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Aspects of time-optimal control of a particle in a dielectrophoretic system

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***Abstract*— The Dielectrophoretic (DEP) force arises as a result of the interaction between a nonuniform electric field and a neutrally charged particle. As the effect of the electric field is more dominant than other forces at the micro/nano scale, this force can be effectively used to manipulate and control particles on this scale. We consider the motion of a particle on an invariant line with the suspending medium being a fluid with low Reynolds number. The system is described by a set of ordinary differential equations with a quadratic term in the control variable (control being the applied voltage on the electrodes which induces the electric field) making the system non-affine. For this system, we address certain aspects of existence and uniqueness of the time-optimal trajectories.**

I. INTRODUCTION

An electrically neutral particle placed in a non-uniform electric field gets polarized and experiences a translational force that can cause the motion of the particle. This resultant motion is termed as *dielectrophoresis* (DEP) by Pohl [1]. The differential equations describing the dynamics of a neutrally buoyant and neutrally charged particle in a DEP system were derived in [2] as

$$\dot{x} = yu + \alpha u^2, \tag{1}$$

$$\dot{y} = -cy + u, \tag{2}$$

with state $(x, y) \in \mathbb{R}^2$, the control $u \in U \subseteq \mathbb{R}$ and real constant parameters α and c. The parameter c is always positive, but the sign of the parameter α depends on the electric characteristics of the particle and the suspending medium. Notice that there exists a quadratic term in the control making the system non-affine in control.

The differential equations in (1)-(2) describe the vertical motion of a neutrally buoyant and neutrally charged particle with the suspending medium being a fluid with low Reynolds number. The variable x is a nonlinear function of the particle position, while the variable y describes the exponentially decaying part of the induced dipole moment. The control u is the voltage on the electrodes which induces the electric field. In this paper, we address certain issues of time-optimal control for this system.

Earlier work [3] has addressed the time-optimal problem for the case of $y(0) = 0$ with $\alpha < 0$. This paper considers the case $y(0) \neq 0$.

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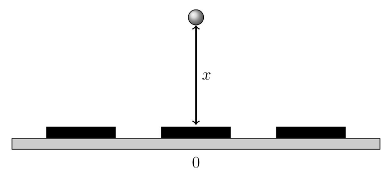


Fig. 1. Front view of the arrangement of a linear electrode array with a neutrally charged particle suspended in the medium.

II. PROBLEM STATEMENT

Consider the DEP system (1)-(2) with the boundary constraints:

$$x(0) = x_0 = \text{ given }, \qquad y(0) = y_0 = \text{ given }, \tag{3}$$

$$x(T) = x_f = \text{ given }, \qquad y(T) = \text{ free }, \tag{4}$$

$$|u| \leq 1, \tag{5}$$

where the constant parameters α and c satisfy

$$\alpha < 0, \quad c > 0. \tag{6}$$

From the system equations (1)-(2), it can be easily verified that there exists a discrete symmetry with respect to the state y and the control u. Therefore, in the rest of the paper we assume the initial condition $y_0 > 0$, as $y_0 < 0$ follows from symmetry.

In Section III, the original time-optimal problem is reduced to an equivalent problem with fewer variables for the case when $x_f \geq x_0$. We use Pontryagin's maximum principle (PMP) [4] to deduce the necessary conditions for time-optimality of the system trajectory. This gives rise to the following three sub-cases of $\alpha < 0$: $1+\alpha c > 0$, $1+\alpha c = 0$ and $1 + \alpha c < 0$. In particular, we address in detail the existence and uniqueness of the time-optimal trajectories for the sub-case $1+\alpha c > 0$. Similar analysis can be carried out for the other two sub-cases which will be given in a journal version of the paper. In Section IV, we extend our analysis for the case when $x_f < x_0$.

III. CASE WHERE $x_f \geq x_0$

For $x_f = x_0$, the control $u = 0$ with $T = 0$ is trivially the time-optimal control. When $x_f > x_0$, we have $x(0) - x(T) = -\int_0^T \dot{x}dt = -\int_0^T (yu + \alpha u^2)dt$. Therefore, reaching $x_f > x_0$ in minimum time is equivalent to minimizing $-\int_0^T (yu + \alpha u^2)dt$ for a fixed T. Hence the original time-optimal control problem with $y_0 > 0$ is equivalent to the following:

$$\underset{u \in U}{\text{minimize}} \int_0^T (-yu - \alpha u^2)dt, \tag{7}$$

subject to

$$\dot{y} = -cy + u,$$
$$y(0) = y_0 > 0, \quad y(T) = \text{ free}, \quad T > 0.$$

