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Representation of a general composition of Dirac structures

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Abstract— **We provide explicit representations for the Dirac structure obtained from an arbitrary number of component Dirac structures coupled by means of another interconnecting Dirac structure. Our work generalizes the results in [1] in two aspects. First, the interconnecting structure is not limited to the simple feedback case considered there, and this opens new possibilities for designing control systems. Second, the number of simultaneously interconnected systems is not limited to two, which allows for extra flexibility in modeling, particularly in the case of electrical networks. Several relevant particular cases are presented, and the application to the interconnection of port-Hamiltonian systems is discussed by means of an example.**

I. INTRODUCTION

Dirac structures [2][3][4][5] provide a very elegant encoding of energy conservation for interconnected systems. In particular, they allow the formulation of generalized port-Hamiltonian systems (PHS), which have been shown to describe a wide variety of both lumped parameter and distributed parameter systems [6][7][5]. PHS are not only useful for modeling and system analysis, but also for control, since passivity-based control techniques with a natural interpretation can be applied to them [8][9][10]. Dirac structures and PHS are also the rigorous mathematical framework underlying bond graph theory [11][5].

PHS yield themselves to the description of complex systems made of many open subsystems, since it can be shown that the appropriate (essentially, power preserving) interconnection of PHS is again a PHS. However, even if the individual PHS subsystems are given as a set of differential equations, the resulting PHS constitutes, in general, a set of differential-algebraic equations (DAE). The practical issue of obtaining the actual representation of the resulting PHS, that is, of its Dirac structure, was addressed in [1] for a simple case, and the generalization of the results presented there is the main objective of this work.

The paper is organized as follows. Section II presents the basic definitions concerning Dirac structures and their representations, and how a PHS can be defined from a Dirac structure, and also reviews one of the fundamental results in [1], which provides explicit representations for the Dirac structure resulting from the interconnection of two of them by means of a simple feedback law. Sections III, IV and V contain the original contributions of the paper. In Section III a general interconnection of an arbitrary number of Dirac structures by means of another one is introduced, and the resulting object is shown to be a Dirac structure, using the same technique as in [1]. In Section IV, explicit kernel and image representations of the resulting structure are constructed. This is the main result of the paper, generalizing the special case treated in [1]. Section V provides several particular examples of general interest, and the application of the results of the paper to the interconnection of several port-Hamiltonian systems is presented by means of an example in Section VI. Finally, Section VII summarizes the results and discusses some of the avenues for future work.

Throughout the paper vectors are understood as column vectors, and ^{T} denotes the transpose. $A \in \mathbb{R}^{m\times n}$ means that A is a real matrix with m rows and n columns. If A and B are matrices with the same number of rows, expressions like $(A \;\; B)$ denote the matrix obtained by putting them together along the columns. $\mathcal{N}(A)$ and $\mathcal{R}(A)$ stand for the kernel and column spaces of matrix A. An identity matrix of appropriate dimension is denoted by $\mathbb{I}$, but zero matrices are written simply as 0. $\partial_x H$ is a row vector containing the partial derivatives of $H(x)$, while individual partial derivatives are written as $\partial_{x_i} H$.

II. DIRAC STRUCTURES

Consider $\mathcal{F}$, a finite-dimensional vector space, called the space of *flows*, and $\mathcal{F}^*$, its dual space, called the space of *efforts*. We denote by $\langle e|f\rangle$ the action of the form $e \in \mathcal{F}^*$ on the vector $f \in \mathcal{F}$. e and f are known as power variables, since $\langle e|f\rangle$ has dimensions of power.

An indefinite, non-degenerate symmetric bilinear form can be defined on $\mathcal{F}\times\mathcal{F}^*$ by means of

$$\lessdot(f^a,e^a)|(f^b,e^b)\gtrdot = \langle e^a|f^b\rangle + \langle e^b|f^a\rangle. \tag{1}$$

A constant *Dirac structure* on $\mathcal{F}\times\mathcal{F}^*$ is a subspace $\mathcal{D}\subset \mathcal{F}\times\mathcal{F}^*$ such that

$$\mathcal{D} = \mathcal{D}^\perp, \tag{2}$$

with $^\perp$ the orthogonal complement with respect to $\lessdot\ |\ \gtrdot$. If the components of vectors and forms are given with respect to dual basis, then $\langle e|f\rangle = e^T f$, and we can write

$$\mathcal{D}^\perp = \left\{(\tilde f,\tilde e)\in\mathcal{F}\times\mathcal{F}^* \mid \tilde e^T f + e^T\tilde f = 0 \;\forall\; (f,e)\in\mathcal{D}\right\}. \tag{3}$$

