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# The Minimum Principle for Time-Varying Hybrid Systems with State Switching and Jumps

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*Abstract*— **The hybrid minimum principle (HMP) is extended to hybrid systems with autonomous (internally forced) switching on switching manifolds, controlled (externally forced) switching, jumps of the continuous state when switching, and time-varying functions for specifying the continuous and discrete dynamics. The formulation of the hybrid optimal control problem (HOCP) includes running, switching, and terminal costs. The HMP provides necessary optimality conditions for a solution of the HOCP and can be used as basis for developing numerical optimal control algorithms.**

## I. Introduction

Hybrid systems combine continuous-time dynamics like differential equations with discrete-event dynamics like impulses or discrete events [1, 2]. Examples of hybrid systems can be found in application domains like biological systems, chemical processes, manufacturing, and embedded systems, where continuously controlled systems interact with digital machines or logical decision processes. Much effort has been spent to find optimal controls for hybrid systems, e.g. using mixed-integer programming [3], value function approaches [4, 5], and direct or indirect multiple shooting [6, 7]. Many numeric approaches are based on evaluating the optimality conditions for the underlying hybrid optimal control problem (HOCP), e.g. to check whether a candidate solution is an optimal one. Especially indirect methods use the optimality conditions to find results with high precision.

Optimality conditions were originally developed for continuous state systems and were based on the concept of calculus of variations [8] or needle variations [9], where the latter results in the minimum (or maximum) principle. In particular optimality conditions for hybrid systems based on needle variations were derived in the form of the hybrid minimum principle (HMP) [10–16]. Especially the versions of the HMP formulated in [12] and [16] can handle a large class of hybrid systems. The corresponding HOCP includes autonomous switchings with resets of the continuous state and switching costs and controlled switchings. Resets of the continuous state at autonomous switches occur e.g. when particles collide [2] or when contact or impact situations in robotics are present, as in bipedal locomotion [17] or in juggling tasks like ball dribbling in basketball robotics [18].

However, the versions of the HMP presented there do not consider the following cases, in contrast to the HMP that is described here: (i) Controlled switching with jumps of the continuous state variable and switching costs. (ii) The dynamics, running costs, jump maps, jump costs, and switching manifolds of the HOCP can be time-varying. These effects are important in practice: Controlled switching with resets of the continuous state can be found in manufacturing processes [19]. Another example is gear shifting in vehicles [20], where the dynamics of the gear shift with open clutch can be modeled abstractly by a jump in the time, position, and velocity of the vehicle [21]. Examples for time-dependent functions are the dynamics of sanding vehicles, where the mass changes with time, and supply chains, which also contain controlled switchings [22].

This paper introduces a novel version of the HMP, which can be used to develop (indirect) optimization algorithms. The proof is based on single needle variations. The technique was originally introduced in [23] for optimal control of nonlinear systems and extended in [24] to nonlinear impulsive systems. In [16], the technique was applied to a less general class of hybrid systems, which does not consider time-varying functions and jumps of the continuous state at autonomous and controlled switches. In [25], a version of the HMP is derived with single needle variations for hybrid systems with autonomous switching and intersecting switching manifolds. There, functions are not time-varying and the continuous state does not jump with switches.

The paper is structured as follows: In Sec. II, a class of hybrid systems is defined. Sec. III introduces the novel HMP and its proof. In Sec. IV, a short example is given and Sec. V concludes the paper.

## II. Hybrid System

In the following, the considered class of time-varying hybrid systems is introduced:

*Definition 1:* A hybrid system is an 8-tuple

$$\mathbb{H} := \{\mathcal{Q}, \mathcal{X}, \Gamma, U, \Omega, \mathcal{F}, \mathcal{M}, \Lambda\}. \tag{1}$$

□

*Assumption 1:*

(a) $\mathcal{Q} = \{1, 2, ..., N_q\}$: set of $N_q$ discrete states $q \in \mathcal{Q}$.
(b) $\mathcal{X} = \{\mathcal{X}_q\}_{q\in\mathcal{Q}}$: collection of state spaces $\mathcal{X}_q \subseteq \mathbb{R}^{n_x}$ assigned to every discrete state $q$, where $\dim(x) = n_x$.
(c) $U = \{U_q\}_{q\in\mathcal{Q}}$: collection of compact sets $U_q \subset \mathbb{R}^{n_u}$ of admissible continuous control values $u$ with $\dim(u) = n_u$. $\mathcal{U} = \{\mathcal{U}_q\}_{q\in\mathcal{Q}}$ is the set of all measurable and bounded control trajectories $u : [t_0, t_e] \to U$, where $t_0$ and $t_e < \infty$ mark the intial and final time of an execution of the hybrid system.

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(d) $\mathcal{F} = \{f_q\}_{q\in\mathcal{Q}}$: collection of vector fields $f_q : \mathbb{R}^{n_x} \times U_q \times \mathbb{R} \to \mathbb{R}^{n_x}$ defined for each $q \in \mathcal{Q}$. The vector fields are at least once continuously differentiable with respect to the continuous state $x$ and the continuous control $u$ and continuous with respect to time $t$. They fulfill a uniform Lipschitz condition, i.e. $\exists L < \infty$ such that $\|f_q(x_1, u_q, t) - f_q(x_2, u_q, t)\|_2 \leq L\|x_1 - x_2\|_2$, for any combination of $x_1, x_2 \in \mathbb{R}^{n_x}$, $u_q \in U_q$, $t \in \mathbb{R}$.

(e) $\mathcal{M} = \{m_{i,k}\}_{i,k\in\mathcal{Q},i\neq k}$: collection of time-dependent switching manifolds, where $m_{i,k}$ is at least once continuously differentiable, i.e. $m_{i,k} \in C^1(\mathcal{X}_i \times \mathbb{R}, \mathbb{R})$. An autonomous transition from discrete state $i$ to $k$ occurs at time $t_j$ for $x(t_j)$ on the codimension 1 manifold $m_{i,k}$, that is locally expressed by $m_{i,k}(x(t_j), t_j) = 0$.

