $t \geq 0$

$$(x(t),\xi(t)) \in S$$

(See for the precise definition of a nonlinear simulation relation [7], [8].) From this point of view Proposition 18 can be interpreted as addressing the question when the stability of the autonomous interconnected system $\Sigma_1 \|_{cl} \Xi_2$ with Ξ_2 lossless, implies that Σ_1 is simulated by Ξ_1. Such an interpretation suggests to apply compositional reasoning techniques as recently developed in [4] to this problem. This is currently under investigation.

V. Conclusions

We have proved a converse to the classical passivity theorem: whenever the interconnected system (with external inputs) is passive, then so are the subsystems. This has been also demonstrated for a general power-conserving interconnection of multiple systems. An important consequence is the fact that a passive interconnected system always has an additive storage function. It also allows to say more about the class of storage functions of a passive interconnected system.

Current investigations deal with the extension of these results to closed negative feedback interconnections. Preliminary results in this direction are reported in the last section.

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