If follows immediately from the definition of $\mathcal{D}$ that, for any $(f,e)\in\mathcal{D}$

$$\langle e|f\rangle = 0, \tag{4}$$

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which encodes the energy conservation property of Dirac structures, also known as *power continuity*. In this sense, Dirac structures are a generalization of Kirchoff's laws of circuit theory, which allow to deduce that conforming currents i and voltages v satisfy Tellegen's theorem $v^T i = 0$.

Every Dirac structure $\mathcal{D}$ admits a *kernel representation*

$$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* \mid Ff + Ee = 0\}, \tag{5}$$

for linear maps $F : \mathcal{F} \to \mathcal{V}$ and $E : \mathcal{F}^* \to \mathcal{V}$ satisfying

$$\begin{aligned} EF^* + FE^* &= 0 & (6)\\ \text{rank } (F + E) &= \dim \mathcal{F}, & (7) \end{aligned}$$

where $\mathcal{V}$ is a vector space with the same dimension as $\mathcal{F}$, and the adjoint linear maps $F^* : \mathcal{V}^* \to \mathcal{F}^*$ and $E^* : \mathcal{V}^* \to \mathcal{F}$ are defined by $\langle F^*u|f\rangle = \langle u|Ff\rangle$, and $\langle e|E^*u\rangle = \langle Ee|u\rangle$ for all $f \in \mathcal{F}, e \in \mathcal{F}^*, u \in \mathcal{V}^*$.

Dirac structures can also be given by means of an *image representation*

$$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* \mid \exists u \in \mathcal{V}^* \text{ s.t. } f = E^*u,\ e = F^*u\} \tag{8}$$

with the same maps and spaces of the kernel representation. If $\dim \mathcal{V} > \dim \mathcal{F}$ one has a *relaxed kernel representation* or a *relaxed image representation*.

Given a basis $\{f_1, \ldots, f_n\}$ in $\mathcal{F}$, the corresponding dual basis $\{e_1, \ldots, e_n\}$ in $\mathcal{F}^*$, and any basis in $\mathcal{V}$, with $\dim \mathcal{V} = m \geq n$, the linear maps F and E are represented by $m \times n$ matrices (which we denote by the same symbol as the maps) satisfying

$$\begin{aligned} EF^T + FE^T &= 0, & (9)\\ \text{rank } (F\ E) &= n. & (10) \end{aligned}$$

Kernel and image representations are then given by (5) and (8), with the adjoint maps replaced by the corresponding transpose matrices.

A port-Hamiltonian system can be defined from a Dirac structure if some of the flows and efforts are identified with time derivatives and generalized forces of the states of a system. Consider a lumped-parameter physical system defined on a manifold $\mathcal{M}$, with local coordinates $x \in \mathbb{R}^n$. The total energy of the system is given by the Hamiltonian function $H(x)$, and the system is assumed to have m open ports. For each x we consider $\mathcal{F}_x = T_x\mathcal{M} \times \mathbb{R}^m$ and $\mathcal{F}_x^* = T_x^*\mathcal{M} \times \mathbb{R}^m$, and define a Dirac structure at each $x \in \mathcal{M}$, $\mathcal{D}(x) \subset \mathcal{F}_x \times \mathcal{F}_x^*$. It is assumed that the Dirac structure varies smoothly over $\mathcal{M}$, which means, essentially, that the matrices of any given representation are smooth functions of x (see [3] for more details). We will write the elements of $\mathcal{F}_x \times \mathcal{F}_x^*$ as (f_x, f_b, e_x, e_b), where b stands for "boundary", and the f_x and e_x are power variables associated to state ports, *i.e.* ports connected to energy storing elements.

A port-Hamiltonian system on $\mathcal{M}$ can be defined, pointwise in $\mathcal{M}$, by[1]

$$\left(-\dot{x}, f_b, \partial_x^T H, e_b\right) \in \mathcal{D}(x)\ \forall\ x \in \mathcal{M}. \tag{11}$$

[1]See [5] for a global formulation in terms of Whitney sums.

