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Lyapunov Stability Analysis of a Twisting Based Control Algorithm for Systems with Unmatched Perturbations

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***Abstract*—In this note, we present a result on stabilization of nonlinear strict-feedback systems affected by unknown perturbations when a control based on the so-called twisting algorithm, a second-order sliding mode controller, is applied. The novelty of the note relies in the stability analysis of the closed loop system. It follows along similar lines as for well-established theorems for nonlinear time-varying systems in $cascade$, with continuous right-hand sides. However, the class of systems that we deal with are discontinuous and perturbed. Although the presented analysis is aimed to perturbed second-order systems in strict feedback form, the purpose of this note is to settle the basis for a methodological stability analysis approach for higher-order systems. An illustrative example is provided.**

I. INTRODUCTION

We study strict feedback systems affected by additive, possibly unbounded disturbances. The control goal is to stabilize the origin in finite time. Roughly, the control method consists in a two-loop design: an outer loop in which the control law is designed following the standard backstepping method and an inner loop in which a discontinuous sliding mode controller, called the twisting algorithm –see [1], is used to exactly compensate for the disturbances.

Backstepping makes our control approach methodological while the use of sliding mode control enables high-accuracy tracking and achieves exact compensation of matched perturbations. Backstepping is particularly useful for systems with unmatched perturbations *i.e.*, parameter uncertainties and smooth external disturbances appearing in dynamic equations where there is no control input. 'Classical' sliding-mode controllers have been applied combined with different robust techniques in order to reduce the effect of such perturbations [2]–[3]. However, such controllers could not ensure the exact tracking of output unmatched variables. The controller proposed in [4], [5] which utilizes the so-called quasi-continuous high-order sliding mode algorithm of [6], ensures exact tracking of a smooth signal despite the presence of unmatched perturbations. Nevertheless, the scheme in [4] only guarantees $local$ stability hence, nothing is ensured about the transient phase before, the sliding mode for each virtual control is reached. In [5], the transient stage is handled by using integral HOSM [7] which increases the control complexity.

Backstepping sliding mode control is certainly not original in this note. Our main contribution strives rather in the method of stability analysis. As is well-understood now, at the core of strict-feedback forms one find *cascaded* systems –[8]. These have been thoroughly studied in the literature of (continuous) nonlinear systems for the last 20 years or so. Cascaded systems consist in two subsystems which independently, are stable and are interconnected by a nonlinearity. Under such setting, a necessary and sufficient condition for stability is that the trajectories of the cascaded system remain bounded –see [9].

Backstepping sliding mode control leads to a complex cascaded system described by integral-differential equations and equations with discontinuous right-hand sides. One way to analyze the stability of the integral-differential equation one needs to differentiate however, this leads to ever more complex equations and eventually, to restrictive conditions of boundedness of trajectories, as is assumed in [6]. Backstepping also leads naturally to the analysis of a *cascaded* system. Following that train of thought, in the recent note [10] we relaxed the restrictive hypotheses from [6].

Yet, the results available in the literature of cascaded systems are inapplicable as such in the present setting. A fundamental, common, assumption in the analysis of cascaded systems is that one disposes of a Lyapunov function for the perturbed system, taken independently. In the present setting, this is a true stumbling block as it is tantamount to asking for a (converse) Lyapunov function for an integral-differential equation, having specific growth order properties.

This note continues and improves the main results in [10]. Firstly, we use the twisting controller for which a Lyapunov function has been recently proposed in [11]. Then, a theorem for stability of cascades of systems with discontinuous right-hand sides is established. The direct outcome of the latter, is to settle the basis for a backstepping-based high-order sliding mode control approach for nonlinear systems in strict feedback form, with unmatched uncertainties.

The rest of the paper is organized as follows. In the following section we present the problem statement and our main result, in Section III we revisit the finite-time stabilization of the double integrator; in Section IV we present an illustrative example and we conclude with some remarks in Section V.