(f) $\Lambda = \{\varphi_{i,k}\}_{i,k\in\mathcal{Q},i\neq k}$: collection of reset functions $\varphi_{i,k} \in C^1(\mathbb{R}^{n_x} \times \mathbb{R}, \mathbb{R}^{n_x})$ for the continuous state $x$, which are associated with a controlled or autonomous switching from discrete state $i$ to $k$ for $i, k \in \mathcal{Q}$. In the case of controlled (but not autonomous) switching, it is assumed that $x = \varphi_{k,i}(\varphi_{i,k}(x, t), t)$.

(g) $\Omega : \{\Omega_q\}_{q\in\mathcal{Q}}$: collection of discrete sets $\Omega_q \subseteq \mathcal{Q}$ of admissible discrete controls $\omega_q$. When $\omega_q \in \Omega_q$, such that $\Gamma(q, x, \omega_q) = i$, a controlled or autonomous switching from discrete state $q \in \mathcal{Q}$ to discrete state $i \in \mathcal{Q}$ is triggered. In the case of autonomous switching, when the continuous state trajectory $x$ hits $m_{q,i}$ at time $t_j$, the discrete control $\omega_q$ is forced to be such that $\Gamma(q, x, \omega_q) = i$. A controlled switch with $\omega_q \in \Omega_q$ can be executed whenever and wherever desired independently of any switching manifold. It is assumed that if $\Gamma(q, x, \omega_q) = i$ with $\omega_q \in \Omega_q$ holds, $\Gamma(i, x, \omega_i) = q$ with $\omega_i \in \Omega_i$ is conversely true.

(h) $\Gamma : \mathcal{Q} \times \mathcal{X} \times \Omega \to \mathcal{Q}$: discrete transition map. □

*Assumption 2:* The switching times $t_j$ fulfill $t_0 < t_1 < ... < t_e < \infty$, and $\forall x(t_j) \in m_{i,k}$, $i, k \in \mathcal{Q}$, the vector fields $f_i(x, u_i, t_j)$ and $f_k(x, u_k, t_j)$ are non-vanishing at and transversal to $m_{i,k}$ for the applied $u_i \in U_i$ and $u_k \in U_k$. The number $N$ of autonomous and controlled switchings of an execution of a hybrid system satisfies $N + 2 \leq \bar{N} < \infty$, where $\bar{N}$ is the maximally allowed number of switchings. This implies that no accumulation points of switching (Zeno points) or sliding motions occur. □

*Assumption 3:* At time $t_0$ with given initial conditions $(q(t_0), x(t_0)) = (q_0, x_0) \in \mathcal{Q} \times \mathcal{X}$, for all switching manifolds it is assumed that $m_{q_0,i}(x(t_0), t_0) \neq 0\ \forall i \in \mathcal{Q}$. □

Note that these assumptions are important for the existence of a unique execution of the hybrid system and for the existence of an optimal control.

*Definition 2:* An execution of a hybrid system is given by $\sigma = (\tau, q, x, u, \omega)$, where $\tau = (t_0, ..., t_{N+1} = t_e)$ is a strictly increasing sequence of initial, switching, and final times with $N$ switchings and $q = (q_0, ..., q_N)$ denotes a sequence of discrete states. The term $x = (x_{q_0}, ..., x_{q_N})$ is a sequence of absolutely (left-) continuous state trajectories $x_{q_j} : [t_j, t_{j+1}) \to \mathcal{X}_{q_j}$ evolving according to:

$$\dot{x}_{q_j} = f_{q_j}(x_{q_j}(t), u_{q_j}(t), t) \tag{2}$$

for a.e. $t \in [t_j, t_{j+1})$, $j \in \{0, 1, ..., N\}$, where $x_{q_0}(t_0) = x_0$ and $x_{q_j}(t_j) = \varphi_{q_{j-1},q_j}(x_{q_{j-1}}(t_j^-), t_j)$ with $t_j^- := \lim_{t\to t_j, t<t_j} t$. The continuous control $u = (u_{q_0}, ..., u_{q_N})$ is a sequence of control trajectories $u_{q_j} : [t_j, t_{j+1}) \to U_{q_j}$. Furthermore, $\omega$ specifies a sequence of discrete controls $(\omega_{q_1}, ..., \omega_{q_N})$. □

*Definition 3:* An optimal solution $\sigma^* = (\tau^*, q^*, x^*, u^*, \omega^*)$ minimizes the cost functional

$$J = g(x_{q_e}(t_e)) + \sum_{j=1}^{N} c_{q_{j-1},q_j}(x_{q_{j-1}}(t_j^-), t_j) + \sum_{j=0}^{N} \int_{t_j}^{t_{j+1}} \phi_{q_j}(x_{q_j}(t), u_{q_j}(t), t)\, dt \tag{3}$$

while satisfying initial conditions $x_0$ and $q_0$ and the hybrid system dynamics of $\mathbb{H}$. The terminal cost function $g$ is in $C^1(\mathbb{R}^{n_x}, \mathbb{R})$, the switching cost function $c_{i,k}$ is in $C^1(\mathbb{R}^{n_x} \times \mathbb{R}, \mathbb{R})$, and the running cost functions $\phi_q : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R} \to \mathbb{R}$, $i, k, q \in \mathcal{Q}$, is at least once continuously differentiable with respect to the continuous state $x$ and control $u$ and continuous with respect to time $t$. □

*Remark 1:* In the sequel, the HOCP (3) given in Bolza representation is reformulated w.l.o.g. into Mayer form [26]. This is advantageous for the proof of the main result. □

## III. Optimality Conditions

The novel HMP provides necessary optimality conditions for a (locally) optimal solution minimizing (3). The HMP includes the conditions of the former versions [16, 25], but modifies the adjoint transversality and Hamiltonian value conditions for controlled and autonomous transitions, and the Hamiltonian minimization condition with respect to the discrete control. In the sequel, the dependence of functions on variables is partly and the asterisk '*' of the optimal switching times $t_j^*$ is often omitted for a better readability.

*Definition 4:* For a concise notation, Hamiltonians

$$H_{q_j}(x(t), \lambda(t), u_{q_j}(t), t) = \lambda(t)^T f_{q_j}(x(t), u_{q_j}(t), t) \tag{4}$$

for a.e. $t \in [t_j, t_{j+1})$, $j \in \{0, ..., N\}$, $q \in \mathcal{Q}$, are introduced with the adjoint variable $\lambda(t) : \mathbb{R} \to \mathbb{R}^{n_x}$. □

*Theorem 1:* Consider a hybrid system $\mathbb{H}$ as in *Def.* 1 and related executions $\sigma$ fulfilling *Assum.* 1, 2, and 3. Then all controls $u^*$ and $\omega^*$ (locally) minimizing the cost functional

$$\inf_{u\in\mathcal{U},\omega\in\Omega} J(u, \omega) \tag{5}$$

lead to an optimal execution $\sigma^* = (\tau^*, q^*, x^*, u^*, \omega^*)$, such that the following conditions are satisfied:

1) The differential equations (2) are fulfilled.