The minus sign in $f_x = -\dot{x}$ is consistent with the common convention that power flows from the boundary ports into the system and from the internal network into the energy storing elements. Indeed, using (4),

$$0 = \langle e|f\rangle = \langle e_x|f_x\rangle + \langle e_b|f_b\rangle = -\partial_x H\dot{x} + e_b^T f_b, \tag{12}$$

from which $\dot{H} = e_b^T f_b$.

Equation (11) is, in general, a set of differential and algebraic equations (DAE), and may include the definition of some boundary variables as inputs or outputs. Source or dissipative terms can be added to the system through the boundary ports.

In [3] it was shown that if some of the efforts and flows of two Dirac structures are connected by means of a relationship that is itself a Dirac structure, the resulting subspace is again a Dirac structure. However the proof was not constructive and no general explicit representations of the resulting structure were provided.

In [1] the interconnection of two Dirac structures by means of a simple feedback relationship is considered, and it is shown, using a different approach, that the result is again a Dirac structure, for which explicit kernel/image representations are computed. The remaining of this Section provides a summary of the relevant results in [1].

Given Dirac structures $\mathcal{D}_A$ on the flow space $\mathcal{F}_{AT} = \mathcal{F}_A \times \mathcal{F}_{AI}$ and $\mathcal{D}_B$ on $\mathcal{F}_{BT} = \mathcal{F}_B \times \mathcal{F}_{BI}$, with $\dim \mathcal{F}_{AI} = \dim \mathcal{F}_{BI}$, a *feedback interconnection* is established by means of

$$\begin{aligned} f_{AI} &= -f_{BI},\ \forall\ f_{AI} \in \mathcal{F}_{AI},\ f_{BI} \in \mathcal{F}_{BI},\\ e_{AI} &= e_{BI},\ \forall\ e_{AI} \in \mathcal{F}_{AI}^*,\ e_{BI} \in \mathcal{F}_{BI}^*. \end{aligned} \tag{13}$$

This induces a *feedback composition* of $\mathcal{D}_A$ and $\mathcal{D}_B$ given by

$$\begin{aligned} \mathcal{D}_A || \mathcal{D}_B = \ &\{(f_A, e_A, f_B, e_B) \in \mathcal{F}_A \times \mathcal{F}_A^* \times \mathcal{F}_B \times \mathcal{F}_B^* \mid\\ &\exists\ (f_I, e_I) \text{ such that } (f_A, e_A, f_I, e_I) \in \mathcal{D}_A\\ &\text{and } (f_B, e_B, -f_I, e_I) \in \mathcal{D}_B\ \} \end{aligned} \tag{14}$$

The following result is then proven ([1], Theorem 3):

Theorem 2.1: $\mathcal{D}_A || \mathcal{D}_B$ is a Dirac structure.

The proof takes advantage of the following linear algebra result, which will be also used in this paper.

Lemma 2.2: For any matrix $A \in \mathbb{R}^{m\times n}$ and vector $b \in \mathbb{R}^m$ one has

$$\begin{gathered} \exists \lambda \in \mathbb{R}^n \text{ such that } A\lambda = b\\ \Updownarrow\\ \forall \alpha \in \mathbb{R}^m \text{ such that } A^T\alpha = 0 \Rightarrow \alpha^T b = 0 \end{gathered} \tag{15}$$

The non-trivial part of Lemma 2.2 follows from the orthogonal decomposition

$$\mathcal{R}(A) \oplus \mathcal{N}(A^T) = \mathbb{R}^m, \tag{16}$$

valid for any matrix $A \in \mathbb{R}^{m\times n}$.

Furthermore, explicit kernel/image representations for $\mathcal{D} = \mathcal{D}_A || \mathcal{D}_B$ can be constructed. If $\mathcal{D}_A$ and $\mathcal{D}_B$ have kernel/image representations given by

$$(F_{AT}, E_{AT}) = ((F_A \;\; F_{AI}), (E_A \;\; E_{AI})), \quad (17)$$
$$(F_{BT}, E_{BT}) = ((F_B \;\; F_{BI}), (E_B \;\; E_{BI})), \quad (18)$$

respectively, then ([1], theorem 4) a kernel/image representation of $\mathcal{D}$ is given by the matrices

$$F = (L_A F_A \;\; L_B F_B), \; E = (L_A E_A \;\; L_B E_B), \quad (19)$$

where $L = (L_A \;\; L_B)$, with L any matrix of maximum rank such that $\mathcal{N}(L) = \mathcal{R}(M)$ and[2]

$$M = \begin{pmatrix} E_{AI} & F_{AI} \\ E_{BI} & -F_{BI} \end{pmatrix}. \quad (20)$$