Applying PMP to the problem in (7), the Hamiltonian function is given by

$$H = \alpha u^2 + (y + \lambda)u - cy\lambda, \tag{8}$$

where λ is the co-state corresponding to the state y. Due to the free boundary condition on $y(T)$, the transversality condition becomes

$$\lambda(T) = 0. \tag{9}$$

By PMP, the optimal trajectory should satisfy the ODE:

$$\dot{y} = -cy + u; \qquad \dot{\lambda} = c\lambda - u; \tag{10}$$

where the optimal control u, which maximizes H at every t, is given by

$$u(t) = \begin{cases} -1 & \text{if } \lambda + y \leq 2\alpha, \\ -\frac{\lambda+y}{2\alpha} & \text{if } |\lambda + y| \leq -2\alpha, \\ 1 & \text{if } \lambda + y \geq -2\alpha. \end{cases} \tag{11}$$

We now examine the phase portraits corresponding to each of the regions in the $y\lambda-$plane to study the existence and uniqueness of $\lambda(0) = \lambda_0$ satisfying (10) and (9).

- $\boldsymbol{\lambda + y \leq 2}\alpha$: In this region $u = -1$ and hence the ODE (10) has a saddle point at $(y, \lambda) = (-\frac{1}{c}, -\frac{1}{c})$.
- $\boldsymbol{\lambda + y \geq -2}\alpha$: In this region $u = 1$ and the ODE (10) has a saddle point at $(y, \lambda) = (\frac{1}{c}, \frac{1}{c})$.
- $\boldsymbol{|\lambda + y| \leq -2}\alpha$: In this region $u = \frac{y+\lambda}{-2\alpha}$ and hence the (x, y, λ) dynamics are given by

$$\dot{x} = \frac{y^2 - \lambda^2}{-4\alpha}, \tag{12}$$

$$\begin{bmatrix} \dot{y} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -(c + \frac{1}{2\alpha}) & -\frac{1}{2\alpha} \\ \frac{1}{2\alpha} & (c + \frac{1}{2\alpha}) \end{bmatrix} \begin{bmatrix} y \\ \lambda \end{bmatrix} := A \begin{bmatrix} y \\ \lambda \end{bmatrix}. \tag{13}$$

Note that $\operatorname{tr} A = 0$ and $\det A = c(1 + \alpha c)/(-\alpha)$. Also, the equilibrium point corresponds to $(y, \lambda) = (0, 0)$ and the type of the equilibrium point depends on the sign of $\det A$. The qualitative phase portraits of the (y, λ)-dynamics can be studied under following three different cases:

$$(1 + \alpha c) > 0 \qquad (1 + \alpha c) < 0 \qquad (1 + \alpha c) = 0$$

In this paper, we restrict our analysis for the sub-case $(1 + \alpha c) > 0$. The same analysis can be extended for the remaining two sub-cases.

When $(1 + \alpha c) > 0$, $(y, \lambda) = (0, 0)$ is a fixed point. In this case, depending on the sign of $(1 + 2\alpha c)$ there are two qualitatively different phase portraits: figure 2(a) (figure 2(b)) corresponds to the phase portrait when the λ intercept of the switching line $y + \lambda = -2\alpha$ is greater (less) than $\frac{1}{c}$. In order to satisfy $y_0 > 0$ and $\lambda(T) = 0$, any optimal trajectory should have its (y_0, λ_0) belonging to the shaded region in the figure 2(a) (figure 2(b)). We will first show that for any $T > 0$ and $y_0 > 0$, there exists a unique λ_0 in the shaded region of the first quadrant such that $T = T(y_0, \lambda_0)$. In Section III-B, we will prove that there cannot be any time-optimal trajectory with $\lambda_0 < 0$. Finally in Section III-C, we will further discuss the uniqueness of these time-optimal trajectories.

Discrete symmetry: There exists a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry in the dynamics which is discussed in detail in [3]. The consequences of this symmetry are summarized here for later use in the paper. Define the following maps:

$$S_1 : (x, y, \lambda) \mapsto (x, \lambda, y)$$
$$S_2 : (x, y, \lambda) \mapsto (x, -\lambda, -y)$$
$$S_3 := S_1 \circ S_2 : (x, y, \lambda) \mapsto (x, -y, -\lambda)$$

These mappings are illustrated in figure 3. For the trajectory AB with $u = (y + \lambda)/(-2\alpha)$ in $y\lambda-$plane, we have for $i = 1, 2$

$$\Delta t_{AB} = \Delta t_{S_i(B)S_i(A)} \text{ and } \Delta x_{AB} = -\Delta x_{S_i(B)S_i(A)}.$$

While the above symmetries are true when $u = (y + \lambda)/(-2\alpha)$ (when the trajectory is in the linear region), the following symmetry holds for the case when $u = 1$ or $u = -1$:

$$\Delta t_{AB} = \Delta t_{S_3(A)S_3(B)}, \ \Delta x_{AB} = \Delta x_{S_3(A)S_3(B)}.$$

Also, from (12) we have $\dot{x} < 0$ in the linear region whenever $|y| < |\lambda|$. This region is shown shaded in figure 3.

