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# Generalized incremental homogeneity, incremental observability and global observer design

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***Abstract*— We introduce the notion of incremental generalized homogeneity, giving new results on observer design for systems with bounded trajectories and putting into a unifying framework constructive results for triangular (feedback/feedforward) and homogeneous systems. An asymptotic state observer is designed by dominating the generalized homogeneity degree of the nonlinearities with the degree of the linear approximation of the observation error system. Incremental generalized homogeneity provides a high flexibility in the design of the observer gains and gives a systematic tool for the stabilization of systems with highly complex structures.**

***Index Terms*— Generalized homogeneity, observers, O-immersions.**

## I. Introduction

Homogeneity and homogeneous approximations have been investigated by many authors for the analysis of the stability of an equilibrium point: [12], [6] and [16] to cite few (see [4] for a complete list of references). This theoretical setup has been exploited in the design of homogeneous observers ([17], [14], [13], [3]): the idea is to design an observer for the homogeneous approximation and convergence to zero of the estimation error is preserved under any perturbation which does not change the homogeneous approximation. The most severe limitation of homogeneous observers is that their gains cannot be varied one independently from the other since the ratio between consecutive gains is constant. When dealing with non-homogeneous systems, different restrictions on the system structure have been adopted such as feedback ([18], [9]) or feedforward forms and strong observability assumptions such as complete uniform observability ([15]). The most general result on observers the author is aware of is [1], which however is mainly an *existence* result. The existence of a global observer has been proved for a restricted class of systems (5) ([2]) under the additional hypotheses that system (5) is output-to-state stable (OSS) and uniformly completely observable. However, also in this case the proof relies heavily on the information of the Lyapunov function as well as the non-negative functions which are used to characterize the output-to-state stability of the nonlinear system (5). Recently, constructive results have been obtained for uniformly completely observable and feedback linearizable systems with bounded state trajectories ([11]). In this paper we prove some new results on observer design by introducing the notion of generalized homogeneity, which extends the notion of homogeneity and puts into a unifying framework existing constructive results for triangular (feedforward/feedback) and homogeneous systems. Roughly speaking, any system can be considered homogeneous in the generalized sense with some degrees and weights, but the basic idea is that only if degrees and weights are properly related then the stabilization is feasible. An asymptotic state observer is designed for systems with bounded trajectories by dominating the generalized homogeneity degree of the nonlinearities with the degree of the linear approximation of the estimation error system. A novelty introduced by the notion of generalized homogeneity is the mixed low/high-gain observer structure, in combination with saturated state estimates ([18]). A peculiarity of our notion is that it is defined for increments of functions and this simplifies the design of the observer.

Using generalized homogeneity and weak observability properties such as incremental observability ([8]), we also see how to design a *globally* convergent observer, in the sense that it does not depend on the compact set containing the system trajectories. Our result is rather along the direction of the paper [11], in which the question of the constructive design of a global observer is addressed for the same class of systems considered in [2], and it can be straightforwardly extended to OSS systems by a simple transformation of the time scale (see also [2]) such that the system has bounded trajectories in the new time scale.

## II. Notation

- $\mathbf{R}^n$ (resp. $\mathbf{R}^{n\times n}$) is the set of $n$-dimensional real column vectors (resp. $n \times n$ matrices). $\mathbf{R}^n_0$ is the set of $n$-dimensional real column vectors with non-zero entries. $\mathbf{R}_+$ (resp. $\mathbf{R}^n_+$, $\mathbf{R}^{n\times n}_+$) denotes the set of real non-negative numbers (resp. vectors in $\mathbf{R}^n$, matrices in $\mathbf{R}^{n\times n}$, with real non-negative entries). $\mathbf{R}_>$ (resp. $\mathbf{R}^n_>$) denotes the set of real positive numbers (resp. vectors in $\mathbf{R}^n$ with real positive entries). Set
$$\mathbf{1} := (1, \cdots, 1)^T$$
with $\mathbf{1} \cdot c := (c, \cdots, c)^T$ for any $c \in \mathbf{R}$.
- For any $G \in \mathbf{R}^{p\times n}$ we denote by $G_{ij}$ (or $[G]_{ij}$ to avoid ambiguity) the $(i, j)$–th entry of $G$ and by $G_i$ (or $[G]_i$ to avoid ambiguity) the $i$–th row of $G$. We retain similar notations for functions.
- We denote by $\mathbf{C}^j(X, Y)$, with $j \geq 0$, $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^p$, the set of $j$-times continuously differentiable functions $f : X \to Y$. For any $f \in \mathbf{C}^0(X, Y)$ we write $f = 0$ to say that $f(x) = 0$ for all $x \in X$. We denote by $\mathbf{D}^j(\mathbf{R}^n)$ the set of functions $f \in \mathbf{C}^j(\mathbf{R}^n, \mathbf{R}^n)$ with decoupled components,

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*viz.* $f(x) = (f_1(x_1), \cdots, f_n(x_n))^T$. $\mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R}^n)$ is the set of bounded functions $f \in \mathbf{C}^0(\mathbf{R}_+, \mathbf{R}^n)$ and $\mathbf{L}^1(\mathbf{R}_+, \mathbf{R}^n)$ is the set of $f \in \mathbf{C}^0(\mathbf{R}_+, \mathbf{R}^n)$ such that $\int_0^\infty \|f(\tau)\| d\tau < \infty$. For any $f \in \mathbf{C}^1(\mathbf{R}^n, \mathbf{R}^p)$ we define $(\nabla f)(x) := ((\frac{\partial f_1}{\partial x})^T(x), \cdots, (\frac{\partial f_p}{\partial x})^T(x))^T$.