II. PROBLEM STATEMENT AND ITS SOLUTION

Consider second order nonlinear systems of the form

$$\dot{\xi}_1 = f_1(t,\xi_1) + g_1(t,\xi)\xi_2 + \omega_1(t,\xi) \quad \text{(1a)}$$
$$\dot{\xi}_2 = f_2(t,\xi) + g_2(t,\xi)u + \omega_2(t,\xi) \quad \text{(1b)}$$

where $\xi = [\xi_1, \xi_2]^T$ is the state vector and is assumed to be known, $\xi_1, \xi_2 \in \mathbb{R}$; $u \in \mathbb{R}$ is the control input. For simplicity we assume that f_i and g_i are smooth functions; also, the unknown

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perturbation term ω_1 is taken to be a bounded function, similarly for ω_2. Furthermore it is assumed that ω_1 is once continuously differentiable. For the application of backstepping control, we also assume that $g_i(t,\xi) \neq 0$ for all $(t,\xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$.

The control problem is to design a controller such that the state ξ_1 tracks a desired smooth reference $t \mapsto \xi_d$ in spite of the presence of the unknown bounded perturbations ω_1, ω_2.

A. The algorithm

We assume that the solutions of a dynamical system involving the sign function, are defined in Filippov's sense. The control algorithm is described next.

Step 1. The first sliding surface is defined as $\{\sigma_1 = 0\}$ with $\sigma_1 = \xi_1 - \xi_d$. According to standard backstepping control, we consider ξ_2 in (1a) as the virtual control input,

$$\phi_1(t,\xi_1) = g_1^{-1}(t,\xi)[-f_1(t,\xi_1) + u_{11} + \dot{\xi}_d] \quad (2a)$$

$$\dot{u}_{11} = -\alpha_1 \text{sgn}(\sigma_1) - \beta_1 \text{sgn}(\dot{\sigma}_1) \quad (2b)$$

The right hand side of (2b) is called twisting controller –see [1].

Step 2. The second sliding surface is defined as $\{\sigma_2 = 0\}$ with $\sigma_2 = \xi_2 - \phi_1$. The control input is designed to make $\sigma_2 \to 0$:

$$u = g_2^{-1}(t,\xi)[-f_2 - \alpha_2 \text{sgn}(\sigma_2) + v] \quad (3)$$

Now, using $\xi_2 = \sigma_2 + \phi_1$ and the expressions (2a), (3) in the system's equations (1) we obtain

$$\dot{\sigma}_1 = u_{11} + \omega_1 + g_1(t,\sigma)\sigma_2 \quad (4a)$$

$$\dot{\sigma}_2 = -\alpha_2 \text{sgn}(\sigma_2) + \omega_2 - \dot{\phi}_1 + v \quad (4b)$$

where $g_1(t,\sigma)\sigma_2 = g_1(t,\xi(t,\sigma))\sigma_2$.

The additional control input v is left to be defined. If $\dot{\phi}_1$ is bounded, one can set $v \equiv 0$ and redefine ω_2 to incorporate $\dot{\phi}_1$ as a perturbation in the second equation. Otherwise, $v = \dot{\phi}_1$. The resulting error system dynamics is

$$\dot{\sigma}_1 = -\int [\alpha_1 \text{sgn}(\sigma_1) - \beta_1 \text{sgn}(\dot{\sigma}_1)]\, dt + g_1(t,\sigma)\sigma_2 + \omega_1 \quad (5a)$$

$$\dot{\sigma}_2 = -\alpha_2 \text{sgn}(\sigma_2) + \omega_2 \quad (5b)$$

the integrand above is also to be considered to be evaluated along the trajectories.

To study the stability of system (5) we rewrite (5a) in differential cascaded form. Let $z_1 = \sigma_1$, $z_2 = \omega_1 + u_{11}$. Then, provided that $z_2(t_0) = \omega_1(t_0, x_1(t_0))$, Equation (4a) is equivalent to

$$\underbrace{\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix}}_{\dot{z}} = \underbrace{\begin{bmatrix} z_2 \\ \dot{\omega}_1 - \alpha_1 \text{sgn}(z_1) - \beta_1 \text{sgn}(z_2) \end{bmatrix}}_{F_1(z)} + \underbrace{\begin{bmatrix} g_1(t, \begin{bmatrix} z_1 \\ \sigma_2 \end{bmatrix})\sigma_2 \\ \Phi \end{bmatrix}}_{G(t,z,\sigma_2)} \quad (6)$$

where $\Phi = -\beta_1 \text{sgn}(\dot{z}_1) + \beta_1 \text{sgn}(z_2)$.