III. GENERAL COMPOSITION OF DIRAC STRUCTURES

Consider N Dirac structures $\mathcal{D}_i$ given in (relaxed) image representation

$$\begin{pmatrix} f_i \\ e_i \\ f_{iI} \\ e_{iI} \end{pmatrix} = \begin{pmatrix} E_i^T \\ F_i^T \\ E_{iI}^T \\ F_{iI}^T \end{pmatrix} \lambda_i, \quad i = 1, \ldots, N \quad (21)$$

where each structure has internal (or maybe open port) flow-effort pairs (f_i, e_i), $f_i, e_i \in \mathbb{R}^{n_i}$, and a number of interconnected port flow-effort pairs (f_{iI}, e_{iI}), $f_{iI}, e_{iI} \in \mathbb{R}^{m_i}$, and where $\lambda_i \in \mathbb{R}^{r_i}$ with $r_i \geq n_i + m_i$. The matrices $E_i, F_i \in \mathbb{R}^{r_i \times n_i}$ and $E_{iI}, F_{iI} \in \mathbb{R}^{r_i \times m_i}$ satisfy

$$\text{rang } (F_i \;\; F_{iI} \;\; E_i \;\; E_{iI}) = n_i + m_i, \quad (22)$$

and

$$(E_i \;\; E_{iI})(F_i \;\; F_{iI})^T + (F_i \;\; F_{iI})(E_i \;\; E_{iI})^T = 0. \quad (23)$$

Furthermore, we consider an interconnecting Dirac structure $\mathcal{D}_C$ given in (relaxed) kernel form as

$$(F_{C1} \;\; \cdots \;\; F_{CN}) \begin{pmatrix} f_{1I} \\ \vdots \\ f_{NI} \end{pmatrix} + (E_{C1} \;\; \cdots \;\; E_{CN}) \begin{pmatrix} e_{1I} \\ \vdots \\ e_{NI} \end{pmatrix} = 0 \quad (24)$$

with matrices $F_{Ci}, E_{Ci} \in \mathbb{R}^{r_C \times m_i}$ satisfying

$$\text{rang } (F_{C1} \;\; \cdots \;\; F_{CN} \;\; E_{C1} \;\; \cdots \;\; E_{CN}) = m, \quad (25)$$

where $m = \sum_{i=1}^N m_i$, and

$$\begin{aligned} &(E_{C1} \;\; \cdots \;\; E_{CN})(F_{C1} \;\; \cdots \;\; F_{CN})^T \\ &+ \;\; (F_{C1} \;\; \cdots \;\; F_{CN})(E_{C1} \;\; \cdots \;\; E_{CN})^T = 0. \end{aligned} \quad (26)$$

Associated to this kernel representation of $\mathcal{D}_C$ there is also an image one, given by

$$\begin{pmatrix} f_{1I} \\ e_{1I} \\ \vdots \\ f_{NI} \\ e_{NI} \end{pmatrix} = \begin{pmatrix} E_{C1}^T \\ F_{C1}^T \\ \vdots \\ E_{CN}^T \\ F_{CN}^T \end{pmatrix} \gamma, \quad (27)$$

with $\gamma \in \mathbb{R}^{r_C}$.

Definition The set $(\mathcal{D}_C)_{i=1}^N \mathcal{D}_i$ is made up of those $(f_1, e_1, \ldots, f_N, e_N)$ for which there exist interconnecting variables

$$(f_{1I}, e_{1I}, \ldots, f_{NI}, e_{NI}) \in \mathcal{D}_C$$

such that

$$(f_i, f_{iI}, e_i, e_{iI}) \in \mathcal{D}_i, \; \forall \; i = 1, \ldots, N.$$

Theorem 3.1: $(\mathcal{D}_C)_{i=1}^N \mathcal{D}_i$ is a Dirac structure.