2) There exists an optimal, absolutely continuous adjoint process $\lambda^*$ such that:

$$\dot{\lambda}^* = -\nabla_x H_{q_j^*}^T(t) \quad \text{a.e. } t \in [t_j, t_{j+1}),\ j \in \{0, ..., N\} \tag{6}$$

The following boundary conditions hold for $\lambda^*$:

a) Terminal condition:

$$\lambda^*(t_e) = \nabla_x g^T(x^*(t_e)) \tag{7}$$

b) If at time $t_j$, $j \in \{1, ..., N\}$, an autonomous transition on $m_{q^*_{j-1},q^*_j}(x^*(t_j^-), t_j) = 0$ with $q^*_{j-1}, q^*_j \in \mathcal{Q}$ is triggered, then:

$$\lambda^*(t_j^-) = \nabla_x \varphi^T_{q^*_{j-1},q^*_j}(x^*(t_j^-), t_j)\, \lambda^*(t_j) + \nabla_x m^T_{q^*_{j-1},q^*_j}(x^*(t_j^-), t_j)\, \pi^*_j \tag{8}$$

with constant and optimal multipliers $\pi^*_j \in \mathbb{R}$.

c) If at time $t_j$, $j \in \{1, ..., N\}$, a controlled transition to discrete state $q^*_j \in \mathcal{Q}$ is triggered, then:

$$\lambda^*(t_j^-) = \nabla_x \varphi^T_{q^*_{j-1},q^*_j}(x^*(t_j^-), t_j)\, \lambda^*(t_j)\,. \tag{9}$$

3) The Hamiltonian has to fulfill the following conditions:

a) If at time $t_j$, $j \in \{1, ..., N\}$ the system switches autonomously from $q^*_{j-1}$ to $q^*_j$, then:

$$H_{q^*_{j-1}}(t_j^-) = H_{q^*_j}(t_j) - \pi^*_j \nabla_t m_{q^*_{j-1},q^*_j} - \lambda^{*T}(t_j) \nabla_t \varphi_{q^*_{j-1},q^*_j}\,. \tag{10}$$

b) If at time $t_j$, $j \in \{1, ..., N\}$ the system switches controlled from $q^*_{j-1}$ to $q^*_j$, then:

$$H_{q^*_{j-1}}(t_j^-) = H_{q^*_j}(t_j) - \lambda^{*T}(t_j) \nabla_t \varphi_{q^*_{j-1},q^*_j}\,. \tag{11}$$

c) The minimization condition with respect to $u^*_{q^*_j}$, $q^*_j \in Q$ is:

$$H_{q^*_j}(x^*(t), \lambda^*(t), u^*_{q^*_j}(t), t) \leq H_{q^*_j}(x^*(t), \lambda^*(t), v, t) \tag{12}$$

for a.e. $t \in [t_j, t_{j+1})$, $j \in \{0, ..., N\}$, and for every $v \in U_{q^*_j}$.

d) The minimization condition with respect to $\omega_{q^*_j} \in \Omega_{q^*_j}$, $q^*_j \in Q$, is $\forall k \in \mathcal{Q}$ with $\Gamma(q^*_j, x(t), \omega_{q^*_j}) = k$ and $\Gamma(k, x(t), \omega_k) = q^*_j$ for $\omega_k \in \Omega_k$:

$$H_{q^*_j}(x^*(t), \lambda^*(t), u^*_{q^*_j}(t), t) \leq H_k(x_k(t), \lambda_k(t), v, t) - \lambda^{*T}(t) \nabla_t \varphi_{k,q^*_j}(x_k(t), t) \tag{13}$$

for a.e. $t \in [t_j, t_{j+1})$, $j \in \{0, ..., N\}$, for every $v \in U_k$, and with $x_k(t) = \varphi_{q^*_j,k}(x^*(t), t)$ and $\lambda_k(t) = \nabla_x \varphi^T_{k,q^*_j}(x_k(t), t)\, \lambda^*(t)$. □

In the following, a sketch of the proof is presented. Since the proof is strongly based on the scheme used in [16], only those parts are shown, where a significant deviation from the mentioned work exists. The proof consists of propagating needle variations in the controls at arbitrary time in the tangent space around an optimal solution to the end time. Suppose an optimal solution passes the discrete state sequence $0, 1, 2, ..., N$, where switching occurs in $t_1, t_2, ..., t_N$. The overall structure of the proof is as follows: (a) To prove the Hamilton minimization condition (12), the adjoint differential equation (6), and the adjoint boundary condition (7) in the $N$-th discrete state, a needle variation in that state is performed. The derivation can be found in [16]. (b) Now, step-by-step, one has to go back in the discrete state sequence to show the Hamiltonian minimization condition (12) and the adjoint differential equation (6) in the corresponding discrete states, and the Hamiltonian value condition for autonomous switchings (10) and the adjoint transversality conditions between discrete states (8) and (9). This is proved by propagating small variations in the state trajectory caused by needle variations in the controls in the current discrete state $j$ to the terminal time and state. (c) The Hamiltonian value condition (11) for controlled transitions is derived by a needle variation in the switching time, which is also propagated to the final time. (d) The Hamiltonian minimization condition (13) with respect to the discrete control $\omega_j$ results from applying a controlled needle variation to the discrete control $\omega_j$ and propagating the variations in the continuous state trajectory to the final time.