- the *increment* $\Delta f$ of $f \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R}^p)$ (at $x \in \mathbf{R}^n$) is defined as $(\Delta f)(x) := f(x + \Delta x) - f(x)$ with increments $\Delta x \in \mathbf{R}^n$. For any $f \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R}^p)$, $g \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R}^n)$ and $x \in \mathbf{R}^n$ we have $\Delta(f \circ g)(x) = f(g(x) + (\Delta g)(x)) - (f \circ g)(x)$, where $\circ$ denotes composition of functions. When we consider only increments $[\Delta x]_l \in \mathbf{R}$ along the $l$-th coordinate $x_l$, $l = 1, \ldots, n$, we replace $\Delta$ by $\Delta_l$ and define $\Delta_l f(x) := f(x_1, \ldots, x_{l-1}, x_l + [\Delta x]_l, x_{l+1}, \ldots, x_n) - f(x)$ with $\Delta_0 f(x) := f(x)$.
- For any two vectors $\epsilon, r \in \mathbf{R}^n$, we define $\epsilon^r = (\epsilon_1^{r_1}, \cdots, \epsilon_n^{r_n})^T$ and $\epsilon^r \diamond x = (\epsilon_1^{r_1} x_1, \cdots, \epsilon_n^{r_n} x_n)^T$, *viz.* the dilation of a vector $x$ with weight $r$. Moreover, we write $x \le y$ (resp. $x < y$, $x = y$) if and only if $x_i \le y_i$ (resp. $x_i < y_i$, $x_i = y_i$) for all $i = 1, \ldots, n$. We retain a similar notation for pairs of vectors: we write $(x, y) \le (z, w)$ (resp. $(x, y) = (z, w)$) if and only if $x_i \le z_i$ and $y_i \le w_i$ (resp. $x_i = z_i$ and $y_i = w_i$) for all $i = 1, \ldots, n$.

## III. Incremental Generalized Homogeneity Degree

### A. Definitions

Below we introduce the notion of generalized homogeneity which generalizes along several directions the classical notion of homogeneity.

**Definition** *3.1:* (**Incremental generalized homogeneity degree**). *A function $\phi \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R})$ is said to have incremental generalized homogeneity (i.g.h.) degree $h \in \mathbf{R}^n$ with weights $r \in \mathbf{R}^n_>$ and limit function $\Phi$ if there exist $\Phi \in \mathbf{C}^0(\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^{1\times n})$ such that*

$$\Delta\phi(\epsilon^r \diamond x) = \sum_{l=1}^n \Phi_l(x, \Delta x)\epsilon^{h_l}[\Delta x]_l \qquad (1)$$

*for all $\epsilon > 0$ and $x, \Delta x \in \mathbf{R}^n$. A function $\phi \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R}^n)$ is said to have incremental generalized homogeneity (i.g.h.) degree $(d, h) \in \mathbf{R}^n \times \mathbf{R}^n$ with weights $r \in \mathbf{R}^n_>$ and limit function $\Phi$ if there exist $\Phi \in \mathbf{C}^0(\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^{n\times n})$ such that*

$$\Delta\phi_i(\epsilon^r \diamond x) = \epsilon^{d_i + r_i} \sum_{l=1}^n \Phi_{il}(x, \Delta x)\epsilon^{h_l}[\Delta x]_l \quad (2)$$

*for all $i = 1, \ldots, n$, $\epsilon > 0$ and $x, \Delta x \in \mathbf{R}^n$.*

**Remark** *3.1:* It is easy to see that the limit function $\Phi x$ retains the same generalized homogeneity properties of $\phi$. I.g.h. reduces to the notion of homogeneity when $\Delta x = -x$, $\phi(0) = 0$, $r \in \mathbf{R}^n_>$ (*viz.* positive weights) and $d = \mathbf{1} \cdot d_0$ and $h = \mathbf{1} \cdot h_0$ for some $d_0, h_0 \in \mathbf{R}$. The function $\phi(x) := x_1 + x_2^3$ has i.g.h. degree $h := (r_1, 3r_2)^T$ with weights $r := (r_1, r_2)^T$ and limit function $\left(1, [\Delta x]_2^2 + 3(x_2[\Delta x]_2 + x_2^2)\right)$. Note that $\phi$ is homogeneous if and only if $r_1 = 3r_2$.□

The function $\sin x$ is not homogeneous in the generalized sense but its absolute value is bounded by the absolute value of $x$ which has i.g.h. degree $(0, 0)$. Therefore, a function, although not homogeneous in the generalized sense, may be bounded by some other function which is homogeneous in the generalized sense. This motivates the following definitions.

**Definition** *3.2:* (**Incremental generalized homogeneity degree in the upper bound**). *A function $\phi \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R})$ is said to have incremental generalized homogeneity in the upper bound (i.g.h.u.b.) degree $h \in \mathbf{R}^n$ with weights $r \in \mathbf{R}^n_0$ and bounding function $\Phi$ if there exist $\Phi \in \mathbf{C}^0(\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}_+^{1\times n})$ and $\epsilon_0 > 0$ such that*

$$|\Delta\phi(\epsilon^r \diamond x)| \le \sum_{l=1}^n \Phi_l(x, \Delta x)\epsilon^{h_l}|[\Delta x]_l| \qquad (3)$$

*for all $\epsilon \ge \epsilon_0$ and $x, \Delta x \in \mathbf{R}^n$. A function $\phi \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R}^n)$ is said to have incremental generalized homogeneity in the upper bound (i.g.h.u.b.) degree $(d, h) \in \mathbf{R}^n \times \mathbf{R}^n$ with weights $r \in \mathbf{R}^n_0$ and bounding function $\Phi$ if there exist $\Phi \in \mathbf{C}^0(\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}_+^{n\times n})$ and $\epsilon_0 > 0$ such that*

$$|\Delta\phi_i(\epsilon^r \diamond x)| \le \epsilon^{d_i + r_i} \sum_{l=1}^n \Phi_{il}(x, \Delta x)\epsilon^{h_l}|[\Delta x]_l| \quad (4)$$

*for all $i = 1, \ldots, n$, $\epsilon \ge \epsilon_0$ and $x, \Delta x \in \mathbf{R}^n$.*