Proof: By combining the individual image representations (21) and the kernel representation for the interconnection (24), and reordering the resulting matrices, one gets

$$\left(F_{C1} E_{1I}^T + E_{C1} F_{1I}^T \quad \cdots \quad F_{CN} E_{NI}^T + E_{CN} F_{NI}^T\right) \lambda = 0, \quad (28)$$

where $\lambda = \left(\lambda_1^T \;\; \cdots \;\; \lambda_N^T\right)^T \in \mathbb{R}^r$, and where we have defined $r = \sum_{i=1}^N r_i$. This can be combined with the equations from (21) for the non-interconnecting variables to get that, for any $(f_1, e_1, \ldots, f_N, e_N) \in (\mathcal{D}_C)_{i=1}^N \mathcal{D}_i$, there exist $\lambda_1, \ldots, \lambda_N$ such that

$$\begin{pmatrix} f_1 \\ e_1 \\ \vdots \\ f_N \\ e_N \\ 0 \end{pmatrix} = K\lambda \quad (29)$$

where

$$K = \begin{pmatrix} E_1^T & \cdots & 0 \\ F_1^T & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & E_N^T \\ 0 & \cdots & F_N^T \\ F_{C1} E_{1I}^T + E_{C1} F_{1I}^T & \cdots & F_{CN} E_{NI}^T + E_{CN} F_{NI}^T \end{pmatrix}. \quad (30)$$

Using Lemma 2.2, this is equivalent to the implication that, for any $(\beta_1, \alpha_1, \ldots, \beta_N, \alpha_N, \gamma)$ such that

$$K^T \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \vdots \\ \beta_N \\ \alpha_N \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad (31)$$

one has

$$\beta_1^T f_1 + \alpha_1^T e_1 + \cdots + \beta_N^T f_N + \alpha_N^T e_N = 0. \quad (32)$$

Taking into account the image representation (27) for the interconnecting Dirac structure, the set of equations (31) can be rewritten as

$$\begin{aligned} E_1 \beta_1 + F_1 \alpha_1 + E_{1I} e_{1I} + F_{1I} f_{1I} &= 0, \\ &\vdots \\ E_N \beta_N + F_N \alpha_N + E_{NI} e_{NI} + F_{NI} f_{NI} &= 0. \end{aligned}$$

[2]This M differs from that in [1], but it spans the same column space, which is all that matters.

But these are the kernel representations of the $\mathcal{D}_i$, with β_i the non-interconnecting efforts and α_i the non-interconnecting flows, and with the $(f_{1I}, e_{1I}, \ldots, f_{NI}, e_{NI})$ belonging to $\mathcal{D}_C$, and hence $(\alpha_1, \beta_1, \ldots, \alpha_N, \beta_N) \in (\mathcal{D}_C)_{i=1}^N \mathcal{D}_i$. Thus, we conclude that, if $(f_1, e_1, \ldots, f_N, e_N) \in (\mathcal{D}_C)_{i=1}^N \mathcal{D}_i$, then, for any $(\alpha_1, \beta_1, \ldots, \alpha_N, \beta_N) \in (\mathcal{D}_C)_{i=1}^N \mathcal{D}_i$ one has

$$\beta_1^T f_1 + \alpha_1^T e_1 + \cdots + \beta_N^T f_N + \alpha_N^T e_N = 0,$$

i.e. any element of $(\mathcal{D}_C)_{i=1}^N \mathcal{D}_i$ belongs to $\left((\mathcal{D}_C)_{i=1}^N \mathcal{D}_i\right)^\perp$. The chain of arguments can be easily reversed to show also that any element of $\left((\mathcal{D}_C)_{i=1}^N \mathcal{D}_i\right)^\perp$ is also of $(\mathcal{D}_C)_{i=1}^N \mathcal{D}_i$, and this concludes the proof. ■

IV. KERNEL AND IMAGE REPRESENTATION OF THE COMPOSED STRUCTURE

We rewrite (28) as

$$M^T \lambda = 0, \tag{33}$$

with

$$M^T = (M_1^T \ \cdots \ M_i^T \ \cdots \ M_N^T) \in \mathbb{R}^{r_C \times r}, \tag{34}$$

and where

$$M_i^T = F_{Ci} E_{iI}^T + E_{Ci} F_{iI}^T, \ i = 1, \ldots, N. \tag{35}$$

Equation (33) implies that $\lambda \in \mathcal{N}(M^T)$ which, according to (16), is equivalent to