*Proof:* **1. Propagation of Variations Through Switchings:** Let the sequence $\{\epsilon^i\}_{i=1}^{\infty}$ be monotonically decreasing with $\epsilon^i > 0$ and $\lim_{i\to\infty} \epsilon^i = 0$. Now, it is analyzed how a needle variation in the optimal input trajectory $u^*$ at the Lebesgue point $t_v \in [t_{j-1}, t_j)$ in discrete state $j - 1 < N$ changes the optimal terminal cost $g(x^*(t_e))$. The case that a variation leads to a change in the discrete state sequence is excluded by assumption since the time $\epsilon^i$ of the variation can be chosen arbitrarily small. The variation is set up as follows, compare Fig. 1:

$$u^i(t) = \begin{cases} u^*(t) & t_0 \leq t < t_v - \epsilon^i \\ v & t_v - \epsilon^i \leq t < t_v \\ u^*_{j-1}(t) & t_v \leq t < t_j - \delta^i_j \\ u^*_j(t_j) & t_j - \delta^i_j \leq t < t_j \\ u^*(t) & t_j \leq t \leq t_e \end{cases}, \tag{14}$$

where $t_v \in [t_{j-1}, t_j)$, $v \in U_{j-1}$, and $u^*_j(t_j)$ is assumed to be a regular Lebesgue point (otherwise use the left-continuous replacement of the measurable function $u^*$). It is assumed that the perturbed trajectory hits the switching manifold $m_{j-1,j}$ at time $t_j - \delta^i_j$.[1]

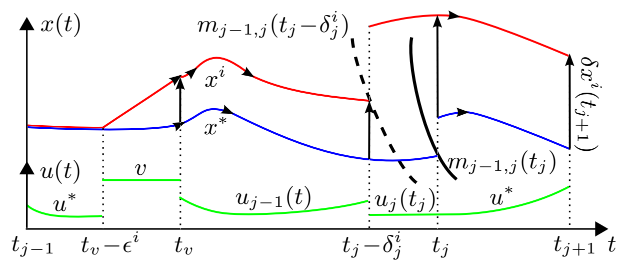


Fig. 1. Optimal and perturbed state trajectory with corresponding perturbed continuous control trajectory.

Having defined the perturbed continuous control trajectory $u^i$, the resulting difference between the perturbed and the optimal trajectory $\delta x^i(t) := x^i(t) - x^*(t)$, $\forall t \in [t_0, t_e]$ and the quantity $\xi(t) := \lim_{i\to\infty} \xi^i(t) := \lim_{i\to\infty} \frac{1}{\epsilon^i} \delta x^i(t)$ are investigated furthermore. Before applying the variation $v$, the disturbed and the optimal trajectory coincide: $\delta x^i(t) = 0$ $\forall t \in [t_0, t_v - \epsilon^i)$. While the variation $v$ of the optimal input $u^*_{j-1}(t)$ is active for the time interval $[t_v - \epsilon^i, t_v)$, the

[1]If the perturbed trajectory arrives $m_{j-1,j}$ at time $t_j + \delta^i_j$, the same results follow. This is not shown here due to space restrictions.

difference between the optimal and the perturbed trajectory at $t = t_v$ is:

$$\delta x^i(t_v) = \int_{t_v-\epsilon^i}^{t_v} f_{j-1}(x^i(t), v, t) - f_{j-1}(x^*(t), u^*_{j-1}(t), t)\, dt\,. \tag{15}$$

Rewriting (15), dividing by $\epsilon^i$ and forming the limit, the equation below follows:

$$\begin{aligned}\xi(t_v) = \lim_{i\to\infty} \frac{1}{\epsilon^i} \int_{t_v-\epsilon^i}^{t_v} & f_{j-1}(x^*(t), v, t) \\ & - f_{j-1}(x^*(t), u^*_{j-1}(t), t)\, dt \\ + \lim_{i\to\infty} \frac{1}{\epsilon^i} \int_{t_v-\epsilon^i}^{t_v} & f_{j-1}(x^i(t), v, t) - f_{j-1}(x^*(t), v, t)\, dt \\ = f_{j-1}(x^*(t_v), v, t_v) & - f_{j-1}(x^*(t_v), u^*_{j-1}(t_v), t_v).\end{aligned} \tag{16}$$

The second integral in (16) is zero as stated in *Lemma 2.1* in [16]. The proof of the lemma makes use of an approximation of $\delta x^i(t)$, $t \in [t_v - \epsilon^i, t_v)$, valid for small $\epsilon^i$:

$$\begin{aligned}\frac{d}{dt}\delta x^i(t) = \nabla_x f_{j-1}(x^*(t), u^*_{j-1}(t), t)\, \delta x^i(t) \\ + \nabla_u f_{j-1}(x^*(t), u^*_{j-1}(t), t)\, \delta u_{j-1}(t) \\ + o(\epsilon^i) + o(\|\delta u_{j-1}\|)\end{aligned} \tag{17}$$

$$\delta x^i(t_v - \epsilon^i) = 0\,, \tag{18}$$

where $\delta u_{j-1}(t) := v - u^*_{j-1}(t)$. Furthermore, the proof of the lemma as well as the proof here need the concept of a state transition matrix $\Phi_j(\tau, \tau_0)$, that transports small deviations from the optimal trajectory $x^*$ along its tangent space from time $\tau_0$ to some time $\tau > \tau_0$:

$$\frac{d}{d\tau}\Phi_j(\tau, \tau_0) = \nabla_x f_j(x^*(\tau), u^*_j(\tau), \tau)\, \Phi_j(\tau, \tau_0) \tag{19}$$

with the initialization $\Phi_j(\tau_0, \tau_0) = I$.

Now, the variation $\delta x^i(t_v)$ is propagated by the transition matrix until the switching manifold $m_{j-1,j}(x(t), t) = 0$ is reached, what happens at time $t_j - \delta^i_j$:

$$\delta x^i((t_j - \delta^i_j)^-) = \Phi_{j-1}(t_j - \delta^i_j, t_v)\, \delta x^i(t_v) + o(\epsilon^i) \tag{20}$$

and the direction of the variation is:

$$\xi(t_j^-) = \lim_{i\to\infty} \frac{1}{\epsilon^i} \delta x^i((t_j - \delta^i_j)^-) = \Phi_{j-1}(t_j, t_v)\, \xi(t_v)\,. \tag{21}$$

Here, the varied state $x^i((t_j - \delta^i_j)^-)$ jumps and evolves further with the dynamics in discrete state $j$ and control $u^*_j(t_j)$ until the optimal switching time $t_j$ is reached:

$$\begin{aligned}x^i(t_j) = \varphi_{j-1,j}(x^i((t_j - \delta^i_j)^-), t_j - \delta^i_j) \\ + \int_{t_j-\delta^i_j}^{t_j} f_j(x^i(t), u^*_j(t_j), t)\, dt\,.\end{aligned} \tag{22}$$

