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# Decentralized Event-triggered Control with Asynchronous Updates

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*Abstract*— **We propose taking event-triggered control actions to implement decentralized control over wireless sensor/actuator networks without requiring synchronized measurement updates. In comparison with the existing results on event-triggered decentralized control, the proposed implementation does not rely on weak coupling between subsystems, nor does it assume the synchronization of local clocks or the existence of a central broadcasting node, and is applicable to nonlinear systems. In addition, higher energy efficiency at the sensors is expected because of the great reduction of the listening times of the sensors. We prove that with asynchronous measurement updates, the event-triggered control actions can guarantee semiglobal practical stability for the sensor/actuator system of interest. We also show that the time between any two consecutive transmissions of measurements at each sensor is bounded from below by a positive constant. Furthermore, asymptotic stability can be achieved when more complicated triggering conditions are introduced. The theoretical analysis is validated by simulations.**

## I. Introduction

The use of aperiodic control techniques like event-triggered [1], [2], [3] and self-triggered control [4], [5], [6] has attracted much attention from the networked control systems community in recent years [7], [8], [9], [10], [11]. The benefits of abandoning the periodic sampling, transmission and control update in favor of event-based techniques lies on the great communication and computation savings that the latter provide. As such they have been suggested as good solutions for embedded systems with shared processors and/or communication buses. There is a natural interest in employing these techniques to enable wireless control systems, in which communications are very power expensive. The added problem on this setting is that of having to design decentralized triggering conditions to decide when communications must take place. Some results are already available along this lines for the case of weakly coupled systems [12], linear systems [13], or requiring synchronized measurements [14]. There are also other results in the literature, in most cases restricted to linear systems, addressing the topic of asynchronous control [15], [16] and applications requiring this kind of control [17].

The present paper provides a solution for a large class of nonlinear systems not requiring any decoupling assumptions, thus applicable to systems more general than those considered in [13] and [12], and removing the requirement of synchronized measurements in [14]. With respect to other work available on asynchronous control we provide a new focus by introducing the asynchronism for energy efficiency. Synchronized measurements means that every sensor in the system needs to be alert for possible updates triggered at other sensors and it may also require the ability to reach all sensors with a single broadcast message from the controller [14] to achieve synchronization, which not always are available possibilities. In wireless networked systems these effects pose a great challenge as keeping sensors with their radio module continuously on implies a drainage of the energy resources. A possible solution is to use sleeping periods for the radio modules, which would introduce delays between the generation of events and the response to them by updating the control signal. These delays can be accommodated at the cost of more conservative triggering conditions which also implies more frequent actuation. We propose the use of asynchronous updates as a way of circumventing the need to keep radios at the sensor nodes listening and thus improve energy efficiency. Combined with low-power wake-up radio technologies [18], [19], [20] the approach we propose has the potential of constructing wireless control systems that only consume energy when updates are necessary.

In the current paper we propose a solution in which the control signal is updated with asynchronous measurements. In plain words, whenever a sensor locally triggers according to some local condition, only this sensor will send measurements to the controller, which will update the control signal using only the measurement obtained from this sensor. Allowing asynchronous updates of the control signal could potentially degenerate into a system with Zeno executions. In order to prevent this effect, at a first stage, we relax the problem providing a solution that renders the closed loop system practically stable. Later we provide conditions and constructions that yield asymptotically stable implementations. This is achieved, in general, at the cost of arbitrarily interspaced synchronized measurements. We also show that in particular cases, as is the case of linear systems, completely asynchronous implementations also exist rendering the system asymptotically stable. All the proposed implementations provide lower bounds for the time elapsing between transmissions at each of the sensors. And finally, we briefly discuss how these techniques can accommodate various practical effects like communication delays or maximum rates of update of the actuators.

The remainder of the paper is organized as follows: In Section II the notation and some preliminary definitions are provided. Section III formalizes the problem, provides the main results of the paper and discusses how to adapt

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to practical issues. Finally, both simulations of linear and nonlinear implementations are presented in Section IV, and the paper is concluded with a discussion in Section V.