Notice that $\Phi \leq 2\beta_1$ and that $G(t,z,0) = 0$. Hence, system (6) together with (5b) is in cascaded form.

B. Main result

Theorem 1: The origin of the system (1) in closed loop with (2) and (3) is globally finite-time stable provided that $\alpha_1 - |\dot{\omega}_1| > \beta_1 > |\dot{\omega}_1|$, $\alpha_2 > |\omega_2|$ and that there exists a non-decreasing function θ such that

$$\left| g_1(t, \begin{bmatrix} z_1 \\ \sigma_2 \end{bmatrix}) \right| \leq \theta(|\sigma_2|)|z_1|. \quad (7)$$

Proof: The proof follows similar arguments as to infer stability of cascaded nonlinear time-varying systems. See for instance [12]. The proof is divided in two steps

1) global finite-time stability of the origin of system (5b). Let $V_2 = \sigma_2^2$, its time derivative along the trajectories of (5b) yields $\dot{V}_2 \leq -2(\alpha_2 - |\omega_2|)|\sigma_2|$ *i.e.*,

$$\dot{V}_2 \leq -2(\alpha_2 - |\omega_2|)V_2^{1/2}$$

choosing α_2 such that $\alpha_2 - |\omega_2| > 0$ finite-time stability follows integrating the previous expression.

2) In this step we invoke:
   (2a) finite-time stability of system $\dot{z} = F_1(z)$;
   (2b) forward completeness of system (6) and
   (2c) finite-time stability of (5b)
   to conclude finite-time stability of the cascade.

In [11] finite-time stability of $\dot{z} = F_1(z)$ is proved with a strict Lyapunov function $V(z)$. In Section III we provide an alternative proof which consists in constructing a function $V(z)$ positive definite and proper, whose total time derivative along the trajectories of $\dot{z} = F_1(z)$ satisfies

$$\dot{V} \leq -c_1 V^{c_2}, \quad c_1 > 0,\ c_2 \in (0,1).$$

and for which there exist $c_3,\ c_4,\ c_5,\ c_6 > 0$ such that

$$|z_1| \geq c_4 \quad \Rightarrow \quad \left| \frac{\partial V}{\partial z_1} \right| |z_1| \leq c_3 V \quad (8)$$

$$\|z\| \geq c_6 \quad \Rightarrow \quad \left| \frac{\partial V}{\partial z_2} \right| \leq c_5 V. \quad (9)$$

Now we use $V(z)$ to prove forward completeness of (6); we have

$$\frac{dV}{dz}[\, F_1(z) + G(t,z,\sigma_2)\,] \leq -c_1 V^{c_2} + \left| \frac{\partial V}{\partial z_1} g_1(t, \begin{bmatrix} z_1 \\ \sigma_2 \end{bmatrix})\sigma_2 \right| + \left| \frac{\partial V}{\partial z_2} 2\beta \right|. \quad (10)$$

Now, since $\sigma_2(t)$ converges to zero in finite time it is globally uniformly bounded hence, for $\sigma_2 = \sigma_2(t)$ we have from (7)

$$\left| g_1(t, \begin{bmatrix} z_1 \\ \sigma_2(t) \end{bmatrix}) \right| |\sigma_2(t)| \leq \theta(|\sigma_2(t)|)|z_1||\sigma_2(t)| \tag{11}$$

$$\leq c_7 |z_1| \tag{12}$$

where c_7 depends only on the size of $\sigma_2(t_0)$. We conclude that the trajectories $z(t)$, $\sigma_2(t)$ satisfy

$$\frac{dV}{dz}[\, F_1(z) + G(t, z, \sigma_2)\,] \leq c_3 c_7 V(z(t)) + 2\beta c_5 V(z(t))$$

for all t such that $|z_1(t)| \geq c_4$ and $\|z\| \geq c_6$. That is,

$$|z_1(t)| \geq c_4, \|z\| \geq c_6 \quad \Rightarrow \quad \dot{V}(z(t)) \leq (c_3 c_7 + 2\beta c_5) V(z(t))\,.$$

Forward completeness follows by integrating the previous inequality to infinity.