**Remark** *3.2:* Without loss of generality one can assume $\epsilon_0 = 1$, otherwise rescale $x$ and $\Delta x$ as $z = \epsilon_0^r \diamond x$ and, respectively, $\Delta z = \epsilon_0^r \diamond \Delta x$ and define new bounding functions $[\Phi']_l(z, \Delta z) := \epsilon_0^{d_i + r_i - r_l + h_l} \Phi_l(\epsilon_0^{-r} \diamond z, \epsilon_0^{-r} \diamond \Delta z)$ (resp. $[\Phi']_{il}(z, \Delta z) := \epsilon_0^{d_i + r_i - r_l + h_l} \Phi_{il}(\epsilon_0^{-r} \diamond z, \epsilon_0^{-r} \diamond \Delta z)$). It is convenient to compare our notion of generalized homogeneity in the upper bound with homogeneous approximations in the $\infty$-limit ([3]). The comparison makes sense with $\Delta x = -x$, $\phi(0) = 0$, $r \in \mathbf{R}^n_>$ (*viz.* positive weights) and $d = \mathbf{1} \cdot d_0$ and $h = \mathbf{1} \cdot h_0$ for some $d_0, h_0 \in \mathbf{R}$ in the above definitions. First of all, in [3] the so-called approximating or limit function $\phi_0(x)$ is an approximation of $\phi(x)$ over compact sets, while in our case $\phi_0(x) := \sum_{l=1}^n \Phi_{il}(x, -x)|x_l|$ is an upper bound of the absolute value of $\phi(x)$. For this reason, in [3] the approximating function $\phi_0(x)$ is homogeneous with the same degree and weights as $\phi(x)$. The function $\sin x$ has i.g.h.u.b. degree $(0, 0)$ with any weights and bounding function 1 but it has no homogeneous approximation in the $\infty$-limit. The function $\phi(x) := (x_2, x_2 + x_2^p)^T$, $p > 1$, is homogeneous of degree $p - 1$ in the $\infty$-limit with weights $(2 - p, 1)^T$ and limit function $(x_2, x_2^p)^T$ *only if* $p < 2$. On the other hand $\phi$ has i.g.h.u.b. degree $((r_2 - r_1, r_2(p - 1))^T, (0, 0)^T)$ with any weights $r \in \mathbf{R}^2_>$ and bounding function $\left((0, 0)^T, (1, 1 + \frac{|(x_2 + [\Delta x]_2)^p - x_2^p|}{|[\Delta x]_2|})^T\right)$ *for all* $p > 1$.

Moreover, in our definition the weights may be also negative. This extension is useful for penalizing large values of some arguments of a function. The function $\phi(x) := x_1 x_2$ has i.g.h.u.b. degree 0 with weights $(1, -1)^T$ and limit function $\Phi := (x_2, x_1 + [\Delta x]_1)$. The same function has i.g.h.u.b. degree 2 with weights $(1, 1)^T$.

The function $\phi = (x_2, x_2^2 \sin x_1)^T$ has i.g.h.u.b. degree $(h, 0)$, with $h = (r_2 - r_1, r_2 + r_1)^T$, weights $r = (r_1, r_2)^T$ and bounding function $\Phi := \left((0, x_2^2 \frac{|(\sin(x_1 + [\Delta x]_1) - \sin x_1|}{|[\Delta x]_1|})^T, (1, |2x_2 + [\Delta x]_2|)^T\right)$.

### B. Some related properties

Throughout the paper we assume $\epsilon_0 \geq 1$ in the above definitions. All the properties and rules listed below and related to functions with i.g.h.u.b. degree can be stated and proved for functions with i.g.h. degree by replacing the inequalities with equalities and omitting the absolute values. We give the following rules without proof. The generalized homogeneity degree of the composition of two functions behaves as follows.

**(P1) (chaining rule)** *For any $\phi \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R}^n)$ with i.g.h.u.b. degree $(d, h)$, weights $l$ and bounding functions $\Phi$ and $\psi \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R}^n)$ with i.g.h.u.b. degree $(-h, p)$, weights $l$ and bounding function $\Psi$, if there exists $\Phi_M \in \mathbf{C}^0(\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}_+^{n\times n})$ such that*

$$\Phi(\epsilon^{-l} \diamond \psi(\epsilon^l \diamond x), \epsilon^{-l} \diamond (\Delta\psi)(\epsilon^l \diamond x)) \leq \Phi_M(x, \Delta x)$$

*for all $\epsilon \geq 1$ and $x, \Delta x \in \mathbf{R}^n$ then $\phi \circ \psi$ has i.g.h.u.b. degree $(d, p)$ with weights $l$ and bounding function $\Phi_M\Psi$.*

**(P2) (shifting rules)** Let $(A, B)$ be in Brunowski canonical form. Note that $A^T$ is the Moore-Penrose pseudoinverse of $A$, viz. $A^TAA^T = A^T$, $AA^TA = A$, $(A^TA)^T = A^TA$ and $(AA^T)^T = AA^T$. Therefore $I - AA^T = BB^T$ is the orthogonal projection onto $(\mathrm{Im}\{A\})^\perp = \mathrm{Im}\{I - AA^T\}$ while $I - A^TA$ is the orthogonal projection onto $(\mathrm{Im}\{A^T\})^\perp = \mathrm{Im}\{I - A^TA\}$ ($\mathrm{Im}\{W\}$ denotes the vector space generated by the columns of the matrix $W$). It is easy to see that

**(P2.1)***for any $w \in \mathrm{Im}\{I - AA^T\}$ (resp. $w \in \mathrm{Im}\{I - A^TA\}$) and $\phi \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R}^n)$, with i.g.h.u.b. degree $(d, h)$, weights $r$ and bounding function $\Phi$, $A\phi$ (resp. $A^T\phi$) has i.g.h.u.b. degree $(A(d + r) - r + w, h)$ (resp. $(A^T(d + r) - r + w, h)$) with weights $r$ and bounding function $A\Phi$ (resp. $A^T\Phi$).*

On the other hand,

**(P2.2)** *for any $w \in \mathrm{Im}\{I - A^TA\}$ (resp. $w \in \mathrm{Im}\{I - AA^T\}$) and $\phi \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R}^n)$ with i.g.h.u.b. degree $(d, h)$, weights $r$ and constant bounding function $\Phi$, $\phi \circ A$ (resp. $\phi \circ A^T$) has i.g.h.u.b. degree $(d, A^T(h - r) + r + w)$ (resp. $(d, A(h - r) + r + w)$) with weights $r$ and constant bounding function $\Phi A$ (resp. $\Phi A^T$).*

## IV. O-IMMERSIONS AND OBSERVER DESIGN

### A. Statement of the main result

Consider the system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + \phi(\mathbf{x}),\ \mathbf{y} = C\mathbf{x} + \psi(\mathbf{x}), \tag{5}$$

with state $x \in \mathbf{R}^n$, inputs $u \in \mathbf{R}^m$, output $y \in \mathbf{R}$, $A$ in Brunowski canonical form with $C = (1, 0, \cdots, 0)^T$, $\phi \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R}^n)$, $\phi(0) = 0$, and $\psi \in \mathbf{C}^0(\mathbf{R}^n, \mathbf{R})$, $\psi(0) = 0$. Throughout the paper we use the notations $\mathbf{x}, \mathbf{u}, \mathbf{y}$ for the functions of time and $x, y, u$ for their values. Also, let $\mathbf{x}(\cdot, \mathbf{u}, x_0)$ denote the state trajectory of (5) with input $\mathbf{u} \in \mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R}^m)$ and ensuing from $x_0$. We limit ourselves to single-output systems (5), leaving the straightforward extension to multi-output systems to the reader. We say that $\sigma \in \mathbf{D}^0(\mathbf{R}^n)$ is a *saturation function with levels* $h \in \mathbf{R}^n_>$ if, for each $i = 1, \ldots, n$, $[\sigma]_i(s) = s$ for all $s : |s| \leq h_i$ and $[\sigma]_i(s) = \mathrm{sign}(s)h_i$ for all $s : |s| > h_i$. The main result of this section is the following.