$$\lambda \perp \mathcal{R}(M). \tag{36}$$

Now we choose an $r_L \times r$ matrix L such that

$$\mathcal{R}(M) = \mathcal{N}(L) \tag{37}$$

or, equivalently,

$$\mathcal{R}(L^T) = \mathcal{N}(M^T). \tag{38}$$

This implies that L is such that $LM = 0$ or, equivalently, $M^T L^T = 0$. Combining (36) and (37) one gets that $\lambda \perp \mathcal{N}(L)$ or

$$\lambda \in \mathcal{R}(L^T). \tag{39}$$

This in turn implies that there exists $\mu \in \mathbb{R}^{r_L}$ such that

$$\lambda = L^T \mu. \tag{40}$$

Notice that, since the r_L columns of L^T must expand the kernel of M^T, one must have

$$r_L \geq \dim \mathcal{N}(M^T). \tag{41}$$

Plugging in (40) into the internal part of the relations (21) one gets, with a common μ,

$$\begin{aligned} \begin{pmatrix} f_i \\ e_i \end{pmatrix} &= \begin{pmatrix} E_i^T \\ F_i^T \end{pmatrix} \lambda_i = \begin{pmatrix} E_i^T \\ F_i^T \end{pmatrix} L_i^T \mu \\ &= \begin{pmatrix} E_i^T L_i^T \\ F_i^T L_i^T \end{pmatrix} \mu, \quad i = 1, \ldots, N, \end{aligned} \tag{42}$$

where L_i^T contains the rows of L^T corresponding to λ_i. Putting together all the relations in (42) yields

$$\begin{pmatrix} f_1 \\ e_1 \\ \vdots \\ f_N \\ e_N \end{pmatrix} = \begin{pmatrix} E_1^T L_1^T \\ F_1^T L_1^T \\ \vdots \\ E_N^T L_N^T \\ F_N^T L_N^T \end{pmatrix} \mu. \tag{43}$$

From this an image representation of the interconnected structure can be read off, with

$$E = (L_1 E_1 \ \cdots L_N E_N), \tag{44}$$

$$F = (L_1 F_1 \ \cdots L_N F_N). \tag{45}$$

Equations (34), (35), (38), (44) and (45) constitute an algorithm to compute a representation of the composed Dirac structure, and it can be easily implemented by using, for instance, standard linear algebra MATLAB functions.

V. SOME BASIC INTERCONNECTIONS

A. General feedback interconnection of two structures

Consider two Dirac structures $\mathcal{D}_1$ and $\mathcal{D}_2$ with external ports $f_{1I}, e_{1I} \in \mathbb{R}^{m_1}$, $f_{2I}, e_{2I} \in \mathbb{R}^{m_2}$, which are interconnected by means of a general static feedback (Figure 1)

$$e_{1I} = K e_{2I}, \tag{46}$$

$$f_{2I} = -K^T f_{1I}, \tag{47}$$

where $K \in \mathbb{R}^{m_1 \times m_2}$. These relations can be put as a Dirac structure in kernel form

$$\begin{pmatrix} K^T & \mathbb{I} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_{1I} \\ f_{2I} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mathbb{I} & -K \end{pmatrix} \begin{pmatrix} e_{1I} \\ e_{2I} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{48}$$

with $r_C = m_1 + m_2$, from which F_{C1}, F_{C2}, E_{C1} and E_{C2} can be read

$$F_{C1} = \begin{pmatrix} K^T \\ 0 \end{pmatrix}, F_{C2} = \begin{pmatrix} \mathbb{I} \\ 0 \end{pmatrix}, E_{C1} = \begin{pmatrix} 0 \\ \mathbb{I} \end{pmatrix}, E_{C2} = \begin{pmatrix} 0 \\ -K \end{pmatrix}. \tag{49}$$

Using then (35) and (34) the matrix M^T whose kernel is spanned by the columns of L^T can be obtained, and one finally gets

$$M = \begin{pmatrix} E_{1I} K & F_{1I} \\ E_{2I} & -F_{2I} K^T \end{pmatrix}, \tag{50}$$

which coincides with (20) for $K = \mathbb{I}$.