Let $t_f < \infty$ be the settling time for $\sigma_2(t)$. From forward completeness, for all t such that $t > t_f$ and observing that $G(t, z, 0) = 0$ we obtain, once more invoking (10)

$$\frac{dV}{dz}[\, F_1(z) + G(t, z, \sigma_2)] \leq -c_1 V^{c_2}$$

for all $t \geq t_f$. Finite-time stability follows integrating the latter. ■

III. Finite-time stabilization of the double integrator

Consider the perturbed double integrator, given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \delta(t, x) + u \qquad (13a) \\ \text{with } u &= -\alpha \operatorname{sgn}(x_1) - \beta \operatorname{sgn}(x_2) \qquad (13b) \end{aligned}$$

where x_1 and $x_2 \in \mathbb{R}$ are scalar state variables, δ is a bounded perturbation and $u \in \mathbb{R}$ is the previously mentioned twisting controller; $\alpha, \beta > 0$ are control parameters.

In [11] it was showed via a strict Lyapunov function that the origin is finite-time stable for sufficiently large gains α and β. The following proposition establishes an alternative proof with the same Lyapunov function as in [11] which fits the backstepping design method from the previous section. More precisely, we provide a proof of inequalities (8) and (9) as well as the other properties used in the previous section, which are not presented in [11].

Proposition 1: Let $|\delta(t, x)| \leq M$ for all $(t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$, $\gamma_1 > 0$, $\gamma_2 > 0$ and consider the function

$$\begin{aligned} V(x_1, x_2) \; &= \alpha^2 \gamma_1 x_1^2 + \gamma_2 |x_1|^{3/2} \operatorname{sgn}(x_1) x_2 \\ &\quad + \alpha \gamma_1 |x_1| x_2^2 + \frac{1}{4} \gamma_1 x_2^4\,. \end{aligned} \tag{14}$$

Then, we have the following.

- Given any $\alpha > 0$ and β such that $\alpha - M > \beta > M$ there always exist parameters γ_1, γ_2 such that V is a strict Lyapunov function for the system (13) that is, it is positive definite, proper and its derivative along the trajectories of (13) is negative definite
- provided that $\beta > M$ there always exist $c_1 > 0$ and $c_2 \in (0, 1)$ such that

$$\frac{dV}{dx_1} \dot{x}_1 + \frac{dV}{dx_2} \dot{x}_2 \leq -c_1 V^{c_2}\,; \tag{15}$$

- (hence) for any pair $(x_{1\circ}, x_{2\circ}) \in \mathbb{R}^2$ all generated solutions satisfying $(x_1(t_0), x_2(t_0)) = (x_{1\circ}, x_{2\circ})$ converge to the origin $(x_1, x_2) = (0, 0)$ in finite time t_f where

$$t_f \leq \frac{4}{c_1} V(x_{1\circ}, x_{2\circ})^{1/4}\,. \tag{16}$$

A. V is positive definite and proper

We proceed to prove the previous statements. Let us show that V is positive definite and proper without any restriction on the control gains, other than $\alpha > 0$. Let $\mu > 0$ and observe that

$$V(x_1, x_2) \; = \mu(|x_1|^{1/2} + |x_2|)^4 + W \tag{17}$$

$$\begin{aligned} W(x_1, x_2) \; &\geq (\alpha^2 \gamma_1 - \mu)|x_1|^2 - (\gamma_2 + 4\mu)|x_1|^{3/2}|x_2| \\ &\quad + (\alpha\gamma_1 - 6\mu)|x_1||x_2|^2 - 4\mu |x_1|^{1/2}|x_2|^3 \\ &\quad + (\frac{1}{4}\gamma_1 - \mu)|x_2|^4\,. \end{aligned} \tag{18}$$

We claim that for any *given* control gain $\alpha > 0$, and an appropriate choice of the parameters γ_1, γ_2 we have $W \geq 0$, thereby implying that V is positive definite and radially unbounded. To see that the claim holds true let

$$\eta_m = \min\left\{(\alpha^2\gamma_1 - \mu),\ \frac{1}{6}(\alpha\gamma_1 - 6\mu),\ (\frac{1}{4}\gamma_1 - \mu)\right\}\,.$$

If $\eta_m > 0$, which implies that $\gamma_1 > 0$ and in turn $\alpha > 0$, then

$$\begin{aligned} W(x_1, x_2) \; &\geq \; -(\gamma_2 + 4\mu)\left[|x_1|^{3/2}|x_2| + |x_1|^{1/2}|x_2|^3\right] \\ &\quad + \eta_m \left[\, |x_1|^2 + 6|x_1||x_2|^2 + |x_2|^4\,\right]. \end{aligned} \tag{19}$$