A. *The case when* $\lambda_0 > 0$

In this subsection and the next, we assume $1 + 2\alpha c \leq 0$. The analysis can be extended to the case of $1 + 2\alpha c > 0$. In order to satisfy $y_0 > 0$ and $\lambda(T) = 0$, any optimal trajectory originating from the shaded region of the first quadrant should have its (y_0, λ_0) belonging to any one the region R_1 through R_4 as in figure 4. The four regions in the first quadrant are defined as follows:

$R_1 : y \geq -2\alpha$, $0 < \lambda < \frac{2\alpha+y}{-1+cy}$

$$R_2 : \begin{cases} 0 < \lambda < \lambda_{\text{up2}}(y) & \text{if } 0 < y \leq -\alpha, \\ 0 < \lambda < -2\alpha - y & \text{if } -\alpha < y < -2\alpha. \end{cases}$$

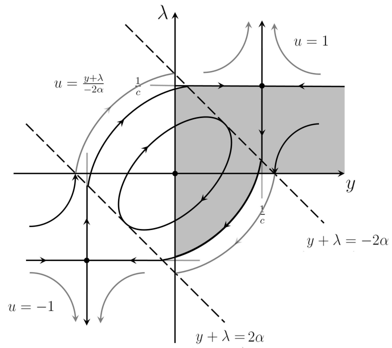
$R_3 : y > -\frac{1+2\alpha c}{c}$, $\lambda > 0$,
$\max\left\{-2\alpha - y, \frac{2\alpha+y}{-1+cy}\right\} \leq \lambda < \frac{1}{c}$.

$R_4 : 0 < y < -\alpha$,
$\lambda_{\text{up2}}(y) \leq \lambda < \min\left\{\lambda_{\text{up4}}(y), -2\alpha - y\right\}$.

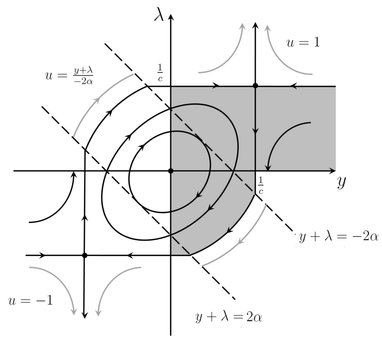
where $\lambda_{\text{up2}}(y)$ and $\lambda_{\text{up4}}(y)$ are respectively the upper boundaries of R_2 and R_4 in the linear region of the $y\lambda-$plane and are given by

$$\lambda_{\text{up2}}(y) = -(1 + 2\alpha c)y + 2\sqrt{\alpha(1 + \alpha c)(\alpha + cy^2)},$$
$$\lambda_{\text{up4}}(y) = -(1 + 2\alpha c)y + 2\sqrt{\frac{\alpha(1 + \alpha c)\left(-1 + c^2y^2\right)}{c}}.$$

Trajectories starting from a point in region R_1**:** Let $(y(t), \lambda(t))$ be a trajectory starting from $(y_0, \lambda_0) \in R_1$. Let T_1 be the time taken such that $\lambda(T_1) = 0$ for the first time.



(a) $(1+2\alpha c)\leq 0$.



(b) $(1+2\alpha c)>0$.

Fig. 2. Phase portrait in $y\lambda-$plane: With $y_0>0$, any trajectory must originate from the shaded region in order to satisfy the transversality condition.

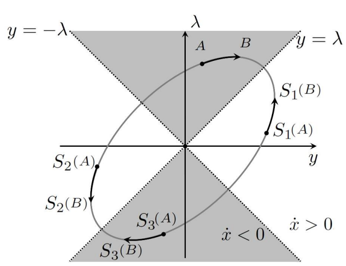


Fig. 3. $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry in the linear region.

Let X_1 be the corresponding increase in state x. Then we have

$$T_1(y_0,\lambda_0)=\int_{\lambda_0}^{0}\frac{d\lambda}{c\lambda-1}=-\frac{1}{c}\log[1-c\lambda_0].$$

It is straightforward from the expression for T_1 that

$$\lim_{\lambda_0\to 0^+}T_1(y_0,\lambda_0)=0.$$

Since $\lambda_0<\frac{1}{c}$, we have

$$\frac{\partial}{\partial\lambda_0}T_1(y_0,\lambda_0)=\frac{1}{1-c\lambda_0}>0,$$

for $(y_0,\lambda_0)\in R_1$. Therefore $T_1(y_0,\lambda_0)$ is a strictly increasing continuous function of λ_0 for each fixed y_0 and is positive valued.

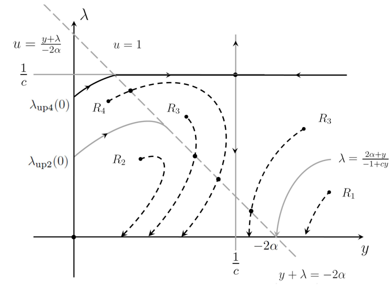


Fig. 4. Regions R_1-R_4 in the phase portrait for the case when $1+2\alpha c\leq 0$.