B. 0-junction (parallel) interconnection of 3 structures

Consider 3 Dirac structures $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, with a common number m_1 of ports connected in parallel

$$f_{1I} + f_{2I} + f_{3I} = 0, \quad e_{1I} = e_{2I} = e_{3I}, \tag{51}$$

with $e_{iI}, f_{iI} \in \mathbb{R}^{m_1}$ $i = 1, 2, 3$. A kernel form is given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{1I} \\ f_{2I} \\ f_{3I} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} e_{1I} \\ e_{2I} \\ e_{3I} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{52}$$

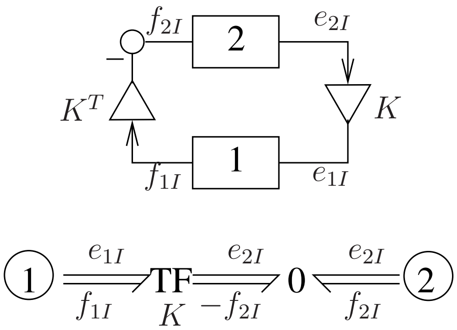


Fig. 1. Block diagram (above) and bond graph (below) of a general feedback interconnection.

Matrices F_{Ci} and E_{Ci} can be then read off and

$$M_1^T = \begin{pmatrix} E_{1I}^T \\ F_{1I}^T \\ 0 \end{pmatrix}, M_2^T = \begin{pmatrix} E_{2I}^T \\ -F_{2I}^T \\ F_{2I}^T \end{pmatrix}, M_3^T = \begin{pmatrix} E_{3I}^T \\ 0 \\ -F_{3I}^T \end{pmatrix}, \tag{53}$$

and finally

$$M = \begin{pmatrix} E_{1I} & F_{1I} & 0 \\ E_{2I} & -F_{2I} & F_{2I} \\ E_{3I} & 0 & -F_{3I} \end{pmatrix}. \tag{54}$$

C. Effort-constraint reduction of PHS

Several methods to truncate the states of a PHS so that the reduced system is again a PHS are presented in [12]. One of them, called effort-constraint reduction, sets to zero the generalized forces associated to the states that are going to be neglected, so that power does not flow due to the variation of those states, and, in order to avoid an inconsistency, the corresponding efforts of the underlying Dirac structure are also set to zero, which is the situation that we will study in our framework.

Consider again the interconnection in Figure 1 but with $K = \mathbb{I}$, so that $m_2 = m_1$, and with system 2 having only interconnecting ports with port variables $f_{2I} \in \mathbb{R}^{m_1}$, $e_{2I} \in \mathbb{R}^{m_1}$ satisfying the trivial Dirac structure

$$\mathcal{D}_2 = \{(f_{2I}, e_{2I}) \mid e_{2I} = 0\}, \tag{55}$$

which, in electrical terms, corresponds to a shorted circuit. The kernel representation of $\mathcal{D}_2$ is given by $F_{2I} f_{2I} + E_{2I} e_{2I} = 0$, with $F_{2I} = 0$ and $E_{2I} = \mathbb{I}$. Using (50) with $K = \mathbb{I}$, one immediately gets (see also [5])

$$M = \begin{pmatrix} E_{1I} & F_{1I} \\ \mathbb{I} & 0 \end{pmatrix}. \tag{56}$$

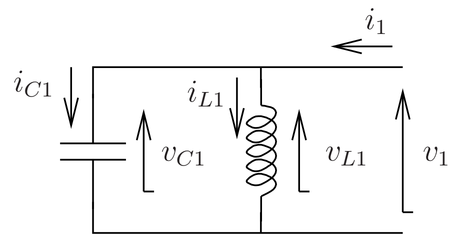
Solving $LM = 0$ with $L = \begin{pmatrix} L_1 & L_2 \end{pmatrix}$ yields $L_1 E_{1I} + L_2 = 0$, $L_1 F_{1I} = 0$, or

$$L_1 F_{1I} = 0, \tag{57}$$
$$L_2 = -L_1 E_{1I}. \tag{58}$$

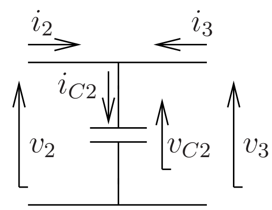
Equation (57), which corresponds to equation (3.13) in [12], is the key condition that must be satisfied by the matrix L_1 which allows the computation of the reduced Dirac structure for subsystem 1, *i.e.* $F_{1R} = L_1 F_1$, $E_{1R} = L_1 E_1$. Equation (58), giving L_2, is irrelevant in our case since $\mathcal{D}_2$ has no remaining Dirac structure after the interconnection.