Furthermore, for any given parameters $\alpha, \gamma_1, \mu > 0$ such that $\eta_m > 0$, pick $\gamma_2 > 0$ such that $\gamma_2 \leq 4\eta_m - 4\mu$. Under such conditions Inequality (19) implies that

$$W(x_1, x_2) \; \geq \eta_m \left[|x_1|^{1/2} - |x_2|\right]^4 \; \geq \; 0\,.$$

We emphasize that there always exists $\gamma_2 > 0$ satisfying $\gamma_2 \leq 4\eta_m - 4\mu$.

We proceed to compute an upper-bound for V. To that end we observe also from (18), that

$$
\begin{aligned}
W(x_1,x_2) \;\le\; & (\alpha^2\gamma_1-\mu)|x_1|^2+(\gamma_2+4\mu)|x_1|^{3/2}|x_2| \\
& +(\alpha\gamma_1-6\mu)|x_1||x_2|^2+4\mu|x_1|^{1/2}|x_2|^3 \\
& +(\frac{1}{4}\gamma_1-4\mu)|x_2|^4 \qquad (20)
\end{aligned}
$$

which together with (17), implies that

$$
V(x_1,x_2) \;\le\; (\mu+\eta_M)\left[|x_1|^{1/2}+|x_2|\right]^4 \qquad (21)
$$

with $\eta_M =$

$$
\max\left\{(\alpha^2\gamma_1-\mu),\ (\alpha\gamma_1-6\mu),\ (\gamma_2+4\mu),\ (\frac{1}{4}\gamma_1-4\mu)\right\}.
$$

B. Derivative of V

Observing that $|x_1|^{3/2}\mathrm{sgn}(x_1)=x_1|x_1|^{1/2}$ we compute

$$
\begin{aligned}
\frac{\partial V}{\partial x_1} &= 2\alpha^2\gamma_1 x_1+\frac{3}{2}\gamma_2|x_1|^{1/2}x_2+\alpha\gamma_1\mathrm{sgn}(x_1)x_2^2 \\
\frac{\partial V}{\partial x_2} &= \gamma_2|x_1|^{3/2}\mathrm{sgn}(x_1)+2\alpha\gamma_1|x_1|x_2+\gamma_1 x_2^3 .
\end{aligned}
$$

Furthermore, let $c_8 := \max\left\{2\alpha^2\gamma_1,\ \frac{3}{2}\gamma_2,\ \alpha\gamma_1\right\}$ then,

$$
\begin{aligned}
\left|\frac{\partial V}{\partial x_1}\right| &\le c_8\left(|x_1|^{1/2}+|x_2|\right)^2 \\
\Rightarrow \quad \left|\frac{\partial V}{\partial x_1}\right||x_1| &\le c_8\left(|x_1|^{1/2}+|x_2|\right)^4 .
\end{aligned}
$$

From this and (21) we obtain

$$
\left|\frac{\partial V}{\partial x_1}\right||x_1| \;\le\; \frac{c_8}{\mu+\eta_M}V(x_1,x_2)
$$

That is, (8) holds for V with $z=(x_1,x_2)^\top$. In a similar manner let $c_9 := \max\{\gamma_2,\ 2\alpha\gamma_1,\ \gamma_1\}$ then,

$$
\left|\frac{\partial V}{\partial x_2}\right| \;\le\; c_9\left(|x_1|^{1/2}+|x_2|\right)^3
$$

which for sufficiently large $\|x\|$ fulfills

$$
\begin{aligned}
\left|\frac{\partial V}{\partial x_1}\right| &\le c_9\left(|x_1|^{1/2}+|x_2|\right)^4 \\
\Rightarrow \quad \left|\frac{\partial V}{\partial x_1}\right| &\le \frac{c_9}{\mu+\eta_M}\left(|x_1|^{1/2}+|x_2|\right)^4 .
\end{aligned}
$$

That is, (9) holds for V with $z=(x_1,x_2)^\top$.