## II. PRELIMINARIES

We denote the positive real numbers by $\mathbb{R}^+$. We also use $\mathbb{N}$ to denote the natural numbers including zero and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. The usual Euclidean ($l_2$) vector norm is represented by $|\cdot|$. When applied to a matrix $|\cdot|$ denotes the $l_2$ induced matrix norm. A matrix $P \in \mathbb{R}^{n\times n}$ is said to be positive definite, denoted by $P > 0$, whenever $x^T Px > 0$ for all $x \neq 0$, $x \in \mathbb{R}^n$. By $\lambda_m(P), \lambda_M(P)$ we denote the minimum and maximum eigenvalues of $P$ respectively. Given an open set $B \subseteq \mathbb{R}^n$, we say that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous on $B$ if there exists a constant $L \in \mathbb{R}_0^+$ such that: $\|f(x) - f(y)\| \leq L\|x - y\|$, $\forall x, y \in B$. A function $\gamma : [0, a[ \to \mathbb{R}_0^+$, $a > 0$ is of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, $\gamma(0) = 0$ and $\gamma(s) \to \infty$ as $s \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times [0, a[ \to \mathbb{R}_0^+$ is of class $\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class $\mathcal{K}_\infty$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is monotonically decreasing to zero for each $s \geq 0$. A class $\mathcal{KL}$ function $\beta(\tau, s)$ is called exponential if $\beta(s, \tau) \leq \sigma s e^{-c\tau}$, $\sigma > 0$, $c > 0$. Given an essentially bounded function $\delta : \mathbb{R}_0^+ \to \mathbb{R}^m$ we denote by $\|\delta\|_\infty$ the $\mathcal{L}_\infty$ norm, *i.e.* $\|\delta\|_\infty = \sup_{t\in\mathbb{R}_0^+}\{|\delta(t)|\} < \infty$.

In the following we consider systems with inputs $\upsilon : \mathbb{R}_0^+ \to \mathbb{R}^m$ an essentially bounded piecewise continuous function of time:

$$\frac{d}{dt}\xi = f(\xi, \upsilon) \tag{1}$$

in which $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map. For notation compactness we will often write $\dot{\xi}$ to denote $\frac{d}{dt}\xi$. Solutions of (1) with initial condition $x$ and input $\upsilon$, denoted by $\xi_{x\upsilon}$, satisfy: $\xi_{x\upsilon}(0) = x$ and $\frac{d}{dt}\xi_{x\upsilon}(t) = f(\xi_{x\upsilon}(t), \upsilon(t))$ for almost all $t \in \mathbb{R}_0^+$. The notation will be relaxed by dropping the subindex when it does not contribute to the clarity of the exposition. A feedback law for a control system is a smooth map $k : \mathbb{R}^n \to \mathbb{R}^m$; we will refer to such a law as a *controller* for the system.

The notion of Input-to-State stability (ISS), formalized in the following, will be central to our discussion:

*Definition 2.1 (Input-to-State Stability [21]):* A control system $\dot{\xi} = f(\xi, \upsilon)$ is said to be *input-to-state stable* (ISS) with respect to $\upsilon$ if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for any $t \in \mathbb{R}_0^+$ and for all $x \in \mathbb{R}^n$:

$$|\xi_{x\upsilon}(t)| \leq \max\{\beta(|x|, t), \gamma(\|\upsilon\|_\infty)\}.$$

We shall refer to $(\beta, \gamma)$ as the *ISS gains* of the ISS estimate.

Rather than relying on this definition, many of our arguments make use of the following alternative characterization of ISS systems using ISS Lyapunov functions:

*Definition 2.2 (ISS Lyapunov function [21]):* A smooth function $V : \mathbb{R}^n \to \mathbb{R}_0^+$ is said to be an ISS Lyapunov function for the closed-loop system $\dot{\xi} = f(\xi, \upsilon)$ if there exists class $\mathcal{K}_\infty$ functions $\underline{\alpha}$,$\overline{\alpha}$, $\alpha_x$ and $\alpha_e$ such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ the following is satisfied:

$$\begin{aligned} \underline{\alpha}(|x|) \leq \quad V(x) \quad &\leq \overline{\alpha}(|x|) \\ \frac{\partial V}{\partial x} f(x, u) \quad &\leq \quad -\alpha_x(|x|) + \alpha_e(|u|). \end{aligned} \tag{2}$$

A system is ISS if and only if there exists an ISS Lyapunov function.

The main result of the paper establishes the semiglobal practical stabilizability of systems under asynchronous updates. This notion of stability is formalized in the following definition.

*Definition 2.3 (Semiglobal Practical Stabilizability [22]):* A system $\dot{\xi} = f(\xi, \upsilon)$ is said to be *semiglobally practically stabilizable* if for any (arbitrarily large) compact set $K$ and any arbitrarily small compact set $Q$ including the origin, there exists a feedback law $k : \mathbb{R}^n \to \mathbb{R}^m$, which in general depends on $K$ and $Q$, such that every trajectory $\xi_{xk(\xi)}$ with $x \in K$ is defined for all $t \in [0, \infty)$ and there exists $T \in [0, \infty)$ such that $\xi_{xk(\xi)}(t) \in Q$ for all $t \in [T, \infty)$.