**Theorem** *4.1: Assume that*

**(D0)** *there exist two compact sets $R, C \subset \mathbf{R}^n$ such that $\mathbf{x}(t, \mathbf{u}^\circ, x_0) \in C$ for all $t \geq 0$, for all $x_0 \in R$ and for some $\mathbf{u}^\circ \in \mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R}^m)$,*

**(D1)** *$C^T\psi$ has i.g.h.u.b. degree $(-g, g)$ with weights $r \in \mathbf{R}^n_>$ and bounding function $C^T\Psi$ such that $\Psi(0, 0) = 0$,*

**(D2)** *$\phi$ has i.g.h.u.b. degree $((I - AA^T)g + A(r - A^Tr - g), g)$ with weights $r$ and bounding function $\Phi$ such that $\Phi(0, 0) = 0$,*

**(D3)** $A(2g - A^Tg) \leq A(r - A^Tr - g) \leq AA^Tg$.

*There exist $K, h_O > 0$, diagonal positive definite $\Gamma_O \in \mathbf{R}^{n\times n}$, $\epsilon \geq 1$ and a saturation function $\sigma$ with levels $h_O\epsilon^r$ such that*

$$\begin{aligned} &\dot{\xi} = A\xi + B\mathbf{u}^\circ + \phi \circ \sigma(\xi) + K^{(n)}(\mathbf{y} - C\xi - \psi \circ \sigma(\xi)), \\ &\xi(0) = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} &K^{(i)} := \epsilon^{2g} \diamond \{KC^T + A^T\Gamma_O K^{(i-1)}\},\ i = 1, \ldots, n, \\ &K^{(0)} := 0, \end{aligned} \tag{7}$$

*is an asymptotic state observer for each state trajectory $\mathbf{x}(\cdot, \mathbf{u}^\circ, x_0)$ of (5) with $x_0 \in R$.*

**Remark** *4.1:* Assumption (D0) is restrictive even for linear system (5), but it comes naturally into the picture if we think of using the state observer (6) together with a stabilizing state feedback controller for semiglobally stabilizing (5) by output feedback. The state observer (6) depends on $C$. In section D we will see how to design a *globally* convergent observer, in the sense that it does not depend on $C$. □

**Remark** *4.2:* Note that $\Phi(0, 0) = 0$ and $\Psi(0, 0) = 0$ are required. This is the simplest assumption which guarantees that the linear approximation of (5) around the origin is observable. Less restrictive assumptions can be considered as well.□

**Remark** *4.3:* For lower triangular $\phi$ and $C^T\psi$ we can always make a choice of the weights $r$ and degrees $g$ which satisfies (**D1**)-(**D3**) and corresponds to the "maximal" value $AA^Tg$ of $A(r - A^Tr - g)$, the vector of the "gaps" between each pair of consecutive weights (we do not prove this here for lack of space). For example, $\phi(x) := (x_1^p, x_2^q)^T$, $p, q > 1$, satisfies (**D1**)-(**D3**) with any weights $r$ and $g := (r_1p + 2qr_2, qr_2)^T$ (positive degrees and high-gain observer (6)).

On the other hand, for strict upper triangular $\phi$ and $C^T\psi$ we can always make a choice of the weights $r$ and degrees $g$ which satisfies (**D1**)-(**D3**) and corresponds to the "minimal" value $A(2g - A^Tg)$ of $A(r - A^Tr - g)$ in (**D3**) (we do not prove this here for lack of space). For example, $\phi(x) := (x_3^q, 0, 0)^T$, $q > 1$, with $r := (r_1, \frac{3q-1}{2(2q-1)}r_1, \frac{q}{2q-1}r_1)^T$, $g := (-\frac{q-1}{4(2q-1)}r_1, -\frac{q-1}{4(2q-1)}r_1, -\frac{q-1}{4(2q-1)}r_1)^T$, any $r_1 > 0$ satisfies (**D1**)-(**D3**) (decreasing weights, negative degrees and low-gain observer (6)).

The rationale behind (**D1**)-(**D3**) can be explained as follows. If the state trajectories of (5) remain for all times in $C$, any Luenberger-like state observer of (5) has the form $\dot{\xi} = A\xi + B\mathbf{u} + \phi \circ \sigma(\xi) + L(\mathbf{y} - C\xi - \psi \circ \sigma(\xi))$ with $L \in \mathbf{R}^n$ and $\sigma \in \mathbf{D}^0(\mathbf{R}^n)$ a saturation function with levels such that the restriction of $\sigma$ to $C$ is the identity function. If $\Delta x := \xi - x$ then the observation error system is $\dot{\Delta\mathbf{x}} = g(\Delta\mathbf{x},\mathbf{x}) := (A - LC)\Delta\mathbf{x} + \Delta(\phi\circ\sigma)(\mathbf{x}) - L\Delta(\psi\circ\sigma)(\mathbf{x})$. For semiglobally stabilizing the observation error system for some choice of $L$ it is sufficient to stabilize the linear approximation $(A - LC)\Delta\mathbf{x}$ of $g(\Delta\mathbf{x},\mathbf{x})$ around the origin $\Delta\mathbf{x} = 0$ with some $L$ and dominate the incremental degree of $\Delta(\phi\circ\sigma)(\mathbf{x}) - L\Delta(\psi\circ\sigma)(\mathbf{x})$ by the incremental degree of $(A - LC)\Delta\mathbf{x}$. Theorem 4.1 states that this is possible if the degree of $\phi$ is in between $((I - AA^T)g + A(2g - A^Tg), g)$ (which corresponds to a "minimal" gap between each pair of weights) and $(g, g)$ (which corresponds to a "maximal" gap between each pair of weights). (**D1**)-(**D3**) can be interpreted as a trade-off condition for non-triangular $\phi$ and $C^T\psi$. For example, $\phi(x) = (x_1^q x_3, 0, 0)^T$, $p > 1$, is non-triangular and satisfies (**D1**)-(**D3**) with $r := (r_1, (p+1)r_1, r_1)^T$, $g := (\frac{3pr_1}{2}, -\frac{pr_1}{2}, -\frac{pr_1}{2})^T$ and any $r_1 > 0$ (negative/positive degrees and low/high-gain observer (6)).□