Next, we compute the total time derivative of V along the trajectories of (13). We have

$$
\begin{aligned}
\dot V(x_1,x_2) \;=\; & 2\alpha^2\gamma_1 x_1 x_2 \\
+\; & \gamma_2|x_1|^{3/2}\Big(-\alpha-\beta\mathrm{sgn}(x_1x_2)+\delta\Big) \\
+\; & \frac{3}{2}\gamma_2|x_1|^{1/2}x_2^2+\alpha\gamma_1 x_2^2\mathrm{sgn}(x_1)x_2 \\
+\; & 2\alpha\gamma_1|x_1|x_2\Big(-\alpha\mathrm{sgn}(x_1)-\beta\mathrm{sgn}(x_2)+\delta\Big) \\
+\; & \gamma_1 x_2^3\Big(-\alpha\mathrm{sgn}(x_1)-\beta\mathrm{sgn}(x_2)+\delta\Big). \qquad (22)
\end{aligned}
$$

and after straightforward algebraic simplifications we obtain

$$
\begin{aligned}
\dot V(x_1,x_2) \;\le\; & -2\alpha\gamma_1(\beta-M)\,|x_1||x_2|-\gamma_1(\beta-M)|x_2|^3 \\
& -\gamma_2\,(\alpha-\beta-M)\,|x_1|^{3/2}+\frac{3}{2}\gamma_2 x_2^2|x_1|^{1/2} \qquad (23)
\end{aligned}
$$

Let $\kappa>0$ and add κ times

$$
(|x_1|^{1/2}+|x_2|)^3-[|x_1|^{3/2}+3|x_1||x_2|+3|x_1|^{1/2}|x_2|^2+|x_2|^3]=0
$$

to the right hand side of (23). We obtain

$$
\begin{aligned}
\dot V(x_1,x_2) \;\le\; & -[\gamma_2\,(\alpha-\beta-M)-\kappa]|x_1|^{3/2} \\
& +\frac{3}{2}[\gamma_2+3\kappa]x_2^2|x_1|^{1/2} \\
& +[3\kappa-2\alpha\gamma_1(\beta-M)]|x_1||x_2| \\
& -[\gamma_1(\beta-M)-\kappa]|x_2|^3-\kappa(|x_1|^{1/2}+|x_2|)^3 .
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\dot V(x_1,x_2) \;\le\; & -[\gamma_2\,(\alpha-\beta-M)-\kappa]|x_1|^{3/2} \\
& -|x_2|\begin{bmatrix}|x_1|^{1/2}\\|x_2|\end{bmatrix}^\top N\begin{bmatrix}|x_1|^{1/2}\\|x_2|\end{bmatrix} \qquad (24)
\end{aligned}
$$

where

$$
N=\begin{bmatrix}2\alpha\gamma_1(\beta-M)-3\kappa & -\frac{3}{2}[\gamma_2+3\kappa]\\ -\frac{3}{2}[\gamma_2+3\kappa] & \gamma_1(\beta-M)-\kappa\end{bmatrix}
$$

is positive semidefinite for sufficiently large values of γ_1 and $\beta>M$. Additionally from (24) we obtain the next restriction $\gamma_2(\alpha-\beta-M)-\kappa>0$. Combining the above inequalities we obtain $\alpha-M>\beta>M$.

C. Tuning and settling time

Next, we proceed to find c_1 and c_2 such that (15) holds.

$$
\dot V(x_1,x_2)\le-\kappa(|x_1|^{1/2}+|x_2|)^3 .
$$

Using (20) we obtain

$$-V(x_1,x_2)^{3/4} \;\geq\; -(\mu+\eta_M)^{3/4}\left[|x_1|^{1/2}+|x_2|\right]^3$$

hence

$$\dot V(x_1,x_2) \leq -\frac{\kappa}{(\mu+\eta_M)^{3/4}} V(x_1,x_2)^{3/4}\,. \qquad (25)$$

In particular, (15) holds for $\alpha - M > \beta > M$, with $\gamma_1 > 0$ and

$$c_1 = \frac{\kappa}{(\mu+\eta_M)^{c_2}}, \quad c_2 = \frac{3}{4}\,.$$

Finally, an upper bound for time convergence of the trajectories to zero, for the perturbed case, may be computed by integrating (25) along the trajectories generated by (13) from any pair of initial conditions $(x_{1\circ}, x_{2\circ}) \in \mathbb{R}^2$

$$t_f \leq \frac{4}{c_1} V(x_{1\circ},x_{2\circ})^{1/4}\,. \qquad (26)$$

Remark 1: It is clear from the previous proof that the convergence rate or, more precisely the settling time t_f is directly related to the control parameters. Indeed $c_1 = \mathcal{O}(\gamma_1^{1/4})$ that is, c_1 is "slowly" increasing for large values of γ_1. Yet, as μ and η_M are independent of β, the settling time is inversely proportional to this control gain.