The optimal continuous state just after the switching time is

$$x^*(t_j) = \varphi_{j-1,j}(x^*(t_j^-), t_j)\,, \tag{23}$$

which gives the variation $\delta x^i(t_j) = x^i(t_j) - x^*(t_j)$. By using (23) and by observing that

$$\begin{aligned}x^i((t_j - \delta^i_j)^-) = x^*(t_j^-) + \delta x^i((t_j - \delta^i_j)^-) \\ - \int_{t_j-\delta^i_j}^{t_j} f_{j-1}(x^*(t), u^*_{j-1}(t), t)\, dt + o(\epsilon^i)\end{aligned} \tag{24}$$

holds, the equivalence

$$\begin{aligned}&\lim_{i\to\infty} \frac{1}{\epsilon^i} \int_{t_j-\delta^i_j}^{t_j} f_j(x^i(t), u^*_j(t_j), t)\, dt = \\ &\quad \lim_{i\to\infty} \frac{1}{\epsilon^i} \Big( \int_{t_j-\delta^i_j}^{t_j} f_j(x^i(t), u^*_j(t_j), t) \\ &\qquad - f_j(\varphi_{j-1,j}(x^i((t_j - \delta^i_j)^-), t_j - \delta^i_j), u^*_j(t_j), t)\, dt \\ &\qquad + \int_{t_j-\delta^i_j}^{t_j} f_j(\varphi_{j-1,j}(x^i((t_j - \delta^i_j)^-), t_j - \delta^i_j), u^*_j(t_j), t)\, dt \Big) \\ &= \lim_{i\to\infty} \frac{1}{\epsilon^i} \int_{t_j-\delta^i_j}^{t_j} f_j(\varphi_{j-1,j}(x^i((t_j - \delta^i_j)^-), t_j - \delta^i_j), u^*_j(t_j), t)\, dt \\ &\overset{(24)}{=} \lim_{i\to\infty} \frac{\delta^i_j}{\epsilon^i} f_j(\varphi_{j-1,j}(x^*(t_j^-), t_j), u^*_j(t_j), t_j)\end{aligned} \tag{25}$$

is shown with *Lemma 2.1* in [16]. This leads to:

$$\begin{aligned}\xi(t_j) = &\lim_{i\to\infty} \frac{1}{\epsilon^i} \delta x^i(t_j) \\ \overset{(22),(23),(25)}{=} &\lim_{i\to\infty} \frac{1}{\epsilon^i} \Big( \varphi_{j-1,j}(x^i((t_j - \delta^i_j)^-), t_j - \delta^i_j) \\ &- \varphi_{j-1,j}(x^*(t_j^-), t_j) \\ &+ \int_{t_j-\delta^i_j}^{t_j} f_j(\varphi_{j-1,j}(x^i((t_j - \delta^i_j)^-), t_j - \delta^i_j), u^*_j(t_j), t)\, dt \Big) \\ \overset{(23),(24),(25)}{=} &\lim_{i\to\infty} \frac{1}{\epsilon^i} \Big( \nabla_x \varphi_{j-1,j}(x^*(t_j^-), t_j) \\ &\big[\delta x^i((t_j - \delta^i_j)^-) - \int_{t_j-\delta^i_j}^{t_j} f_{j-1}(x^*(t), u^*_{j-1}(t), t)\, dt\big] \\ &+ \nabla_t \varphi_{j-1,j}(x^*(t_j^-), t_j)\, (-\delta^i_j) \\ &+ \delta^i_j\, f_j(x^*(t_j), u^*_j(t_j), t_j) \Big) \\ \overset{(21)}{=} &\nabla_x \varphi_{j-1,j}(x^*(t_j^-), t_j)\, \xi(t_j^-) \\ &+ \lim_{i\to\infty} \frac{\delta^i_j}{\epsilon^i} \Big( f_j(x^*(t_j), u^*_j(t_j), t_j) \\ &- \nabla_x \varphi_{j-1,j}(x^*(t_j^-), t_j) f_{j-1}(x^*(t_j^-), u^*_{j-1}(t_j), t_j) \\ &- \nabla_t \varphi_{j-1,j}(x^*(t_j^-), t_j) \Big)\,.\end{aligned} \tag{26}$$

The time shift $\delta^i_j$ in the switching time $t_j$ is found by the following Taylor approximation:

$$\begin{aligned}m_{j-1,j}(x^i((t_j - \delta^i_j)^-), t_j - \delta^i_j) = m_{j-1,j}(x^*(t_j^-), t_j) \\ + \nabla_x m_{j-1,j}(x^*(t_j^-), t_j)\, \big(\delta x^i((t_j - \delta^i_j)^-) \\ - \delta^i_j\, f_{j-1}(x^*(t_j^-), u^*_{j-1}(t_j), t_j)\big) \\ + \nabla_t m_{j-1,j}(x^*(t_j^-), t_j)\, (-\delta^i_j) + o(\epsilon^i)\,,\end{aligned} \tag{27}$$

which leads to

$$\delta^i_j = \eta_j \nabla_x m_{j-1,j}(x^*(t_j^-), t_j)\, \delta x^i((t_j - \delta^i_j)^-) \tag{28}$$

with $m_{j-1,j}(x^*(t_j^-),t_j) = m_{j-1,j}(x^i((t_j-\delta_j^i)^-), t_j-\delta_j^i) = 0$, $\eta_j = \frac{1}{\nabla_x m_{j-1,j} f_{j-1}(t_j) + \nabla_t m_{j-1,j}}$, and the short notations $f_{j-1}(t_j) := f_{j-1}(x^*(t_j^-), u_{j-1}^*(t_j), t_j)$ and $m_{j-1,j} := m_{j-1,j}(x^*(t_j^-), t_j)$. Defining $\varphi_{j-1,j} := \varphi_{j-1,j}(x^*(t_j^-), t_j)$ and $f_j(t_j) := f_j(x^*(t_j), u_j^*(t_j), t_j)$ and inserting (28) in (26), the direction of the variation is finally derived at $t_j$:

$$\xi(t_j) = \Big(\nabla_x\varphi_{j-1,j} + \eta_j\Big[f_j(t_j) - \nabla_x\varphi_{j-1,j} f_{j-1}(t_j) - \nabla_t\varphi_{j-1,j}\Big]\nabla_x m_{j-1,j}\Big)\xi(t_j-)\,. \tag{29}$$

The term $\xi(t_j)$ is transported to the next switching time $t_{j+1}$:

$$\xi(t_{j+1}^-) = \Phi_j(t_{j+1}, t_j)\,\xi(t_j)\,, \tag{30}$$

and further until the final time $t_e$ is reached with the direction of the variation $\xi(t_e)$.