Since $u(t)=1$ and $y(t)=e^{-ct}y_0+\frac{1}{c}(1-e^{-ct})$ in R_1, we have

$$\begin{aligned}X_1(y_0,\lambda_0)&=\int_0^{T_1}(y(t)+\alpha)dt\\&=\frac{\lambda_0(-1+cy_0)}{c}-\frac{(1+ac)}{c^2}\log[1-c\lambda_0].\end{aligned}$$

Note that

$$\lim_{\lambda_0\to 0^+}X_1(y_0,\lambda_0)=0.$$

Since $0<\lambda_0<\frac{y_0+2\alpha}{-1+cy_0}$ and $y_0\geq\frac{1}{c}$, we have

$$\frac{\partial}{\partial\lambda_0}X_1(y_0,\lambda_0)=\frac{(1+\alpha c)}{c(1-c\lambda_0)}+\frac{(-1+cy_0)}{c}>0,$$

for $(y_0,\lambda_0)\in R_1$. Therefore the function $X_1(y_0,\lambda_0)$ is a strictly increasing continuous function of λ_0 for each fixed y_0 and is positive valued.

Trajectories starting from a point in region R_2: Let $(y(t),\lambda(t))$ be a trajectory with $(y_0,\lambda_0)\in R_2$. Since, in this region $u(t)=\frac{y(t)+\lambda(t)}{-2\alpha}$, we obtain

$$\begin{aligned}y(t)&=y_0\cos(\omega t)-\frac{(\lambda_0+y_0+2\alpha cy_0)\sin(\omega t)}{2\alpha\omega},\\\lambda(t)&=\lambda_0\cos(\omega t)+\frac{(\lambda_0+2\alpha c\lambda_0+y_0)\sin(\omega t)}{2\alpha\omega},\end{aligned}$$

where $\omega=\sqrt{-\frac{c(1+\alpha c)}{\alpha}}$. Then the expression for $T_2(y_0,\lambda_0)$ is obtained by solving $\lambda(T_2)=0$:

$$T_2(y_0,\lambda_0)=\frac{1}{\omega}\tan^{-1}\left(\frac{-2\alpha\omega\lambda_0}{\lambda_0+2\alpha c\lambda_0+y_0}\right).$$

It is easy to verify

$$\lim_{\lambda_0\to 0^+}T_2(y_0,\lambda_0)=0\text{ and }\frac{\partial}{\partial\lambda_0}T_2(y_0,\lambda_0)>0.$$

Hence, the function $T_2(y_0,\lambda_0)$ is a strictly increasing continuous function of λ_0 for each fixed y_0 and is positive valued.

The increment in x, due to the control $u=\frac{y+\lambda}{-2\alpha}$ is given by

$$X_2(y_0,\lambda_0)=\int_0^{T_2}\frac{y(t)^2-\lambda(t)^2}{-4\alpha}dt=\frac{1}{2}y_0\lambda_0.$$

Notice that $X_2(y_0, \lambda_0)$ is a strictly increasing continuous function of λ_0 for each fixed $y_0 > 0$. At boundary points of R_2, $X_2(y_0, 0) = 0$ and

$$\lim_{\lambda_0 \to -2\alpha - y_0} X_2(y_0, \lambda_0) = -\frac{1}{2} y_0 (2\alpha + y_0)$$

which is positive but finite.

Trajectories starting from a point in region R_3: Let $(y(t), \lambda(t))$ be a trajectory with $(y_0, \lambda_0) \in R_3$. In this region $u(t) = 1$ till trajectory hits the switching line $y + \lambda = -2\alpha$ after which $u(t) = \frac{y(t)+\lambda(t)}{-2\alpha}$ till the transversality condition is satisfied. Let (y_{S3}, λ_{S3}) be the intersection of the orbit in R_3 and the switching line $y + \lambda = -2\alpha$, which is given by

$$y_{S3}(y_0, \lambda_0) = -\alpha + \frac{\sqrt{(1+\alpha c)^2 - (-1 + c\lambda_0)(-1 + cy_0)}}{c},$$
$$\lambda_{S3}(y_0, \lambda_0) = -\alpha - \frac{\sqrt{(1+\alpha c)^2 - (-1 + c\lambda_0)(-1 + cy_0)}}{c}.$$

Let $T_{S3}(y_0, \lambda_0)$ be the switching time. Then we obtain the expression for $T_{S3}(y_0, \lambda_0)$ as

$$T_{S3}(y_0, \lambda_0) = \int_{\lambda_0}^{\lambda_{S3}} \frac{d\lambda}{c\lambda - 1} = -\frac{1}{c} \log \left[\frac{-1 + c\lambda_0}{-1 + c\lambda_{S3}} \right].$$