## III. ASYNCHRONOUS EVENT-TRIGGERED CONTROL

### A. Problem Formulation

The problem we aim at solving is that of controlling, on wireless networked settings, systems of the form:

$$\dot{\xi}(t) = f(\xi(t), \upsilon(t)), \qquad \forall t \in \mathbb{R}_0^+, \tag{3}$$

where $\xi : \mathbb{R}_0^+ \to \mathbb{R}^n$ and $\upsilon : \mathbb{R}_0^+ \to \mathbb{R}^m$. In particular, we are interested in finding stabilizing sample-and-hold implementations of a controller $k$ such that updates can be performed with asynchronous measurements of the different state variables. This problem can be formalized as follows:

*Problem 3.1:* Given system (3) and a controller $k : \mathbb{R}^n \to \mathbb{R}^m$ find sequences of update times $\{t^i_{k_i}\}$, $k_i \in \mathbb{N}$ for each sensor $i = 1, \dots, n$ such that an asynchronous sample-and-hold controller implementation:

$$\begin{aligned} \upsilon_j(t) = \quad & k_j(\xi_1(t^1_{k_1}), \xi_2(t^2_{k_2}), \dots, \xi_n(t^n_{k_n})),\ j = 1, \dots, m \\ & t \in [\max_{i=1,\dots,n}\{t^i_{k_i}\}, \min_{i=1,\dots,n}\{t^i_{k_i+1}\}[, \end{aligned} \tag{4}$$

renders the closed-loop system practically (or asymptotically) stable.

### B. Main Results

Before providing the main result of the paper, let us introduce the following assumption:

*Assumption 3.2 (ISS with respect to measurement errors):* Given system (3) there exists a controller $k : \mathbb{R}^n \to \mathbb{R}^m$ such that the closed-loop system

$$\dot{\xi}(t) = f(\xi(t), k(\xi(t) + \varepsilon(t)), \qquad \forall t \in \mathbb{R}_0^+ \tag{5}$$

is ISS with respect to measurement errors $\varepsilon$.

Representing the effect of the sample-and-hold as a measurement error

$$\varepsilon_i(t) = \xi_i(t^i_{k_i}) - \xi_i(t),\ t \in [t^i_{k_i}, t^i_{k_i+1}[,\ i = 1, \dots, n$$

we propose rules of the following form:

$$t^i_{k_i} = \min\{t > t^i_{k_i-1} \,|\, \epsilon_i^2(t) \geq \eta_i\}, \tag{6}$$

where $\eta_i > 0$ are design parameters, defining implicitly the sequences of update times $\{t^i_{k_i}\}$ for each sensor $i$.

The following theorem shows how relying on the rule (6) to update the controller results in semiglobal practically stable implementations of a control system.

*Theorem 3.3 (Semiglobal Practical Stability):* If the assumption 3.2 holds, the system (3) is semiglobally practically stabilizable using asynchronous measurements. In particular, any controller rendering the closed-loop system ISS implemented as prescribed by (4) with update rules of the form (6) renders the closed loop system practically stable. Moreover, the time between transmissions of measurements at each sensor is bounded from below by some $\tau_i^* > 0$, $i = 1, \ldots, n$.

*Proof:* Let us start proving that the system is semiglobally practically stabilizable using asynchronous measurements. For that matter, we will actually prove that the solution proposed actually renders the system semiglobally practically stable. Assumption 3.2 provides with the existence of a controller such that the following bound is satisfied:

$$\dot{V} \leq -\alpha_x(|\xi|) + \alpha_e(|\varepsilon|).$$

If the following inequality is enforced: $\alpha_e(|\varepsilon|) \leq \eta$, then $\dot{V} \leq -\alpha_x(|\xi|) + \eta$. Thus, the last inequality implies, invoking the ISS estimate, that: $|\xi_{xv}(t)| \leq \max\{\beta(|x|, t), \gamma(\alpha_e^{-1}(\eta))\}$. Note that for every compact set $K$ there exists some $\rho_K \in \mathbb{R}^+$ such that every $x \in K$ is such that $|x| \leq \rho_K$; and similarly for every compact $Q$ containing the origin there exists some $\rho_Q \in \mathbb{R}^+$ such that $|x| \leq \rho_Q$ implies $x \in Q$. Thus, for every compact set $Q$ there exists sufficiently small $\eta$ and a sufficiently large $T$, such that $\beta(\rho_K, T) \leq \gamma(\alpha_e^{-1}(\eta)) \leq \rho_Q$, which implies that imposing $\alpha_e(|\varepsilon|) \leq \eta$ the desired semiglobal practical stability is attained. Therefore, selecting $\eta_i$ such that $\sum_{i=1}^n \eta_i = \left(\alpha_e^{-1}(\eta)\right)^2$ and noting that the update rule (6) enforces $\epsilon_i^2(t) \leq \eta_i$ $\forall t \in \mathbb{R}_0^+$ semiglobal practical stability is proven.