**2. Hamiltonian Minimization Condition:** Here, for simplicity, the details of the propagation of variations are restricted to the case of one switching time, but the conditions can be extended to multiple switchings straightforwardly. Thus, $t_2 = t_e$ and the single switching time is denoted by $t_1$. Since $x^*$ is an optimal trajectory, any perturbed trajectory $x^i$ in a certain neighborhood has to lead to greater or equal terminal costs $g(x^i(t_2)) \geq g(x^*(t_2))$. Since the terminal costs $g(x(t_2))$ are assumed to be at least continuously differentiable, the operation below is well defined:

$$\begin{aligned}\lim_{i\to\infty}\frac{1}{\epsilon^i}&\left[g(x^*(t_2)+\epsilon^i\xi(t_2)) - g(x^*(t_2))\right]\\ &= \nabla_x g(x^*(t_2))\,\xi(t_2)\\ &= \nabla_x g(x^*(t_2))\,\Phi_1(t_2,t_1)\,\xi(t_1) \geq 0\,.\end{aligned} \tag{31}$$

Considering (29) and (31) and setting

$$\lambda^T(t_1) := \nabla_x g(x^*(t_2))\,\Phi_1(t_2,t_1) \tag{32}$$

$$\pi_1 := \eta_1\nabla_x g(x^*(t_2))\,\Phi_1(t_2,t_1)\Big(f_1(t_1) - \nabla_x\varphi_{0,1}f_0(t_1) - \nabla_t\varphi_{0,1}\Big)\,, \tag{33}$$

the adjoint transversality condition (8) for autonomous switching is found:

$$\lambda(t_1^-) = \nabla_x\varphi_{0,1}^T\,\lambda(t_1) + \nabla_x m_{0,1}^T\,\pi_1\,. \tag{34}$$

This condition specializes to (9) in case of controlled switching, since the varied trajectory $x^i$ does not lead to a change $\delta_1^i$ in the controlled switching time $t_1$. Inserting (16), (21), and (29) in (31) and defining $\lambda(t_v) = \Phi_0^T(t_1,t_v)\,\lambda(t_1^-)$, the adjoint differential equation (6) is derived

$$\begin{aligned}\dot\lambda(t_v) &= \frac{d}{dt_v}\Phi_0^T(t_1,t_v)\,\lambda(t_1^-)\\ &= -\nabla_x f_0^T(t_v)\,\Phi_0^T(t_1,t_v)\,\lambda(t_1^-)\\ &= -\nabla_x f_0^T(t_v)\,\lambda(t_v)\end{aligned} \tag{35}$$

and the Hamiltonian minimization condition for $t_v \in [t_0, t_1)$ results:

$$\lambda^T(t_v)\,f_0(x^*(t_v), v, t_v) \geq \lambda^T(t_v)\,f_0(x^*(t_v), u_0^*(t_v), t_v)\,. \tag{36}$$

**3. Hamiltonian Value Condition:** In the case of autonomous switching, the Hamiltonian value condition (10) can immediately be shown by applying the definitions of $\eta_1$, $\lambda(t_1^-)$, and $\lambda(t_1)$ and regrouping terms in (33). In the case of controlled switching, an approach is required, which differs from the previous one with a needle variation in the continuous control. Restricting w.l.o.g. the investigation to one controlled switching time in $[t_0, t_e]$, a needle variation in the optimal switching time $t_1$ is applied, such that the perturbed switching time is $t_1 - \epsilon^i$ and the corresponding continuous control is:

$$u^i(t) = \begin{cases} u_0^*(t) & t_0 \leq t < t_1 - \epsilon^i \\ u_1^*(t_1) & t_1 - \epsilon^i \leq t < t_1 \\ u_1^*(t) & t_1 \leq t \leq t_2 \end{cases}\,. \tag{37}$$

With $x^i((t_1-\epsilon^i)^-) = x^*((t_1-\epsilon^i)^-)$, this variation leads to:

$$x^i(t_1) = \varphi_{0,1}(x^i((t_1-\epsilon^i)^-), t_1-\epsilon^i) + \int_{t_1-\epsilon^i}^{t_1} f_1(x^i(t), u_1^*(t_1), t)\,dt \tag{38}$$

$$x^*(t_1) = \varphi_{0,1}(x^*(t_1^-), t_1)\,. \tag{39}$$

Expressing the varied switching point $x^i((t_1-\epsilon^i)^-)$ by $x^*(t_1^-) - \int_{t_1-\epsilon^i}^{t_1} f_0(x^*(t), u_0^*(t), t)\,dt$ and using again *Lemma 2.1* in [16], the direction of the variation $\xi(t_1)$ is obtained:

$$\begin{aligned}\xi(t_1) = &-\nabla_x\varphi_{0,1}(x^*(t_1^-), t_1)\,f_0(x^*(t_1^-), u_0^*(t_1), t_1)\\ &-\nabla_t\varphi_{0,1}(x^*(t_1^-), t_1) + f_1(x^*(t_1), u_1^*(t_1), t_1)\,.\end{aligned} \tag{40}$$

This results in the final value $\xi(t_2) = \Phi_1(t_2,t_1)\,\xi(t_1)$. Due to the optimality of $x^*$, we again have $g(x^i(t_2)) - g(x^*(t_2)) \geq 0$. Dividing the relation by $\epsilon^i$ and passing to the limit, the following holds:

$$\begin{aligned}&\nabla_x g\,\Phi_1(t_2,t_1)\,\nabla_x\varphi_{0,1}\,f_0(t_1)\\ &\qquad\leq \nabla_x g\,\Phi_1(t_2,t_1)\left(f_1(t_1) - \nabla_t\varphi_{0,1}\right)\,,\end{aligned} \tag{41}$$

with $\lambda^T(t_1^-) = \nabla_x g\,\Phi_1(t_2,t_1)\,\nabla_x\varphi_{0,1}$ and $\lambda^T(t_1) = \nabla_x g\,\Phi_1(t_2,t_1)$. When the same steps are repeated with a perturbed switching time $t_1 + \epsilon^i$, a similar relation with opposite signs is obtained:

$$\lambda^T(t_1^-)\,f_0(t_1) \geq \lambda^T(t_1)\,f_1(t_1) - \lambda^T(t_1)\,\nabla_t\varphi_{0,1}\,. \tag{42}$$