Then $T_3(y_0, \lambda_0)$ is given by

$$T_3(y_0, \lambda_0) = T_{S3}(y_0, \lambda_0) + T_2(y_{S3}, \lambda_{S3}).$$

Therefore, we have

$$\frac{\partial}{\partial \lambda_0} T_3(y_0, \lambda_0) > 0.$$

Therefore, $T_3(y_0, \lambda_0)$ is strictly increasing continuous function of λ_0. It is easy to verify that at the boundary points of R_3

$$T_3\left(y_0, \frac{2\alpha + y_0}{-1 + cy_0}\right) = T_1\left(y_0, \frac{2\alpha + y_0}{-1 + cy_0}\right) > 0,$$
$$T_3(y_0, -y_0 - 2\alpha) = T_2(y_0, -y_0 - 2\alpha) > 0,$$
$$\lim_{\lambda_0 \to \frac{1}{c}} T_3(y_0, \lambda_0) = \infty.$$

Hence, $T_3(y_0, \lambda_0) > 0$ for every $(y_0, \lambda_0) \in R_3$.

Let $X_3(y_0, \lambda_0)$ be the distance traveled in time $T_3(y_0, \lambda_0)$. Then $X_3(y_0, \lambda_0)$ is given by

$$\begin{aligned} X_3(y_0, \lambda_0) &= \int_0^{T_{S3}(y_0, \lambda_0)} (y(t) + \alpha) dt + X_2(y_{S3}, \lambda_{S3}) \\ &= -\frac{1 + \alpha c}{c^2} \log \left[\frac{-1 + c\lambda_0}{-1 + c\lambda_{S3}} \right] \\ &+ \frac{(-\lambda_0 + \lambda_{S3})(-1 + cy_0)}{c(-1 + c\lambda_{S3})} + \frac{1}{2} y_{S3} \lambda_{S3}. \end{aligned}$$

A straightforward computation yield

$$\frac{\partial}{\partial \lambda_0} X_3(y_0, \lambda_0) > 0.$$

Hence, $X_3(y_0, \lambda_0)$ is a strictly increasing continuous function of λ_0. It can be verified that at the boundary points

$$X_3\left(y_0, \frac{2\alpha + y_0}{-1 + cy_0}\right) = X_1\left(y_0, \frac{2\alpha + y_0}{-1 + cy_0}\right) > 0,$$
$$X_3(y_0, -2\alpha - y_0) = X_2(y_0, -2\alpha - y_0) > 0,$$
$$\lim_{\lambda_0 \to \frac{1}{c}} X_3(y_0, \lambda_0) = \infty.$$

Hence, $X_3(y_0, \lambda_0) > 0$ for every $(y_0, \lambda_0) \in R_3$.

Trajectories starting from a point in region R_4: Let $(y(t), \lambda(t))$ be a trajectory with $(y_0, \lambda_0) \in R_4$. In this region $u(t) = \frac{y+\lambda}{-2\alpha}$ till trajectory hits the switching line $y + \lambda = -2\alpha$ at (y_{S4}, λ_{S4}) for the first time, after which trajectory is same as that in R_3 with $y_0 = y_{S4}$ and $\lambda_0 = \lambda_{S4}$. Expression for (y_{S4}, λ_{S4}) is given by

$$y_{S4}(y_0, \lambda_0) = -\alpha - \sqrt{\alpha\left(\alpha + \frac{1}{c}\right) - \lambda_0 y_0 - \frac{(\lambda_0 + y_0)^2}{4\alpha c}}, \tag{14}$$

$$\lambda_{S4}(y_0, \lambda_0) = -\alpha + \sqrt{\alpha\left(\alpha + \frac{1}{c}\right) - \lambda_0 y_0 - \frac{(\lambda_0 + y_0)^2}{4\alpha c}}. \tag{15}$$

Let $T_{S4}(y_0, \lambda_0)$ be the switching time. Because of S_1 symmetry, we obtain the expression for $T_4(y_0, \lambda_0)$ as

$$\begin{aligned} &T_4(y_0, \lambda_0) = T_{S4}(y_0, \lambda_0) + T_3(y_{S4}, \lambda_{S4}) \\ &= \frac{1}{\omega}\left(\tan^{-1}\left(\frac{2y_0\alpha\omega}{2cy_0\alpha + y_0 + \lambda_0}\right) + 2\tan^{-1}\left(\frac{y_{S4}\omega}{1 - cy_{S4}}\right)\right) \\ &+ \frac{1}{c}\log\left(\frac{1 - cy_{S4}}{cy_{S4} + 2c\alpha + 1}\right). \end{aligned}$$

Straightforward computation yields

$$\frac{\partial}{\partial \lambda_0} T_4(y_0, \lambda_0) > 0.$$

Therefore $T_4(y_0, \lambda_0)$ is a strictly increasing continuous function of λ_0. It can be verified at the boundary points of R_4 that

$$T_4\Big(y_0, \lambda_{\mathrm{up2}}(y_0)\Big) = T_2(y_0, \lambda_{\mathrm{up2}}(y_0)) > 0, \tag{16}$$
$$T_4(y_0, -2\alpha - y_0) = T_3(y_0, -2\alpha - y_0) > 0, \tag{17}$$
$$\lim_{\lambda_0 \to \lambda_{\mathrm{up4}}(y_0)^-} T_4(y_0, \lambda_0) = \infty.$$

The positive monotonicity of T_4 together with (16) and (17) implies that $T_4(y_0, \lambda_0) > 0$ for every $(y_0, \lambda_0) \in R_4$.