**Remark** *4.4:* The observer gain matrix $K^{(n)}C$ in (6) is homogeneous with weights $r$ only if $g_j = g_0\mathbf{1}$, which, since $AA^T\mathbf{1} = A\mathbf{1}$, corresponds to the choice $2A\mathbf{1}\cdot g_0 = A(r - A^Tr)$ in (**A2**). Therefore, the gaps between each pair of consecutive weights are all equal to $2g_0$. Under this restriction, (**D2**) simply says that $\phi$ (resp. $\psi$) be homogeneous (in the upper bound) with degree $2g_0$ (resp. 0) and weights $r$, which is the condition required in [13] under the additional assumption that $\psi = 0$. Under this regard, it is important to say that the saturation function $\sigma$ is a crucial design issue only when $\phi$ and $\psi$ are not homogeneous in the classical sense and the saturation levels are directly proportional to the maximal gap between two consecutive weights. If $\phi$ and $\psi$ are homogeneous the saturation function can be set equal to the identity function and (6) is linear and homogeneous.□

### *B. Constructive procedure for the observer (6)*

The construction of the state observer (6) is accomplished according to the following steps:

(**iv**) Find a diagonal positive definite $\Gamma_O \in \mathbf{R}^{n\times n}$ and $K > 0$ such that

$$T_O := -2(KC^TC + A^T\Gamma_O A) + (A + A^T\Gamma_O^2)\cdot \cdot(I - A^T\Gamma_O)^{-1} - (I - A^T\Gamma_O)^{-T}(A + A^T\Gamma_O^2)^T < 0. \quad (8)$$

It is easy to see by direct calculations that the matrix $\Gamma_O$ always exists.

(**v**) Find $h_O > 0$ such that

$$R_O := -2(KC^TC + A^T\Gamma_O A) + \Omega_O + \Omega_O^T < 0 \quad (9)$$

where

$$\Omega_O := \Big(2(I + A^T\Gamma_O)\{\max_{\substack{z:\|z\|\le\sqrt{n}h_O,\\ \Delta z:\|\Delta z\|\le 2\sqrt{n}h_O}} \Phi(z,\Delta z)\} +2KC^T\{\max_{\substack{z:\|z\|\le\sqrt{n}h_O,\\ \Delta z:\|\Delta z\|\le 2\sqrt{n}h_O}} \Psi(z,\Delta z)\} + A + A^T\Gamma_O^2\Big)\cdot \cdot(I - A^T\Gamma_O)^{-1}. \quad (10)$$

This $h_O$ always exists on account of (8) and since $\Phi$ and $\Psi$ are continuous and $\Phi(0,0) = 0$ and $\Psi(0,0) = 0$.

(**vi**) Pick $\epsilon \ge 1$ such that $\max_{x:x\in C}|x_i| \le h_O\epsilon^{r_i}$ for all $i = 1,\ldots,n$.

### *C. Proof of the main result*

We prove theorem 4.1 by showing how steps (**iv**)-(**vi**) of section IV-B lead to establish that (6) is an asymptotic state observer for the state trajectories of (5). A key point of the proof relies on the following notion of O-immersion, which is *per se* a useful tool for observer design.

**Definition** *4.1: A system $\Sigma(\mathbf{x},\mathbf{u},\mathbf{y}) : \dot{\mathbf{x}} = F(\mathbf{x},\mathbf{u},\mathbf{y})$, with state $x \in \mathbf{R}^n$, input $u \in U \subset \mathbf{R}^m$ and output $y \in \mathbf{R}^p$, is said to be O-immersed under mappings $(X_O, \Pi_O, Y_O^u)$ into $\Sigma_O(\mathbf{x}_O,\mathbf{u},\mathbf{y}_O) : \dot{\mathbf{x}}_O = F_O(\mathbf{x}_O,\mathbf{u},\mathbf{y}_O)$, with state $x_O \in \mathbf{R}^n$, input $u \in U$ and output $y_O \in \mathbf{R}^{\hat p}$, $\hat p \ge p$, if there exist $X_O \in \mathbf{C}^1(\mathbf{R}^n,\mathbf{R}^n)$, $\Pi_O \in \mathbf{C}^0(\mathbf{R}^n,\mathbf{R}^{\hat p\times p})$ and $Y_O^u \in \mathbf{C}^0(\mathbf{R}^n,\mathbf{R}^{\hat p})$ such that*

(**E1**) *the mapping $x_O \mapsto x = X_O(x_O)$ is a diffeomorphism and $\mathrm{rank}\{\Pi_O(x_O)\} = p$ for each $x_O$,*

(**E2**) *$F_O(\cdot,u,Y_O^u(\cdot))$ and $F(\cdot,u,(\Pi_O^TY_O^u)\circ X_O^{-1}(\cdot))$ are $X_O$-related for each $u \in U$.*

The meaning of an O-immersion is the following: the state $x$ of $\Sigma$ is mapped diffeomorphically by $X_O$ onto the state $x_O$ of $\Sigma_O$ and, for each $x_O$, $\Pi_O(x_O)$ is (up to isomorphisms) the canonical immersion of $\mathbf{R}^p$ (the output space of $\Sigma$) into $\mathbf{R}^{\hat p}$ (the output space of $\Sigma_O$). A particular O-immersion of $\dot{\mathbf{x}} = A\mathbf{x} + \phi(\mathbf{x})$, $\mathbf{y} = C\mathbf{x} + \psi(\mathbf{x})$ corresponds to consider $C^Ty + A^T\dot x = x + A^T\phi(x)$ as the new outputs. The idea of using an augmented output vector for stabilizing a lower triangular system is not new ([18]) but our O-immersion can be defined also for non-triangular systems.