The minimum time that can elapse between two consecutive transmissions from a sensor is given by the time it takes for $|\varepsilon_i|$ to go from 0 (at the update instances $\varepsilon_i(t^i_{k_i}) = 0$) to the value $\sqrt{\eta_i} > 0$. To prove the existence of such a minimum time it is sufficient to note that $\dot{\varepsilon_i} = -f_i(\xi, k(\xi + \varepsilon))$ and thus, from the continuity of $\varepsilon_i$ such a minimum time is guaranteed to be strictly greater than zero. ■

For convenience and compactness of the presentation we will denote from here on by

$$\eta = \alpha_e\left(\sqrt{\sum_{i=1}^n \eta_i}\right), \tag{7}$$

and consider $\eta$ as a design parameter that once specified restricts the choices of $\eta_i$ to be used at each sensor.

*Corollary 3.4 (Lower bound for inter-transmission times):* If $V(\xi(0)) > \overline{\alpha} \circ \alpha_x^{-1}(\eta)$, then a lower bound for the minimum time between transmissions of a sensor is given by:

$$t^i_{k_i+1} - t^i_{k_i} \geq \tau_i^* = \frac{\sqrt{\eta_i}}{\overline{f}_i(V(\xi(0)))} \tag{8}$$

where $\overline{f}_i(y) = \max_{(x,e)\in S(y)} |f_i(x, k(x+e))|$ and $S(y) = \{(x,e) \in \mathbb{R}^{n\times n} \,|\, V(x) \leq y,\ |e| \leq \eta\}$.

*Proof:* First recall that the minimum time between events at a sensor is given by the time it takes for $|\varepsilon_i|$ to evolve from the value $|\varepsilon_i(t^i_{k_i})| = 0$ to $|\varepsilon_i(t^{i-}_{k_i+1})| = \sqrt{\eta_i}$. Therefore all that needs to be proved is the existence of an upper bound on the rate of change of $|\varepsilon_i|$. Note that with the proposed controller implementation the following holds: $|x| > \alpha_x^{-1}(\eta) \Rightarrow \dot{V}(x) < 0$. The assumption $V(x) > \overline{\alpha} \circ \alpha_x^{-1}(\eta)$ implies $|x| > \alpha_x^{-1}(\eta)$ and thus the set $S(V(\xi(0)))$ is forward invariant. Finally, one can trivially bound the evolution of $|\varepsilon|$ as:

$$\frac{d}{dt}|\varepsilon_i| \quad \leq \quad |\dot{\varepsilon}_i| = |f_i(\xi, k(\xi + \varepsilon))|,$$

and the maximum rate of change of $|\varepsilon_i|$ is bounded by $\overline{f}_i(V(\xi(0)))$. Note that the existence of such a maximum is guaranteed by the continuity of the maps $f$ and $k$ and the compactness of the set $S(V(\xi(0)))$. ■

If one can assume that the functions $f$ and $k$ are Lipschitz on compacts, then one can further bound $\overline{f}_i(V(\xi(0))) \leq L_{f_i}\left(\underline{\alpha}^{-1} \circ V(\xi(0)) + \alpha_e^{-1}(\eta)\right)$, where $L_{f_i}$ denotes the Lipschitz constant of the function $f_i(x, k(x+e))$. If one further assumes that the local parameters $\eta_i$ are defined through an appropriate scaling $\eta_i = \theta_i^2 \eta^2$ so that the restriction (7) holds, and that $\alpha_e^{-1}(\eta) \leq L_e^{-1}\eta$ for some constant $L_e > 0$, then, selecting as $\eta = \rho\underline{\alpha}^{-1} \circ V(\xi(0))$ the resulting minimum time between sensor transmissions is:

$$\tau_i^* \geq \frac{L_{f_i}^{-1}\theta_i\eta}{\underline{\alpha}^{-1} \circ V(\xi(0)) + \alpha_e^{-1}(\eta)} \geq \frac{L_{f_i}^{-1} L_e \rho\theta_i}{L_e + \rho}. \tag{9}$$

Note that, as $\frac{x}{ax+b}$, $a > 0$, $b > 0$ is an increasing function on $x$, in practice one can use values of $\eta > \underline{\alpha}^{-1}(V(x))\rho$ and the same lower bound for the inter-transmission times would hold.