Let $X_4(y_0, \lambda_0)$ be the distance traveled in time $T_4(y_0, \lambda_0)$. Then $X_4(y_0, \lambda_0)$ is given by

$$\begin{aligned} X_4(y_0, \lambda_0) &= X_2(\lambda_0, y_0) - X_2(\lambda_{S4}, y_{S4}) + X_3(y_{S4}, \lambda_{S4}) \\ &= \frac{\lambda_0 y_0}{2} + \frac{(-\lambda_{S4} + y_{S4})}{c} \\ &- \frac{(1 + \alpha c)}{c^2} \log\left[\frac{-1 + c\lambda_{S4}}{-1 + cy_{S4}}\right]. \end{aligned}$$

Straightforward computation yields

$$\frac{\partial}{\partial \lambda_0} X_4(y_0, \lambda_0) = \frac{y_0}{2} + \frac{(\lambda_0 + y_0 + 2\alpha c y_0)(\alpha + y_{S4})}{2\alpha(-1 + c\lambda_{S4})(-1 + cy_{S4})} > 0.$$

Therefore $X_4(y_0, \lambda_0)$ is a strictly increasing continuous function of λ_0. It can be verified at the boundary points of R_4

$$X_4\Big(y_0, \lambda_{\mathrm{up2}}(y_0)\Big) = X_2\left(y_0, \lambda_{\mathrm{up2}}(y_0)\right) > 0, \quad (18)$$

$$X_4(y_0, -2\alpha - y_0) = X_3\left(y_0, -2\alpha - y_0\right) > 0. \quad (19)$$

$$\lim_{\lambda_0 \to \lambda_{\mathrm{up4}}(y_0)^-} X_4(y_0, \lambda_0) = \infty.$$

The positive monotonicity of X_4 together with (18) and (19) implies that $X_4(y_0, \lambda_0) > 0$ for every $(y_0, \lambda_0) \in R_4$.

We now put together the analysis in the four separate regions R_1, R_2, R_3 and R_4 for the case when $1 + 2\alpha c \leq 0$. Let us define $\lambda_{\mathrm{up}}(y)$ for $y > 0$ as follows

$$\lambda_{\mathrm{up}}(y) = \begin{cases} \lambda_{\mathrm{up4}}(y) & \text{if } 0 < y \leq -\frac{1+2\alpha c}{c}, \\ \frac{1}{c} & \text{if } y > -\frac{1+2\alpha c}{c}. \end{cases}$$

Note that the $\lambda_{\mathrm{up}}(y)$ is the upper λ boundary of the region $R_1 \cup R_2 \cup R_3 \cup R_4$. Define the time function and the corresponding distance function as follows:

$$T(y_0, \lambda_0) = T_i(y_0, \lambda_0) \quad \text{if } (y_0, \lambda_0) \in R_i \quad i = 1, 2, 3, 4.$$

$$X(y_0, \lambda_0) = X_i(y_0, \lambda_0) \quad \text{if } (y_0, \lambda_0) \in R_i \quad i = 1, 2, 3, 4.$$

We have shown that in each region, *the time T and the corresponding distance function X are positive continuous functions and are strictly increasing functions of $\lambda_0 \in (0, \lambda_{\mathrm{up}}(y_0))$ for each fixed $y_0 > 0$.* Also, we have shown that $\lim_{\lambda_0 \to 0^+} T(y_0, \lambda_0) = \lim_{\lambda_0 \to 0^+} X(y_0, \lambda_0) = 0$, while

$$\lim_{\lambda_0 \to \lambda_{\mathrm{up}}(y_0)^-} T(y_0, \lambda_0) = \infty,$$

$$\lim_{\lambda_0 \to \lambda_{\mathrm{up}}(y_0)^-} X(y_0, \lambda_0) = X(y_0, \lambda_{\mathrm{up}}(y_0)) = \infty.$$

With this analysis we can conclude the following :

Theorem III.1 *Given any $T > 0$ and $y_0 > 0$, there exists a unique one-shot extremal with $\lambda_0 \in (0, \lambda_{\mathrm{up}}(y_0))$ such that $T = T(y_0, \lambda_0)$. Also, given any $y_0 > 0$ and $0 < x < \infty$, there exists a unique one-shot extremal with $\lambda_0 \in (0, \lambda_{\mathrm{up}}(y_0))$ such that $x = X(y_0, \lambda_0)$.*

For formal definitions of one-shot and multi-shot extremals, we refer [3].