*Proof of theorem 4.1* (Sketch). The proof relies on the following fact which is a consequence of the notion of O-immersion: for a given O-immersion $\Sigma_O$ of $\Sigma$ under mappings $(X_O, \Pi_O, Y_O^u)$, if for each fixed input $\mathbf{u}$ the outputs $\mathbf{y}_O := Y_O^u(\mathbf{x}_O)$ are such that the equilibrium $\mathbf{x}_O = 0$ of $\dot{\mathbf{x}}_O = F_O(\mathbf{x}_O,\mathbf{u},\mathbf{y}_O)$ is locally (resp. globally) stable then for each fixed input $\mathbf{u}$ the outputs $\mathbf{y} := (\Pi_O^TY_O^u)\circ X_O^{-1}(\mathbf{x})$ are such that the equilibrium $\mathbf{x} = 0$ of $\dot{\mathbf{x}} = F(\mathbf{x},\mathbf{u},\mathbf{y})$ is locally (resp. globally) stable. By virtue of the augmented number of outputs in $\Sigma_O$, we easily construct the outputs $\mathbf{y}_O := Y_O^u(\mathbf{x}_O)$ from which we obtain the outputs $\mathbf{y} := (\Pi_O^TY_O^u)\circ X_O^{-1}(\mathbf{x})$.

Let $I \in \mathbf{R}^{n\times n}$ be the identity matrix, $G_O \in \mathbf{R}^{n\times n}$ a diagonal positive definite matrix and $H \in \mathbf{R}_>$ and identify (whenever necessary) $G_O, I, HC^TC$ and $A$ with linear maps $G_O, I, HC^TC \in \mathbf{D}^1(\mathbf{R}^n)$ and $A \in \mathbf{C}^1(\mathbf{R}^n,\mathbf{R}^n)$. In particular, select $G_O$ with i.g.h. degree $(Ag, Ag)$, weights $r$ and diagonal positive definite limit $\Gamma_O \in \mathbf{R}^{n\times n}$ and $H$ such that $HC^TC$ has i.g.h. degree $(g, g)$, weights $r$ and constant limit function $KC^TC$, where $K$ and $\Gamma_O$ are chosen as pointed out in (8) (constructive step (**iv**)). On account of (8) and using continuity of $\Phi$ and $\Psi$ and the fact that $\Phi(0,0) = 0$ and $\Psi(0,0) = 0$, find $h_O > 0$ as pointed out in (9) (constructive steps (**v**)). Moreover, for each $\epsilon \ge 1$ let $\sigma$ be a saturation function $\sigma$ with levels $h_O\epsilon^r$.

For each $\epsilon \ge 1$ satisfying the constructive step (**vi**) of section IV-B we have $\sigma(x) = x$ for all $x \in C$ and on account of (**D0**) the trajectories $\mathbf{x}(\cdot,\mathbf{u}^\circ,x_0)$ of (5) with $x_0 \in R$ satisfy

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}^\circ + (\phi\circ\sigma)(\mathbf{x}),\ \mathbf{y} = C\mathbf{x} + (\psi\circ\sigma)(\mathbf{x}). \quad (11)$$

With (6), the observation error $\Delta\mathbf{x} := \xi - \mathbf{x}$ satisfies

$$\dot{\Delta\mathbf{x}} = A\Delta\mathbf{x} + \Delta(\phi\circ\sigma)(\mathbf{x}) - K^{(n)}\Delta\mathbf{y} \quad (12)$$

with $\Delta\mathbf{y} := C\Delta\mathbf{x} + \Delta(\psi\circ\sigma)(\mathbf{x})$ and $K^{(n)}$ defined in (7). By straightforward calculations we find out that $K^{(n)} = (I - A^T G_O)^{-1}HC^T$. It easy to see that the system (12) with state $\Delta x$, input $x \in C$ and output $\Delta y$ is O-immersed under mappings $(X_O, \Pi_O, Y_O^x)$:

$$X_O := (I - A^T G_O)^{-1},\ Y_O^x := I + \Delta\psi_O(x_O),\ \Pi_O := C^T, \quad (13)$$

where

$$\psi_O := [C^T(\psi\circ\sigma) + A^T(\phi\circ\sigma + G_O)]\circ X_O,$$
$$x_O := X_O^{-1}(x), \quad (14)$$

into the system

$$\dot{\Delta\mathbf{x}}_O = \Delta\phi_O(\mathbf{x}_O) - K_O\Delta\mathbf{y}_O, \quad (15)$$

with state $\Delta x_O \in \mathbf{R}^n$, input $x \in C$ and output $\Delta y_O \in \mathbf{R}^n$, where

$$\phi_O = [A + (\phi\circ\sigma)]\circ X_O,\ K_O := HC^T C + A^T G_O A. \quad (16)$$

We will prove that, for each fixed state trajectory $\mathbf{x}(\cdot,\mathbf{u}^\circ,x_0)$ of (11) with $x_0 \in R$, (15) with outputs $\Delta\mathbf{y_O} := Y_O^u(\Delta\mathbf{x_O})$ is globally asymptotically stable and therefore, by the O-immersion property, (12) with outputs $\Delta\mathbf{y} := (\Pi_O^T Y_O^u)\circ X_O^{-1}(\Delta\mathbf{x}) = C\Delta\mathbf{x} + \Delta(\psi\circ\sigma)(\mathbf{x})$ is globally asymptotically stable, *viz.* the observation error $\Delta\mathbf{x}$ tends to zero as $t$ tends to infinity. By repeated use of properties (**P1**), (**P2**) and (**P3**) it can be established that

**(F1)** $K_O$ has i.g.h. degree $(g,g)$ with weights $r + \mathbf{1}\cdot(g_1 - g_n)$ and limit function $KC^TC + A^T\Gamma_O A$,

**(F2)** $\phi_O - K_O\psi_O$ has i.g.h.u.b. $(g,g)$ with weights $r + \mathbf{1}\cdot(g_1 - g_n)$ and bounding function $\Omega_O$ defined as in (10).

From (9) (constructive step (**v**)) it follows that

**(F3)** there exists $\alpha_O > 0$ such that

$$\Delta x_O^T[\omega_O(\Delta x_O) - (KC^TC + A^T\Gamma_O A)\Delta x_O]$$
$$\leq -\alpha_O\|\Delta x_O\|^2 \quad (17)$$

for each $\Delta x_O \in \mathbf{R}^n$ and for all

$$\omega_O(\Delta x_O) \in F_O(\Delta x_O)$$
$$:= \Big\{ w \in \mathbf{R}^n : |w_j| \leq \textstyle\sum_{l=1}^n [\Omega_O]_{jl}|[\Delta x_O]_l| \Big\}. \quad (18)$$