The following corollary proposes an approach to construct asymptotic stabilizing asynchronous implementations by letting the parameter $\eta$ change over time:

*Corollary 3.5 (Asymptotic stability):* If $f$, $k$ and $\alpha_e^{-1}$ are Lipschitz on compacts and there exists a constant $\rho > 0$ such that:

$$\alpha_x \circ \overline{\alpha}^{-1}(y) > \rho\underline{\alpha}^{-1}(y),\ \forall\, y \leq V(\xi(0)), \tag{10}$$

then any divergent sequence of times $\{t_k^c\}$ and any selection of $\eta_i(k) = \theta_i\eta^2(k) > 0$ satisfying (7) with

$$\eta = \eta(k) = \rho\underline{\alpha}^{-1} \circ V(\xi(t_k^c)),\ \forall\, t \in (t_k^c, t_{k+1}^c]$$

renders the system asymptotically stable. Furthermore, the minimum time between transmissions is bounded by the value $\tau_i^*$ given by (9).

*Proof:* The expression of the minimum time between transmissions at a sensor is a direct consequence of substituting the expression of $\eta_i(k)$ and $\eta(k)$ in (8) and considering the instants $t_k^c$ as instants at which the control system is initialized, which let us replace in (9) $V(\xi(0))$ by $V(\xi(t_k^c))$. The condition (10) provides the bound:

$$V(\xi(t_k^c)) > \overline{\alpha} \circ \alpha_x^{-1} \circ \left(\rho\underline{\alpha}^{-1} \circ V(\xi(t_k^c))\right) = \overline{\alpha} \circ \alpha_x^{-1}(\eta(k)),$$

which implies that $V(\xi(t)) < V(\xi(t_k^c))$, $\forall\, t > t_k^c$ and thus ensures that the sequence $\eta(k)$ is decreasing. The parameter $\eta$ determines the radius $\rho_\eta = \gamma(\alpha_e^{-1}(\eta))$ of the ball to which the trajectories of the closed-loop system converge, and thus, as $\eta(k)$ is a decreasing sequence asymptotically converging to zero the system is asymptotically stable. ■

We want to remark that the reason to pick $\eta(k)$ with the specific form proposed is to be able to provide an explicit expression of the minimum time between updates. Also note that, while in general it might be very hard to verify the condition (10), because we are working on compacts, there always exists a constant $\rho$ sufficiently small so that the condition holds. Finally we want to remark that, in order to obtain an asymptotically stable implementation following the recommendation from this corollary, synchronous measurements are required at the instants $t_k^c$. However, these synchronous measurements can be spaced arbitrarily apart in time, as the only requirement on the sequence $\{t_k^c\}$ is that it is divergent. Furthermore, exploiting the fact that using $\eta > \underline{\alpha}^{-1}(V(x))\rho$ respects the inter-transmissions time lower bound, one can construct for linear systems event-triggered asymptotically stabilizing controllers with only asynchronous measurements. This is detailed in the following section.

### C. Linear case

In order to illustrate and clarify the proposed techniques, we apply our approach to linear systems. Note that the resulting implementation for linear systems (or a very similar solution) was already proposed in [13] in the context of consensus problems.

Consider a linear system $\dot{\xi} = A\xi + B\upsilon$, with controller $\upsilon = K\xi$, and the Lyapunov function $V(x) = \sqrt{x^T P x}$, where $P > 0$. Let us denote as $A_c = A + BK$, and $Q = -(PA_c + A_c^T P)$, $Q > 0$. For this Lyapunov function, the functions $\alpha_x$, $\alpha_e$, $\underline{\alpha}$, $\overline{\alpha}$ take linear forms:

$$\alpha_x(x) = \lambda_x x, \qquad \underline{\alpha}(x) = \underline{\lambda} x$$
$$\alpha_e(x) = \lambda_e x, \qquad \overline{\alpha}(x) = \overline{\lambda} x$$

where

$$\lambda_x = \frac{\lambda_m(Q)}{2\sqrt{\lambda_M(P)}}, \qquad \underline{\lambda} = \frac{\lambda_m(P)}{\sqrt{\lambda_M(P)}},$$
$$\lambda_e = \frac{|PBK|}{\sqrt{\lambda_m(P)}}, \qquad \overline{\lambda} = \frac{\lambda_M(P)}{\sqrt{\lambda_m(P)}},$$

and $L_{f_i} = \max\{|[A_c]_i|, |[BK]_i|\}$, where $[M]_i$ denotes the $i$-th row of the matrix $M$. Thus $\rho$ can be selected to satisfy:

$$\rho < \frac{\lambda_x \underline{\lambda}}{\overline{\lambda}}, \tag{11}$$

and

$$\eta(k) = \rho \frac{V(\xi(t_k^c))}{\underline{\lambda}} \tag{12}$$

which results in the minimum time between updates (9). Actually, using values of $\eta(k) > \rho \frac{V(\xi(t_k^c))}{\underline{\lambda}}$ results in the same lower bound for the inter-transmission times, as previously indicated. This property can be exploited to remove the need of synchronized measurements to achieve asymptotic stability by using $\eta(k) = \rho \frac{\tilde{V}(\xi(t_k^c))}{\underline{\lambda}}$ where $\tilde{V}$ is a decreasing function upper bounding the evolution of $V$. Note that obtaining closed-form estimates upper-bounding the evolution of $V$ is a trivial problem in the linear case, which means that for linear systems fully asynchronous decentralized event-triggered controller implementations exist that render the closed-loop system asymptotically stable. In fact, a trivial implementation of this form, with $\{t_k^c\} = \{k\tau_\eta \mid k \in \mathbb{N}\}$, is provided by the update rule:

$$\begin{aligned} \eta(k+1) &= g(\tau_\eta)\eta(k) \\ \eta(0) &= \frac{\rho}{\underline{\lambda}} V(\xi(0)) \\ g(\tau_\eta) &= e^{-\frac{\lambda_x}{\overline{\lambda}}\tau_\eta}\left(1 - \frac{\rho\overline{\lambda}}{\underline{\lambda}\lambda_x}\right) + \frac{\rho\overline{\lambda}}{\underline{\lambda}\lambda_x}. \end{aligned} \tag{13}$$

### D. Practical issues

So far, the solutions we have proposed only guarantee that there exists some time greater than zero between events triggering the transmission of measurements. However, in a real implementation it is also desirable that the controller does not receive measurements arbitrarily close to each other, which would require computing updates of the control signal arbitrarily fast[1]. In order to cope with this problem one can set a period $\tau_c$ determining the instants at which the controller is recomputed if measurements were received in the past $\tau_c$ units of time. This procedure imposes a minimum time between transmissions from the controller to the actuators of $\tau_c$ seconds, but also introduces a delay $\Delta_c \leq \tau_c$ between the transmission of a measurement and the use of it in the control action. This is not the only delay that might appear in the system, as communications and the computation of the control will introduce some delays too. Let us assume that both communication delays and computation delays are bounded by $\Delta_{tx}^{ij}$, $\Delta_\mu$ respectively. Denote the accumulated delay $\Delta_c + \Delta_\mu + \Delta_{tx}^{ij}$ by $\Delta^{ij}$, and $\Delta^i = \max_{j=1,\ldots,m} \Delta^{ij}$. The value $\Delta^i$ denotes the maximum time that can elapse between an event is generated at sensor $i$ and the acquired measurement is reflected in all the actuators that depend on the measured state variable. Bounded delays, such that $\Delta^i < \tau_i^*$, are easy to accommodate in the proposed strategy, in a similar way as was proposed in [3] and subsequent works [12], by using more conservative triggering conditions. More precisely, replacing $\eta_i$ in the triggering law (6) by $\tilde{\eta}_i$ as given by: $\sqrt{\tilde{\eta}_i} = \sqrt{\eta_i} - L_{f_i}\left(\underline{\alpha}^{-1} \circ V(\xi(t_k^c)) + \alpha_e^{-1}(\eta(k))\right)\Delta^i$, the system is still rendered practically (asymptotically) stable. Furthermore, it is easy to see that a new lower bound for the inter transmissions time is given by $\tilde{\tau}_i^* = \tau_i^* - \Delta^i$. Alternatively, one can see the effect of the delays as increasing the effective value of the upper bound $\eta$ that the implementation manages to enforce for $|\varepsilon(t)|$. This means, that for a fixed design parameter $\eta$, the proposed implementation converges to a ball around the origin that increases in radius with increasing values of the delay's bounds. Finally, it is worth noting that this same reasoning provides bounds for the maximum allowed delays and minimum update rates necessary at the actuators.

[1] Notwithstanding this, Zeno executions are prevented because of the existence of minimum inter-transmission times at the (finite number) of sensors linked to the actuators

## IV. Simulation Results

### A. Nonlinear example

We first illustrate the proposed techniques on a nonlinear example used in [5], and originally borrowed from [23]. The nonlinear system

$$\begin{aligned}\dot{\xi}_1 &= \upsilon_1\\ \dot{\xi}_2 &= \upsilon_2\\ \dot{\xi}_3 &= \xi_1\xi_2\end{aligned} \tag{14}$$

models the control of the angular velocity of a rigid body, after a preliminary feedback and normalization. We remark that the use of this systems is purely as an academic exercise and as such we do not claim there would be any interest on applying our techniques to this particular system in practice. In [23], the following control law is proposed

$$\upsilon_1 = -\xi_1\xi_2 - 2\xi_2\xi_3 - \xi_1 - \xi_3 \tag{15}$$

$$\upsilon_2 = 2\xi_1\xi_2\xi_3 + 3\xi_3^2 - \xi_2 \tag{16}$$

which can be proven to render the system asymptotically stable, as certified by the Lyapunov function:

$$V(x) = \frac{1}{2}(x_1 + x_3)^2 + \frac{1}{2}(x_2 - x_3^2)^2 + x_3^2. \tag{17}$$