B. The case when $\lambda_0 < 0$

In this section we will show that the trajectories with $\lambda_0 < 0$ are not time-optimal. This will allow us to extend the validity of Theorem III.1 for $\lambda_0 < 0$. Before we claim this and prove, we will state a lemma which will be used later in the proof.

Lemma III.2 *Let $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ be two solutions for the states x and y in (1)-(2) with the control $u = 1$ and initial conditions (x_0, y_{01}) and (x_0, y_{02}) respectively, such that $y_{01} > y_{02}$. Then $x_1(T) > x_2(T)$, for any $T > 0$.*

Proof: The result follows from the fact that $y_1(t) > y_2(t)$, $\forall t \geq 0$. ■

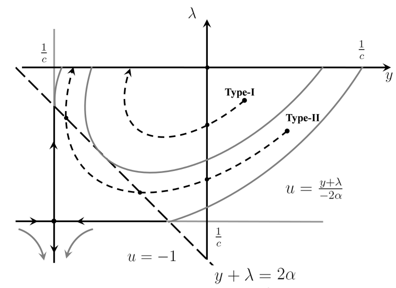


Fig. 5. $\lambda_0 < 0$: Type-I and Type II trajectories for the case when $1+2\alpha c \leq 0$.

Claim III.1 *There cannot be any time-optimal extremal with $y_0 > 0$ and $\lambda_0 < 0$.*

Proof: With $\lambda_0 < 0$, we can have qualitatively two different types of trajectories for the case when $1+2\alpha c \leq 0$, based on the region from which the trajectory originates.This has been illustrated in figure 5 .

Type-I trajectories: A type-I trajectory originates from the linear region of the fourth quadrant in the $y\lambda-$plane and continue to remain in the linear region until it satisfies the transversality condition (see figure 5). Note that by S_1 and S_2 symmetries, the total increment in x is negative. Hence, type-I trajectories cannot be time-optimal.

Type-II trajectories: A type-II trajectory originates from the linear region of the fourth quadrant in the $y\lambda-$plane, but switch to the nonlinear region in the third quadrant and then switches back to the linear region in the same quadrant before it satisfies the transversality condition (see figure 5). Let $ABCDE$ be a typical type-II trajectory with the initial condition (y_0, λ_0^-), as in figure 6(a) or figure 6(b). Assume that this trajectory is time-optimal with the control u in (11) for $x_f > x_0$. Let $B'C'D'E'$ be the trajectory such that $S_3(BCDE) = B'C'D'E'$. Hence, the increment in x due to control u is given by

$$\begin{aligned}\Delta x_{ABCDE} &= \Delta x_{AB} + \Delta x_{BCDE} \\ &= \Delta x_{AB} + \Delta x_{B'C'D'E'} < \Delta x_{B'C'D'E'}, \quad (20)\end{aligned}$$

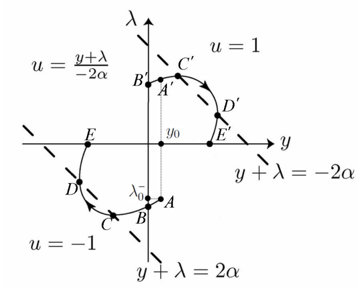
where the inequality is true because $\Delta x_{AB} < 0$ (as illustrated in figure 3). Let $y_{C'}$ be the value of the state y when the trajectory is at C' (we will use this convention in the rest of the paper to denote the value of a state at any particular point on the trajectory). Then we have two possibilities: $y_0 < y_{C'}$ or $y_0 \geq y_{C'}$. The phase portraits for these two cases are shown in figure 6(a) and figure 6(b).

Suppose $y_0 < y_{C'}$ (see figure 6(a)). Let A' be the point on the trajectory $B'C'D'E'$ when the trajectory intersects with $y = y_0$ line. Since $\dot{x} < 0$ along $B'C'$,

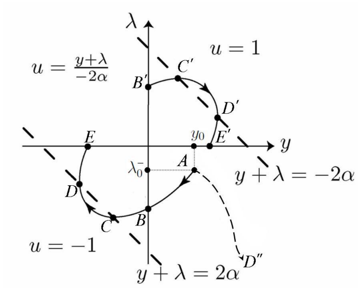
$$\Delta x_{B'C'D'E'} < \Delta x_{A'C'D'E'}. \quad (21)$$

From (20) and (21), we have

$$\Delta x_{ABCDE} < \Delta x_{A'C'D'E'}. \quad (22)$$



(a) $y_0 < y_{C'}$.



(b) $y_0 \geq y_{C'}$.

Fig. 6. $\lambda_0 < 0$: Type-II trajectories are not time-optimal.

It is also straightforward to show that

$$\Delta t_{ABCDE} = \Delta t_{AB} + \Delta t_{B'C'D'E'} > \Delta t_{A'C'D'E'}. \quad (23)$$

From (22) and (23), the trajectory $A'C'D'E'$ with $\lambda_0 = \lambda_{A'}$ reaches x_f before the trajectory $ABCDE$, which contradicts the time-optimality of the trajectory $ABCDE$ with $\lambda_0 = \lambda_0^-$. Hence, type-II trajectories with $y_0 < y_{C'}$ cannot be time-optimal.