Let $\epsilon \geq 1$ and $V_O(\Delta x_O) := \frac{1}{2}\|\epsilon^{-s}\diamond\Delta x_O\|^2$. By virtue of (**F1**)-(**F3**)

$$\nabla V_O(\Delta x_O)\Big(\Delta\phi_O(x_O) - K_O Y_O^x(\Delta x_O)\Big)$$
$$\leq -\alpha_O\|\epsilon^{g-s}\diamond\Delta x_O\|^2 \quad (19)$$

for each $x \in C$, $x_O = X_O^{-1}(x)$ and for all $\Delta x_O$. As a consequence of the definition of O-immersion, $\Delta\phi_O(x_O) - K_O Y_O^x$ and $A + \Delta(\phi\circ\sigma)(x) - K^{(n)}\Pi_O^T Y_O^x \circ X_O^{-1}$ are $X_O$-related for each $x \in C$. Let $W := V_O \circ X_O^{-1}$. It follows that

$$\nabla W(\Delta x)\Big(A\Delta x + \Delta(\phi\circ\sigma)(x)$$
$$-K^{(n)}[C\Delta x + \Delta(\psi\circ\sigma)(x)]\Big)$$
$$= (\nabla V_O)\circ X_O^{-1}(\Delta x)\Big(\Delta\phi_O\circ X_O^{-1}(x)$$
$$-K_O Y_O^x \circ X_O^{-1}(\Delta x)\Big) \leq -\alpha_O\|\epsilon^{g-s}\diamond X_O^{-1}(\Delta x)\|^2$$

for all $\Delta x \in \mathbf{R}^n$ and $x \in C$. By selecting $\epsilon \geq 1$ as in the constructive step (**iv**) of section IV-B and on account of (**D0**) it follows that (7) is an asymptotic state observer for each state trajectory $\mathbf{x}(\cdot,\mathbf{u}^\circ,x_0)$ of (5) with $x_0 \in R$. □

### D. Globally convergent observers

The state observer (6) depends on $C$. In this section we will see how to design a *globally* convergent observer, in the sense that it does not depend on $C$. This can be achieved at the price of an additional "incremental" observability condition on (5). Denote by $\mathbf{x}(\cdot,x_0)$ (or simply $\mathbf{x}(\cdot)$ when there is no ambiguity) and $\mathbf{y}(\cdot,x_0)$ (or simply $\mathbf{y}(\cdot)$) the state and, respectively, the output trajectory of (5) with input $\mathbf{u} = 0$ and ensuing from $x_0$ (resp. $y_0 := Cx_0 + \psi(x_0)$).

**Theorem** *4.2: Assume (**D1**)-(**D3**) and, in addition,*

**(G1)** $\mathbf{x}(\cdot,x_0) \in \mathbf{L}^\infty(\mathbf{R}_+,\mathbf{R}^n)$ *for all* $x_0 \in \mathbf{R}^n$,

**(G2)** *(5) with* $\mathbf{u} = 0$ *is incrementally observable, viz. for any* $x_0, z_0 \in \mathbf{R}^n$ *we have* $\mathbf{y}(t,x_0) = \mathbf{y}(t,z_0)\forall t \geq 0 \Rightarrow \mathbf{x}(t,x_0) = \mathbf{x}(t,z_0)\forall t \geq 0$.

*There exist* $K, h_O > 0$, *diagonal positive definite* $\Gamma_O \in \mathbf{R}^{n\times n}$ *and a saturation function* $\sigma$ *such that*

$$\dot{\xi} = A\xi + \phi\circ\sigma(\xi) + K^{(n)}(\mathbf{y} - C\xi - \psi\circ\sigma(\xi)),$$
$$\xi(0) = 0, \quad (20)$$
$$\dot{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}\{|\boldsymbol{\epsilon}^{-s_1+g_1}[\sigma]_1(\mathbf{y} - C\xi - \psi\circ\sigma(\xi))|^2$$
$$+\|\boldsymbol{\epsilon}^{-s+g}\diamond\sigma(\xi - \sigma(\xi))\|^2\},\ \boldsymbol{\epsilon}(0) = 1, \quad (21)$$

*with* $K^{(n)}$ *defined in (7) and* $s := r + \mathbf{1}\cdot(g_1 - g_n)$, *is an asymptotic state observer for each state trajectory* $\mathbf{x}(\cdot,x_0)$ *of (5).*

**Remark** *4.5:* The matrix $\Gamma_O$ and $K > 0$ are selected as in (**iv**) and, with $\Omega_O$ defined in (10):

**(v')** $h_O > 0$ is such that

$$-2(KC^TC + A^T\Gamma_O A) + [A^T\Gamma_O(I - A^T\Gamma_O)^{-1}$$
$$+(I - A^T\Gamma_O)^{-T}\Gamma_O A]2nh_O^2 + \Omega_O + \Omega_O^T < 0. \quad (22)$$

Moreover, the saturation function $\sigma$ has levels $h_O\boldsymbol{\epsilon}^r$, where $\boldsymbol{\epsilon}$ is provided by (21). □

The Van Der Pol oscillator $\dot{\mathbf{x}}_1 = \mathbf{x}_2,\ \dot{\mathbf{x}}_2 = -\mathbf{x}_1 + \mathbf{x}_2(1 - \mathbf{x}_1^2)$ with output $\mathbf{y} = \beta(\mathbf{x}_1)$, $\beta \in \mathbf{C}^0(\mathbf{R},\mathbf{R})$ any injective globally Lipschitz continuous function, satisfies the assumptions of theorem 4.2.

(*Proof of theorem 4.2*) (Sketch). Let $G_O$, $\Gamma_O$, $H$, $K$, $\Gamma_O$, $h_O > 0$ and $\sigma$ be as in the proof of theorem 4.1. Denote by $\mathbf{x}(t), \boldsymbol{\epsilon}(t), \xi(t)$ (resp. $\mathbf{y}(t)$) or simply $\mathbf{x}, \boldsymbol{\epsilon}, \xi$ (resp. $\mathbf{y}$) the state (resp. output) trajectories of (5)-(20)-(21) with input