From the two relations, we can conclude the Hamiltonian value condition (11) for controlled switching:

$$\lambda^T(t_1^-)\,f_0(t_1) = \lambda^T(t_1)\,f_1(t_1) - \lambda^T(t_1)\,\nabla_t\varphi_{0,1}\,. \tag{43}$$

**4. Hamiltonian Minimization with Respect to the Discrete Control:** To show condition (13) in discrete state $j$, a needle variation in the optimal discrete control $\omega^*$ is performed. The needle variation consists of switching from discrete state $j$ to $k \in \mathcal{Q}$ with control $\omega_j \in \Omega_j$ at time $t_k - \epsilon^i$ and back to $j$ at time $t_k$. This is possible since by assumption there exists $\omega_k \in \Omega_k$, such that $\Gamma(k, x(t), \omega_k) = j$, and

$N+2 \leq \bar{N} < \infty$. Let the varied discrete control sequence be $\omega^i = (..., \omega_k, \omega_j, ...)$ and the continuous control:

$$u^i(t) = \begin{cases} u^*(t) & t_0 \leq t < t_k - \epsilon^i \\ v & t_k - \epsilon^i \leq t < t_k \\ u^*(t) & t_k \leq t \leq t_e \end{cases} . \quad (44)$$

The varied state at time $t_k$ is:

$$x^i(t_k) = \varphi_{k,j}\Big(\varphi_{j,k}(x^*(t_k - \epsilon^i), t_k - \epsilon^i) + \int_{t_k-\epsilon^i}^{t_k} f_k(x^i(t), v, t)\, dt,\ t_k\Big) . \quad (45)$$

The optimal state at time $t_k$ can be written as:

$$x^*(t_k) = \varphi_{k,j}\big(\varphi_{j,k}(x^*(t_k - \epsilon^i), t_k - \epsilon^i), t_k - \epsilon^i\big) + \int_{t_k-\epsilon^i}^{t_k} f_j(x^*(t), u_j^*(t), t)\, dt . \quad (46)$$

Dividing the variation $\delta x^i(t_k) = x^i(t_k) - x^*(t_k)$ by $\epsilon^i$ and forming the limit, the direction of the variation is derived:

$$\begin{aligned} \xi(t_k) = &\lim_{i\to\infty}\Big(\nabla_x \varphi_{k,j}(\varphi_{j,k}(x^*(t_k - \epsilon^i), t_k - \epsilon^i), t_k - \epsilon^i) \\ &\frac{1}{\epsilon^i}\int_{t_k-\epsilon^i}^{t_k} f_k(\varphi_{j,k}(x^*(t_k - \epsilon^i), t_k - \epsilon^i), v, t)\, dt \\ &+ \nabla_t \varphi_{k,j}(\varphi_{j,k}(x^*(t_k - \epsilon^i), t_k - \epsilon^i), t_k - \epsilon^i) \\ &- \int_{t_k-\epsilon^i}^{t_k} f_j(x^*(t), u_j^*(t), t)\, dt\Big) \\ = &\nabla_x \varphi_{k,j}(\varphi_{j,k}(x^*(t_k), t_k), t_k)\, f_k(\varphi_{j,k}(x^*(t_k), t_k), v, t_k) \\ &+ \nabla_t \varphi_{k,j}(\varphi_{j,k}(x^*(t_k), t_k), t_k) - f_j(x^*(t_k), u_j^*(t_k), t_k). \end{aligned} \quad (47)$$

Following the derivation in part 2 and 3 before and assuming w.l.o.g. no further switchings, the relation below is achieved:

$$\nabla_x g\, \Phi_0(t_e, t_k)\Big(\nabla_x \varphi_{k,0}\, f_k(\varphi_{0,k}, v, t_k) + \nabla_t \varphi_{k,0}\Big) \geq \nabla_x g\, \Phi_0(t_e, t_k)\, f_0(t_k) . \quad (48)$$

With $\lambda^T(t_k^-) = \nabla_x g\, \Phi_0(t_e, t_k)\, \nabla_x \varphi_{k,0}$ and $\lambda^T(t_k) = \nabla_x g\, \Phi_0(t_e, t_k)$, it leads to the Hamiltonian minimization condition (13) with respect to the discrete control. ∎

## IV. Example

**1. Hybrid Optimal Control Problem**: An example with controlled switching and resets is given for illustration. A car with longitudinal dynamics and two gears is modeled, where $q \in \{1, 2\}$ denotes gear, $y$ position, $v$ velocity, and $u$ control and engine torque. The HOCP in Mayer form consists in finding a solution with at most one gear shift, which minimizes the terminal costs:

$$g(x(t_e)) = \alpha(t_e) - \gamma y(t_e) \quad (49)$$

with the accumulated running costs $\alpha$ and $\gamma \in \mathbb{R}^+$.

The continuous dynamics of the hybrid system is

$$\dot{\alpha} = au + (b-q)v \quad (50)$$

$$\dot{y} = v \quad (51)$$

$$\dot{v} = (b-q)u \quad (52)$$

$$\dot{\tau} = 1 ,$$

where $u \in U_q$, $a \in \mathbb{R}^+$, $b \in \mathbb{N}$, $b > N_q = 2$, and $\tau$ is the physical time added as state variable. A gear shift, which is a controlled switch, is modeled abstractly by a jump in the continuous state $x = (\alpha, y, v, \tau)^T$ at the switching time $t_1$:

$$x(t_1) = \varphi_{1,2}(x(t_1^-)) = \begin{pmatrix} \alpha(t_1^-) + \Delta\alpha \\ y(t_1^-) + \frac{\epsilon}{d} v(t_1^-) \\ (1-\epsilon)v(t_1^-) \\ \tau(t_1^-) + \Delta\tau \end{pmatrix} \quad (53)$$

with $\epsilon := 1 - e^{-d\Delta\tau}$ and $\Delta\alpha, \Delta\tau, d \in \mathbb{R}^+$. The initial conditions are: $q(t_0) = 1$, $\alpha(t_0) = y(t_0) = v(t_0) = \tau(t_0) = 0$, and $t_0 = 0$. The final time $t_e$ is set to $1 - \Delta\tau$ in the case of one switch and 1 if no switch occurs, such that $\tau(t_e) = 1$.