Suppose $y_0 \geq y_{C'}$ (see figure 6(b)). From (20), we have

$$\Delta x_{ABCDE} < \Delta x_{B'C'D'E'} = \Delta x_{C'D'}. \quad (24)$$

Let us construct another control v as follows

$$v(t) = 1, \qquad t \in [0, \Delta t_{C'D'}],$$

where $\Delta t_{C'D'}$ is the time taken by the trajectory to reach point D' from C'. Let AD'' be the trajectory corresponding to the control v. Let $\Delta x_{AD''}$ be the corresponding increment in x . Since $y_0 \geq y_{C'}$, we have from Lemma III.2

$$\Delta x_{AD''}(v) \geq \Delta x_{C'D'}. \quad (25)$$

From (24) and (25), we have $\Delta x_{ABCDE} < \Delta x_{AD''}(v)$. Also, we have $\Delta t_{ABCDE} > \Delta t_{CD} = \Delta t_{C'D'}$. Therefore, the trajectory AD'' with control v reaches x_f before the trajectory $ABCDE$, which contradicts the time-optimality of the trajectory $ABCDE$. Hence, type-II trajectories with $y_0 \geq y_{C'}$ cannot be time-optimal. ∎

C. *Multi-shot trajectories and uniqueness*

By the discussions in Section III-A and Section III-B, it is clear that for a given $x_f > x_0$ and $y_0 > 0$, there is a unique one-shot extremal with $(y_0, \lambda_0) \in R_1 \cup R_2 \cup R_3 \cup R_4$ such that $x_f - x_0 = X(y_0, \lambda_0)$. When $1 + 2\alpha c > 0$, we can similarly show that there exists a unique one-shot extremal with $\lambda_0 > 0$ for a given $x_f > x_0$ and $y_0 > 0$. Here we do not discuss this in detail since the methodology is very similar to that of the case when $1 + 2\alpha c \leq 0$. In either case ($1 + 2\alpha c \leq 0$ or $1 + 2\alpha c > 0$), we can show that no multi-shot extremals are time-optimal if $3 + 4\alpha c \geq 0$. The proof is similar to the case when $y_0 = 0$ discussed in [3]. If $3 + 4\alpha c < 0$ we have no general proof that only one-shot extremals are time-optimal. With this we state the following result for the time-optimal control problem stated in Section II for the DEP system with $1 + \alpha c > 0$:

Theorem III.3 *If $3 + 4\alpha c \geq 0$, then given $y_0 > 0$, a one-shot extremal with $0 < \lambda_0 < \lambda_{\text{up}}(y_0)$ as stated in Theorem III.1 is the unique time-optimal trajectory.*

IV. Case where $x_f < x_0$

Similar to the case when $x_f > x_0$, by application of the PMP, one can solve the time-optimal control problem for the case where $x_f < x_0$ and $y_0 > 0$ as follows:

Theorem IV.1 *Let*

$$T(y_0, \lambda_0) = \frac{1}{c} ln\Big(\frac{1}{1 - c\lambda_0}\Big),$$

$$X(y_0, \lambda_0) = \frac{1 + \alpha c}{c^2} ln\left(\frac{1}{1 - c\lambda_0}\right) - \frac{1 + cy_0}{c}\lambda_0,$$

where $\lambda_0 \in (0, \lambda_{\text{up}}(y_0))$, with $\lambda_{\text{up}}(y) = \frac{y}{cy+1}$. If $x_f - x_0 < X(y_0, \lambda_{\text{up}}(y_0))$, then there is no optimal control. If $x_f - x_0 \geq X(y_0, \lambda_{\text{up}}(y_0))$, then there is a unique optimal control given by

$$u(t) = \begin{cases} -1 & \text{if } \lambda - y \leq 0, \\ 1 & \text{if } \lambda - y \geq 0. \end{cases}$$

where y and λ are the solutions of the ODE

$$\dot{y} = -cy + u; \qquad \dot{\lambda} = c\lambda + u.$$

with $\lambda_0 \in (0, \lambda_{\text{up}}(y_0))$ uniquely computed by solving $x_f - x_0 = X(y_0, \lambda_0)$. Then the optimal time $T = T(y_0, \lambda_0)$.

V. CONCLUSIONS AND FUTURE WORK

We addressed certain aspects of existence and uniqueness of the time-optimal trajectories of a particle in a dielectrophoretic system for the case where $\alpha < 0$ and $c > 0$ with non-zero y_0. Original time-optimal control problem was reduced to an equivalent problem with fewer variable which simplified our analysis. Discrete symmetry in the dynamics further simplified the analysis. As a future work, we plan to consider the time-optimal control of several particles suspended in a medium.

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