VI. COMPOSITION OF PHS: AN EXAMPLE

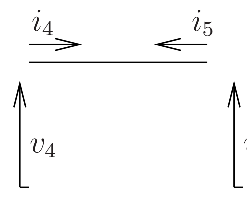
Consider the three circuits displayed in Figure 2.



(a) A circuit with 2 state ports and one boundary port.



(b) A circuit with one state port and two boundary ports.



(c) A system with just two boundary ports.

Fig. 2. Three circuits to be connected in parallel through ports 1, 2 and 4, leaving 3 and 5 open. The third circuit is just a stub, introduced in order to get an open port at the point of connection of the other two circuits.

Kernel representations (they are not unique) are given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -i_{C1} \\ -v_{L1} \\ i_1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_{C1} \\ i_{L1} \\ v_1 \end{pmatrix} = 0, \tag{59}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -i_{C2} \\ i_2 \\ i_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} v_{C2} \\ v_2 \\ v_3 \end{pmatrix} = 0, \tag{60}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} i_4 \\ i_5 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_4 \\ v_5 \end{pmatrix} = 0. \tag{61}$$

Connecting the three circuits in parallel at the ports (i_1, v_1), (i_2, v_2), (i_4, v_4) separates the ports into internal or open ports on one side and interconnected ports on the other, and from the above kernel representations one immediately reads F_1, F_{1I}, E_1, E_{1I}, F_2, F_{2I}, E_2, E_{2I}, F_3, F_{3I}, E_3 and E_{3I}. For instance

$$F_{1I} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{62}$$

From these and (54) M^T can be computed

$$M^T = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{63}$$

Using, for instance, the MATLAB command `null` with the `'r'` option, a matrix L such that $M^T L^T = 0$ can be obtained

$$L = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{64}$$

From L and E_i, F_i, $i = 1, 2, 3$, one can compute the matrices of the kernel representation of the composed structure

$$F = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{65}$$

$$E = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{66}$$

The kernel representation $Ff + Ee = 0$, with $f = (-i_{C1}\ -v_{L1}\ -i_{C2}\ i_3\ i_5)^T$ and $e = (v_{C1}\ i_{L1}\ v_{C2}\ v_3\ v_5)^T$, yields the network equations of the composed structure

$$\begin{aligned} -v_{C1} + v_{L1} &= 0, \\ v_{C2} - v_3 &= 0, \\ -v_{C1} + v_{C2} &= 0, \\ -i_{L1} - i_{C1} - i_{C2} + i_3 + i_5 &= 0, \\ v_{C1} - v_5 &= 0. \end{aligned} \tag{67}$$

Finally, putting $i_{C1} = \dot{q}_1$, $i_{C2} = \dot{q}_2$, $v_{L1} = \dot{\lambda}$, $v_{C1} = \partial_{q_1} H$, $v_{C2} = \partial_{q_2} H$, and $i_{L1} = \partial_\lambda H$, with $H = H(q_1, q_2, \lambda)$, yields the DAE of the interconnected system, with i_3 and i_5 as inputs and v_3 and v_5 as outputs:

$$\begin{aligned} \dot{q}_1 + \dot{q}_2 &= -\partial_\lambda H + i_3 + i_5, \\ \dot{\lambda} &= \partial_{q_1} H, \\ \partial_{q_1} H &= \partial_{q_2} H, \\ v_3 &= \partial_{q_2} H, \\ v_5 &= \partial_{q_1} H. \end{aligned} \tag{68}$$

VII. CONCLUSIONS

An algorithm to compute kernel/image representations for the Dirac structure resulting from the interconnection of several Dirac structures has been provided, generalizing a simple special case previously published in the literature. As a byproduct, an new proof of the basic result concerning the interconnection of Dirac structures has been obtained. Several particular but relevant cases have been worked out in detail, and an example of application to the interconnection of PHS has been presented.

Besides the immediate application to the modeling of complex systems, the results in the present paper can be useful in several related fields, such as optimal control and model order reduction of PHS systems [12], using the techniques developed for general nonlinear DAE systems [13][14][15][16]. Also, from the point of view of the theory of Dirac structures and control by interconnection, some of the results in [1][5] about the achievability of a closed-loop Dirac structure could be further investigated using the generalized framework presented here.

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