It is easy to show that $V(x) \leq \frac{1}{2}|x|^4 + |x|^3 + 2|x|^2$, and applying a sum-of-squares program one can arrive at the bound $V(x) \geq 5.6 \cdot 10^{-12}|x|^2$. In [5] a bound for the derivative of $V$, along the trajectories of the closed-loop system with measurement errors, is provided in the form:

$$\dot{V} \leq -91446|\xi|^4 + 147190|\varepsilon|^2|\xi|^2, \tag{18}$$

which by noting that $2ab|x|^2|e|^2 \leq a^2|x|^4 + b^2|e|^4$ can be further bounded to obtain:

$$\dot{V} \leq -|\xi|^4 + 6 \cdot 10^4|\varepsilon|^4. \tag{19}$$

To select the value of $\eta$ for the design we shall respect the condition $V(\xi(0)) \geq \overline{\alpha} \circ \alpha_x^{-1}(\eta)$ as prescribed in corollary 3.4. Thus we shall select $\eta$ satisfying $\eta \leq (\overline{\alpha}^{-1}(V(\xi(0))))^4$, where $\overline{\alpha}(x) = \frac{1}{2}x^4 + x^3 + 2x^2$. If we restrict the system to take initial conditions in the set $\{x \in \mathbb{R}^3 \,|\, |x| \leq 15\}$, then $V(\xi(0)) \leq \overline{\alpha}(15)$, and thus we can select any $\eta < 15^4$. Let us pick a rather small $\eta = 10$ which with the same $\eta_i = \frac{1}{3}\left(\alpha_e^{-1}(\eta)\right)^2$ at all sensors results in an $\eta_i = 0.136$. With this parameter selection one can compute the minimum time between sensor transmissions which for this particular case seem to be quite conservative: $\tau_1^* = 4.72 \cdot 10^{-17}s$, $\tau_2^* = 2.72 \cdot 10^{-32}s$, $\tau_3^* = 1.41 \cdot 10^{-16}s$, while the minimum times observed in the simulation are $\tau_1 = 5 \cdot 10^{-3}s$, $\tau_2 = 4 \cdot 10^{-4}s$, $\tau_3 = 6 \cdot 10^{-3}s$. However conservative, mainly due to the very non-symmetric Lyapunov function, these times reflect how fast each of the sensors will trigger in practice. This information could be used to more smartly adjust the way in which $\eta$ is split among sensors, in the spirit of [14]. Figure 1 depicts the different inter-transmission times for each of the sensors. It can also be appreciated the gain with respect to a periodic implementation with a period of $4.5 \cdot 10^{-5}s$, as reported in [5]. In figure 2 the stabilization of the system is illustrated both by showing the Lyapunov function evolution and the trajectory of the system, in which it can be appreciated how the system is only practically stable.

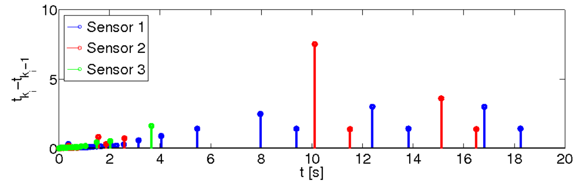


Fig. 1. Times between transmissions at each of the sensors for the nonlinear example.

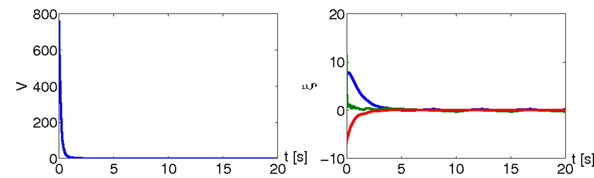


Fig. 2. Lyapunov function and state trajectory for the nonlinear example.