$\mathbf{u}=0$ and initial conditions $x_0$, $\epsilon_0 := 1$, $\xi_0 := 0$ (resp. $y_0 := Cx_0 + \psi(x_0)$). Let $s := r + \mathbf{1}\cdot(g_1 - g_n)$ and $V_O(\Delta x_O, \epsilon) := \frac{1}{2}\|\epsilon^{-s} \diamond \Delta x_O\|^2$. Note that $A^T G_O X_O$ has i.g.h.u.b. degree $(-g, g)$ with weights $s$ and b.f. $A^T\Gamma_O(I - A^T\Gamma_O)^{-1}$ and $|\epsilon^{-s_1+g_1}[\sigma]_1(v)|^2 + \|\epsilon^{-s+g} \diamond \sigma(w)\|^2 \leq \epsilon^{2g_n} 2nh_O^2$ for all $v, w$ and $\epsilon \geq 1$ (since $\sigma$ is a saturation function with levels $h_O\epsilon^r$ and $\{g_j\}_{j=1,\ldots,n}$ is a decreasing sequence). Using the constructive step (**v'**) and following the same steps for the proof of (19) with $\Delta y_O := Y_O^x \circ X_O(\Delta x_O)$

$$\dot{V}_O \leq -\frac{\alpha_O}{2}\|\boldsymbol{\epsilon}^{g-s} \diamond \Delta\mathbf{x}_O\|^2 - \min_i\{s_i\} V_O(\Delta\mathbf{x}_O, \boldsymbol{\epsilon})\frac{\dot{\boldsymbol{\epsilon}}}{\boldsymbol{\epsilon}}$$
$$+\frac{2}{\alpha_O}\|\boldsymbol{\epsilon}^{-s-g} \diamond \{(I - K_O A^T)((\phi\circ\sigma - \phi)\circ X_O)(\mathbf{x}_O)$$
$$-K_O C^T((\psi\circ\sigma - \psi)\circ X_O)(\mathbf{x}_O)\}\|^2. \quad (23)$$

**Claim** $\#1$. $\boldsymbol{\epsilon} \in \mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R})$. Since the right-hand part of $\dot{\boldsymbol{\epsilon}}$ is non-negative, $\boldsymbol{\epsilon}(\cdot)$ is monotonically increasing and there exists $T \in (0, \infty]$ such that $\lim_{t\uparrow T}\boldsymbol{\epsilon}(t) = \epsilon_\infty \leq \infty$ and $[0, T)$ is the maximal right extension interval of $\boldsymbol{\epsilon}(\cdot)$. Assume that $\epsilon_\infty = \infty$. By (**G1**) there exists $T' < T$ such that $|\mathbf{x}_i(t)| \leq h_O\boldsymbol{\epsilon}^{r_i}(t)$ for all $i = 1, \ldots, n$ and for all $t \in [T', T)$. Therefore, along such trajectories $(\sigma \circ X_O)(\mathbf{x}_O(t)) = \sigma(\mathbf{x}(t)) = \mathbf{x}(t) = X_O(\mathbf{x}_O(t))$ for all $t \in [T', T)$. Since $\dot{\boldsymbol{\epsilon}}(t)$ and $\boldsymbol{\epsilon}(t)$ are non-negative for all $t \in [T', T)$, by integrating the equation of $\dot{\boldsymbol{\epsilon}}$ over $[T', t]$ and on account of (23),

$$\ln\boldsymbol{\epsilon}(t) \leq \ln\boldsymbol{\epsilon}(T') + \frac{k_0}{\alpha_O} V_O(\Delta\mathbf{x}_O(T'), \boldsymbol{\epsilon}(T')) < \infty \quad (24)$$

for all $t \in [T', T)$ and for some $k_0 > 0$ which is a contradiction since $\lim_{t\uparrow T}\boldsymbol{\epsilon}(t) = \infty$. This proves the claim.

**Claim** $\#2$. $\lim_{t\to\infty}(\mathbf{y}(t) - C\xi(t) - \psi\circ\sigma(\xi(t))) = 0$ *and* $\lim_{t\to\infty}(\sigma(\xi(t)) - \xi(t)) = 0$. By integrating (21) over $[0, t]$, $t \geq 0$, on account of (23) and since $\mathbf{x} \in \mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R}^n)$ (by (**G1**)) and $\boldsymbol{\epsilon} \in \mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R})$ (by claim $\#1$) , it follows that $V_O(\Delta\mathbf{x}_O, \boldsymbol{\epsilon}) \in \mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R})$ and, therefore,

$$\xi, \boldsymbol{\epsilon}^{-s+g} \diamond \sigma(\xi - \sigma(\xi)) \in \mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R}^n),$$
$$\boldsymbol{\epsilon}^{-s_1+g_1}[\sigma]_1(\mathbf{y} - C\xi - \psi\circ\sigma(\xi)) \in \mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R}). \quad (25)$$

Moreover, $\boldsymbol{\epsilon}^{-s_1+g_1}[\sigma]_1(\mathbf{y} - C\xi - \psi\circ\sigma(\xi))$ and $\boldsymbol{\epsilon}^{-s+g}\diamond\sigma(\xi - \sigma(\xi))$ are uniformly continuous on $\mathbf{R}_+$, since $\dot{\mathbf{x}}, \mathbf{x}, \dot{\xi}, \xi \in \mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R}^n), \dot{\boldsymbol{\epsilon}}, \boldsymbol{\epsilon} \in \mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R})$ by (**G1**), claim $\#1$ and (25) and $\psi\circ\sigma$, $\sigma$ and $\psi$ are uniformly continuous on any compact set of $\mathbf{R}^n$. Our claim follows from Barbalat's lemma.

Since $\mathbf{x}, \xi \in \mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R}^n)$ (by (**G1**) and (25)) and $\boldsymbol{\epsilon} \in \mathbf{L}^\infty(\mathbf{R}_+, \mathbf{R})$ (by claim $\#1$) the $\Omega$-limit set of $(\mathbf{x}, \xi, \boldsymbol{\epsilon})$ is non-empty, compact and invariant and, by virtue of claim $\#2$, it is contained in the set of points $(x, \xi, \epsilon)$ such that $Cx + \psi(x) = C\xi + \psi\circ\sigma(\xi)$ and $\sigma(\xi) = \xi$. Inside this set we have $\dot{\mathbf{x}} = A\mathbf{x} + \phi(\mathbf{x})$, $\dot{\xi} = A\xi + \phi(\xi)$ and $C\mathbf{x} + \psi(\mathbf{x}) = C\xi + \psi(\xi)$, which by (**G2**) implies that $\mathbf{x} = \xi$. Therefore, $\lim_{t\to\infty}(\mathbf{x}(t) - \xi(t)) = 0$, *viz* the observation error tends to zero as $t \to \infty$ whatever is $x_0$.□

## V. CONCLUSIONS

We introduced a notion of incremental generalized homogeneity, giving new constructive results on observer design. Using incremental generalized homogeneity, we also point out the procedure for designing a globally convergent observer for systems with bounded trajectories. Future work will be devoted to the global observer design for systems with unbounded trajectories.

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