Hence, not only the previous stability proof for the twisting algorithm provides a strict Lyapunov function but it provides a simple rule of thumb relating the control gains to the settling time. Otherwise, there exist no restrictions on the gains, other than to dominate over the disturbance.

D. Extension to unbounded perturbations

For the sake of clarity, we have assumed that the disturbance terms contained in $\delta(t,x)$ are bounded. However, this assumption may be relaxed if we admit modifications to the control law. Consider again the system (13) and assume that there exist a positive constant M and a continuous non-decreasing function Δ such that

$$|\delta(t,x)| \leq \Delta(|x|) + M \qquad (27)$$

then, the control algorithm (13b) may be modified to the following

$$u = -\alpha \mathrm{sgn}(x_1) - \beta'[\,\Delta(|x|)+M\,]\mathrm{sgn}(x_2) \qquad (28)$$

For which all claims of Proposition 1 hold with $\beta' > 1$ and $\beta = \beta'[\,\Delta(|x|)+M\,]$.

As the setting of the double integrator (13) is reminiscent of control of mechanical systems under bounded perturbations (modulo a prior feedback linearizing feedback) the latter setting may be related to the problem of mechanical systems under the influence of non-dissipative forces, as for instance in the early work [13] where a result of semiglobal asymptotic stability was obtained.

IV. Example

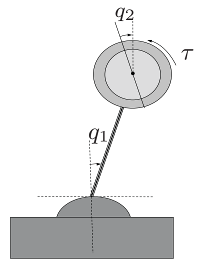

Fig. 1. Inertial wheel pendulum

Consider the inertia-wheel pendulum illustrated in Figure IV. The parameters J_i of the inertia matrix and parameter h of the gravitational term –see Eqs. (29) below, are computed from an experimental benchmark manufactured by Quanser Inc. We have $J_1 = 4.572\times10^{-3}$, $J_2 = 2.495\times10^{-5}$, and $h = 0.3544$. A global coordinate transformation reported in [14] is applied to change the Lagrangian equations

$$\begin{bmatrix} J_1 & J_2 \\ J_2 & J_2 \end{bmatrix}\begin{bmatrix} \ddot q_1 \\ \ddot q_2 \end{bmatrix} - \begin{bmatrix} h\sin(q_1) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\tau \qquad (29)$$

into a system in strict-feedback form,

$$\begin{aligned} \dot z_1 &= -h\sin(q_1) \\ \dot q_1 &= J_1^{-1}z_1 - J_1^{-1}J_2 z_2 + \omega_1 \\ \dot z_2 &= \frac{h\sin(q_1)}{J_1-J_2} + \frac{J_1}{J_2(J_1-J_2)}\tau + \omega_2 \\ \dot q_2 &= z_2 \end{aligned} \qquad (30)$$

into which we introduced the disturbances ω_1, ω_2 after the transformation. These 'disturbances' (may) account for unmodelled dynamics and parametric uncertainties. The controller is constructed according to Section II –*cf.* [4].

Step 1. The first sliding surface is $\sigma_1 = q_1 - q_d$ and the virtual controller is

$$\begin{aligned} \phi_1(q_1) &= J_1 J_2^{-1}\{J_1^{-1}z_1 + u_{1,1}\} \\ \dot u_{1,1} &= -\alpha_1 \mathrm{sgn}(\sigma_1) - \beta_1 \mathrm{sgn}(\dot\sigma_1) \end{aligned}$$

The derivative $\dot\sigma_1$ is calculated by means of the next robust differentiator [15]

$$\begin{aligned} \dot s_0 &= -\lambda_2 L^{1/2}|s_0-\sigma_1|^{1/2}\mathrm{sgn}(s_0-\sigma_1) + s_1 \\ \dot s_1 &= -\lambda_1 L \mathrm{sgn}(s_0 - \dot s_0) \end{aligned}$$