### B. Linear example

For a linear example we borrow the Batch Reactor model from [24], used as a benchmark by several authors [25], [6], with state space description:

$$\dot{\xi} = \begin{bmatrix} 1.38 & -0.20 & 6.71 & -5.67\\ -0.58 & -4.29 & 0 & 0.67\\ 1.06 & 4.27 & -6.65 & 5.89\\ 0.04 & 4.27 & 1.34 & -2.10 \end{bmatrix}\xi + \begin{bmatrix} 0 & 0\\ 5.67 & 0\\ 1.13 & -3.14\\ 1.13 & 0 \end{bmatrix}\upsilon$$

A state feedback controller placing the poles of the closed loop system at $\{-3 + 1.2i, -3 - 1.2i, -3.6, -3.9\}$ is:

$$K = \begin{bmatrix} 0.1006 & -0.2469 & -0.0952 & -0.2447\\ 1.4099 & -0.1966 & 0.0139 & 0.0823 \end{bmatrix}.$$

The matrix $Q$ selected was the identity, and the matrix $P$ determining the Lyapunov function results from solving the associated Lyapunov equation $Q = -(PA_c + A_c^T P)$.

A fully asynchronous decentralized event-triggered implementation rendering the system asymptotically stable was designed as suggested in Section III-C. The design resulted in a value of $\rho = 0.072$ and $\theta_i = \frac{1}{2\lambda_e}$, $i = 1, \ldots, 4$. The resulting lower bounds for the inter-transmission times of $\tau_1^* = 41.6\mu s$, $\tau_2^* = 64.38\mu s$, $\tau_3^* = 35.89\mu s$ and $\tau_4^* = 77.13\mu s$. To avoid actuator updates too close to each other we allowed only the update of the actuation signals at times $t = \frac{\tau_3^*}{2}r$, $r \in \mathbb{N}$, which introduces a delay of $\frac{\tau_3^*}{2}\mu s$. We only consider this delay in the system and we compensate for it by reducing $\eta_i$ accordingly to $\tilde{\eta}_i = \frac{\eta_i}{2}$. The results shown in the experiment were obtained using $\tau_\eta = 1s$ and a resulting $g(\tau_\eta) = 0.9161$. The new lower bounds for the inter-transmission times are $\frac{\tau_i^*}{2}$ and the actual minimum times observed in the simulations are $\tau_1 = 1.6ms$, $\tau_2 = 635\mu s$, $\tau_3 = 789\mu s$ and $\tau_4 = 2.5ms$ illustrating how conservative the bounds for the inter-transmissions time are. In this case, a smarter selection of the Lyapunov function or LMI methods [25] might suffice to obtain tighter bounds for the inter-transmission times. While the actuators would have required, in the worst point of the simulation, to be recomputed after solely $1\mu s$, by implementing our periodic actuation update strategy this time was enlarged to $18\mu s$. As in the previous example we illustrate the results in figures 3 and 4 through the inter-transmission times and Lyapunov and state trajectory plots. It is worth noting that, although not a fair comparison because of the different controllers used, the network usage is well in the range of the output-feedback solution in [25]: inter-transmission times between $0.01s$ to $0.07s$ for this same system. In particular, the average inter-transmission times in this simulation were $\bar{\tau}_1 = 0.121s$, $\bar{\tau}_2 = 0.073s$, $\bar{\tau}_3 = 0.056s$ and $\bar{\tau}_4 = 0.057s$.

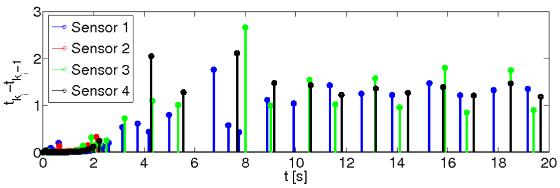


Fig. 3. Times between transmissions at each of the sensors for the linear example.

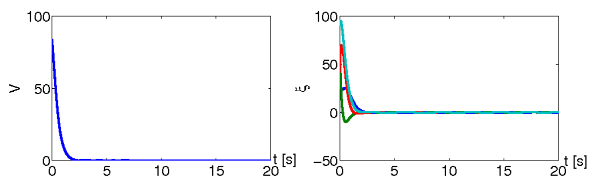


Fig. 4. Lyapunov function and state trajectory for the linear example.

## V. Discussion

We have shown that it is possible to practically stabilize (Input-to-State stabilizable) control systems relying on asynchronous aperiodic measurements of the state variables. The proposed implementations are based on local event generators that trigger the transmission of measurements of each sensor individually. We have also shown that asymptotic stability can be easily attained if one is willing to add synchronous measurements at arbitrarily placed instants. Furthermore, we discussed how in the case of linear systems asymptotic stability can be attained without requiring any synchronous measurements. While the theory presented provides with lower bounds for the minimum inter-transmission times, there is room for improvement of these conservative bounds, possibly by using numerical approaches as in [25]. A more detailed analysis is also necessary to provide explicit performance measures in the spirit of [6]. In the future we also plan to extend the results to the case of output feedback systems, along the lines of [26], and include adaptive techniques in the sensor thresholds assignments, as in [14].

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