**2. Model Derivation**: The terminal costs (49) require a combination of minimizing fuel consumption and maximizing the traveled distance. The running costs (50) penalize engine torque and driving with high engine speeds, such that driving in a higher gear is more efficient. The possible acceleration in (52) decreases with the gear number. A gear shift works as follows: It is assumed that a gear shift takes the time $\Delta\tau$, such that $\tau$ jumps at time $t_1$. In the interval $\Delta\tau$, the dynamics of the car cannot be controlled since the clutch is open. Consequently, only friction with friction coefficient $d$ is acting on the car according to

$$\dot{v} = -dv . \quad (54)$$

Solving (54) analytically, the velocity after the gear shift is obtained: $v(t_1) = e^{-d\Delta\tau} v(t_1^-)$. The parameter $\epsilon := 1 - e^{-d\Delta\tau}$ is the relative decrease in the velocity with a gear shift. Inserting the solution of (54) into (51), the distance $\frac{\epsilon}{d} v(t_1^-)$, that the car drives in time span $\Delta\tau$, is determined. While shifting gears, the engine is in idle speed and torque, such that a constant increase $\Delta\alpha$ in the running costs occurs.

**3. Solution**: In the sequel, the optimality conditions (2), (6), (7), (9), (11), and (12) are evaluated and solved for the given HOCP. The optimality conditions (8) and (10) are not considered due to the lack of autonomous switching. The Hamiltonian minimization condition with respect to the discrete control (13) is not valid in this example. The proof of (13) needs the assumption $x(t) = \varphi_{1,2}(\varphi_{2,1}(x(t)))$, see *Assum.* 1(f), which is not satisfied here. The reason is that the costs, distance, velocity, and time jump at each gear shift according to (53) independently of the choice of gears.

The adjoint conditions (6), (7), and (9) for one switch are:

$$\dot{\lambda}(t) = \Big(0 \quad 0 \quad -(b-q)\lambda_\alpha(t) - \lambda_y(t) \quad 0\Big)^T$$

$$\lambda(t_e) = \begin{pmatrix} 1 \\ -\gamma \\ 0 \\ 0 \end{pmatrix}, \quad \lambda(t_1^-) = \begin{pmatrix} \lambda_\alpha(t_1) \\ \lambda_y(t_1) \\ (1-\epsilon)\lambda_v(t_1) + \frac{\epsilon}{d}\lambda_y(t_1) \\ \lambda_\tau(t_1) \end{pmatrix}$$

with $\lambda = (\lambda_\alpha\ \lambda_y\ \lambda_v\ \lambda_\tau)^T$. Evaluating the conditions, one obtains for all $t \in [t_0, t_e]$ that $\lambda_\alpha(t) = 1$, $\lambda_y(t) = -\gamma$, $\lambda_\tau(t) = 0$, and $\lambda_v(t) = \lambda_v(t_s) + (b - q - \gamma)(t_s - t)$ with $t_s = t_e$ if $q = 2$ and $t_s = t_1^-$ if $q = 1$. Since the control $u(t)$ enters the Hamiltonian

$$H_q = \lambda_\alpha(au + (b-q)v) + \lambda_y v + \lambda_v(b-q)u + \lambda_\tau$$

only linearly, the minimum of $H_q(t)$ is obtained if the control $u(t)$ takes values on the boundaries of the admissible sets $U_q = [0,1]$, $q \in \{1,2\}$, compare (12):

$$u(t) = 0 \qquad \text{for} \quad \lambda_v(t) \geq -\frac{a}{b-q}$$
$$u(t) = 1 \qquad \text{for} \quad \lambda_v(t) < -\frac{a}{b-q}\,.$$

Assuming constant controls $u_q$, $q \in \{1,2\}$, the state trajectories $y$ and $v$ are given by:

$$\begin{aligned} y(t) &= y(t_r) + \big(v(t_r) - t_r(b-q)u_q\big)(t - t_r) \\ &\quad + \frac{1}{2}(b-q)(t^2 - t_r^2)u_q \\ v(t) &= v(t_r) + (b-q)(t-t_r)u_q \end{aligned}$$

with $t_r = t_1$ if $q = 2$ and $t_r = t_0$ if $q = 1$. Finally, the Hamiltonian value condition $H_1(t_1^-) = H_2(t_1)$ from (11) for switching from $q = 1$ to $q = 2$ is solved numerically for the switching time $t_1$. With $a = \Delta\tau = \Delta\alpha = 0.1$, $d = 1$, $b = 5$, and $\gamma = 3$, the optimal switching time $t_1 = 0.26$ and the optimal costs $g(x(t_e)) = -0.0857$ are obtained with $u_1(t) = 1$ for $t \in [0, 0.26]$ and $u_2(t) = 0$ for $t \in [0.26, 0.9]$, see also Fig. 2. It can be verified that the discrete state sequence $q = (1,2)$ is optimal, when comparing with the optimal costs $g(x(t_e)) = 0 > -0.0857$ in the case that the discrete state $q$ is 1 in the entire time interval $[0,1]$. In this case, the optimal control $u_1$ is zero for $t \in [0,1]$.

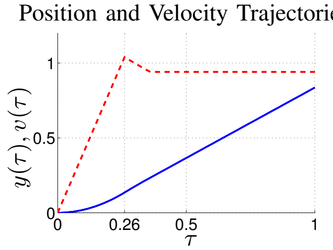


Fig. 2. Optimal position $y(\tau)$ (blue, solid) and velocity $v(\tau)$ (red, dashed) over time $\tau$.

## V. Conclusions

The paper introduces a hybrid minimum principle for time-varying hybrid systems with jumps of the continuous state at autonomous and controlled switchings. The version of the hybrid minimum principle enables us to find optimal controls with indirect methods for robotic scenarios in dynamic environments with impacts, where the velocity jumps and the impulse of the system remains continuous. In future, indirect optimal control algorithms based on the novel hybrid minimum principle will be developed.

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