Step 2. Now for state z_2, $\sigma_2 = z_2 - \phi_1(q_1)$

$$\begin{aligned} u &= J_2 J_1^{-1}\{h\sin(q_1) + (J_1-J_2)u_{2,1}\} \\ u_{2,1} &= -\alpha_2 \mathrm{sgn}(\sigma_2) \end{aligned}$$

As mentioned in previous section, in order to compensate the state norm bounded $\dot\phi$, the gain α_2 should be variable, the next error-dependent gain was chosen

$$\alpha_2(\sigma_2) = \alpha_{2a} e^{\alpha_{2b}|\sigma_2|}$$

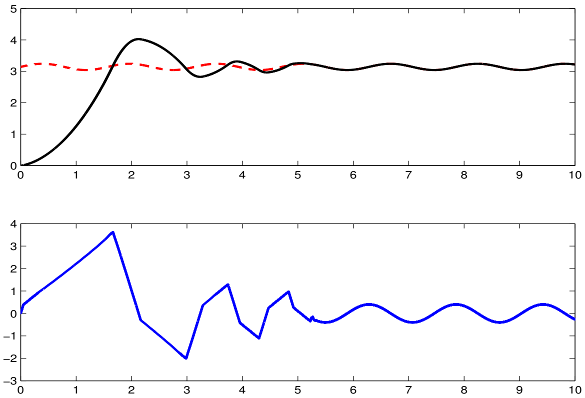

Fig. 2. Position q_1, q_d (red line) (top) and q_2 (bottom)

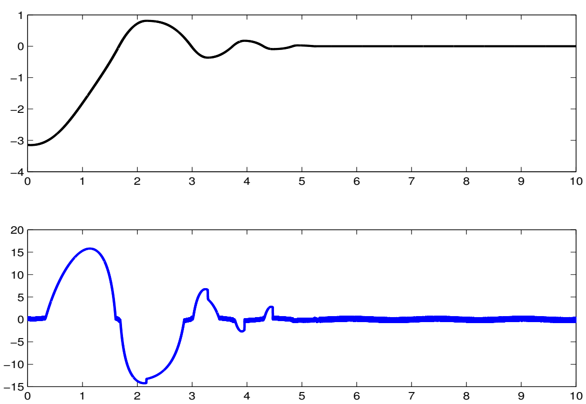

Fig. 3. Errors σ_1 (top) and σ_2 (bottom)

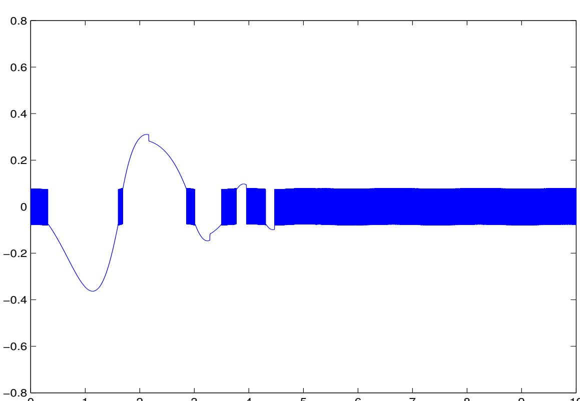

Fig. 4. Control signal u

We ran some simulations to test the performance of our algorithm on this academic set-up. The desired position is set to $0.1\sin(4t)$, the initial conditions for the inertia wheel pendulum where all set to zero *i.e.*, the pendulum is assumed to start off from the downward position. The disturbances are $\omega_1 = 0.1\cos(40t)$ and $\omega_2 = 0.1\sin(40t)$. The controller parameters are set to $\alpha_1 = 5,\ \beta_1 = 3,\ \alpha_{2a} = 3000$ and $\alpha_{2b} = 0.1$ while for the differentiator $\lambda_1 = 1.1,\ \lambda_2 = 1.5,\ L = 10$. The graphs of the system's responses and control input are depicted in Figures 2-4.

V. Conclusion

We set preliminary basis for a methodological approach to control systems in strict feedback form, via a backstepping-like design and high-order sliding modes. The method relies on regarding the closed-loop system as a cascade. This considerably simplifies the analysis. On one hand, the authors have no knowledge of any previous result involving finite-time stability in cascade systems using Lyapunov methods. Further research is being carried out to extend the method beyond the second-order.

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