2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Stability analysis of piezoelectric beams

T. Voß and J.M.A. Scherpen

***Abstract*— Piezoelectric materials are used in many engineering application. When modeling piezoelectric materials the standard assumption is that the electromagnetic field which is used to actuate the piezoelectric material is quasi static. In this paper we show that although the assumption of a quasi static electrical field is valid when one is interested in simulating a piezoelectric material, this assumption renders the system non stabilizable in terms of control. We also show that this issue is caused by the assumption of a quasi static electrical field and therefore can be avoided by modeling a dynamical electrical field.**

I. INTRODUCTION

In this paper we analyze the finite dimensional dynamics of a piezoelectric beam in the port-Hamiltonian (pH) framework. These dynamics have been previously derived using a structure preserving spatial discretization scheme [3], see [9], [10]. We will show that although the method proposed in [3] yields a finite dimensional pH system, it is not guaranteed that this model can then be used for the design of a controller because the system may not satisfy a necessary condition for stabilization. The reason for this could be that the system is uncontrollable itself. However, in our research, it turns out that the system is not stabilizable due to an assumption made when modeling the infinite dimensional system.

We also show two possible problems which render the finite dimensional model non stabilizable during the spatial discretization. These problems can be avoided by modeling the infinite dimensional system in a different manner. The first problem appears if one treats an infinite dimensional system which has states that depend on each other — this is mostly done to avoid non constant interconnection structures. To overcome this problem we propose an infinite dimensional coordinate change which is spatially discretized. The second problem is even more crucial. The standard procedure in engineering is that very small effects are neglected because one assumes that they have hardly any influence on the dynamics of the system. This may be true if one is only interested in the simulation results of the dynamics. But if one is interested in controlling the dynamics, neglecting parts of the dynamics can yield a finite dimensional model which is non stabilizable. In this paper we first show that neglecting the magnetic field in a piezoelectric material, and so treating a quasi static electrical field, results in a non stabilizable system. Furthermore, we also show that remodeling the infinite dimensional system with a dynamical electrical field yields a stabilizable finite dimensional system.

T. Voß is with the Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, t.voss@tue.nl

J.M.A. Scherpen is with the University of Groningen, Faculty of Math. & Natural Sciences, Nijenborgh 4, 9747 AG Groningen, NL, J.M.A.Scherpen@rug.nl

II. SHORT INTRODUCTION TO FINITE DIMENSIONAL PH SYSTEMS

In this section we introduce the pH modeling framework, see [1], [8]. The reason why we use pH systems to do modeling for control is that these systems have specific properties which make them suitable for control design. Moreover, in this framework one can easily model complex finite dimensional systems by modeling subsystems independently and then interconnect the systems. As a consequence, the modeling effort is much smaller when using this "divide and conquer" approach. Next we give a brief introduction to finite dimensional pH systems. For more details we refer the interested reader to [2].

The pH framework was originally developed for modeling finite dimensional systems, but was later on extended to the case of infinite dimensional systems as shown in [4], [5].

A finite dimensional pH system in local coordinates can be described as

$$\begin{aligned} \dot{x} &= (J(x) - R(x)) \frac{\partial H}{\partial x}(x) + B(x)u \qquad (1) \\ y &= B^\top(x) \frac{\partial H}{\partial x}(x) \end{aligned}$$

where

- $x = (x_1, \dots, x_n)$ expresses local coordinates in an n-dimensional state space manifold $\mathcal{X} \subset \mathbb{R}^n$.
- $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ are the inputs and outputs respectively. Together they define the ports of the system.
- $J(x) : \mathcal{X} \to \mathbb{R}^{n\times n}$ is the interconnection matrix and depends smoothly on x. Also $J(x)$ is skew-symmetric ($J(x) = -J^\top(x)$).
- $R(x) : \mathcal{X} \to \mathbb{R}^{n\times n}$ is the resistance matrix and is symmetric positive semidefinite ($R(x) = R^\top(x) \geq 0$). Also $R(x)$ depends smoothly on x.
- $B(x) : \mathcal{X} \to \mathbb{R}^{n\times m}$ is the input matrix and depends smoothly on x.
- $H(x) : \mathcal{X} \to \mathbb{R}$ with $H(x) > c > -\infty \, \forall x \in \mathcal{X}$ is the so called Hamiltonian of the system, $H(x)$ represents the stored energy in the system.

Note that for a pH system (1) the energy-balancing property holds

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial^\top H}{\partial x}\dot{x} = \frac{\partial^\top H}{\partial x}\left((J-R)\frac{\partial H}{\partial x} + Bu\right) \qquad (2) \\ &= -\underbrace{\frac{\partial^\top H}{\partial x} R \frac{\partial H}{\partial x}}_{\geq 0} + \underbrace{\frac{\partial^\top H}{\partial x} B}_{y^\top} u \leq y^\top u. \end{aligned}$$

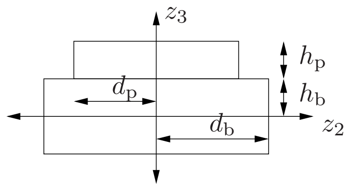

Fig. 1. Cross sectional area of the composite

So, the Hamiltonian is a storage function and therefore a candidate Lyapunov function for the unforced system. Also it follows from (2) that the system is passive.

The last property we would like to mention is that the interconnection of two finite dimensional pH systems yields again a finite dimensional pH system. This property can be exploited for finite dimensional control design which is based on shaping the energy system of the to be controlled system by interconnecting it to another passive system (the controller).

III. Finite dimensional model of the Timoshenko beam with quasi static electrical field

The model we present here was derived in [9], [10] as follows. We first modeled the infinite dimensional dynamics of the piezoelectric beam in the pH framework [10] and then we used the method proposed in [3] to spatially discretize an infinite dimensional nonlinear piezoelectric Timoshenko beam while preserving the pH structure [9]. Different to the very simple system in [3] the model in [9] consists of 8 states and has a non constant interconnection structure. This yields some additional problems which we solve in this paper.

To better understand what kind of beam we discuss, we briefly introduce the geometry of the beam. We consider a piezoelectric composite beam which consists of a base layer to which a piezoelectric layer is bonded. The cross section of the beam is depicted in Figure 1. Moreover, and without loss of generality, we assume that the base layer has a constant thickness ($2d_{\rm b}$) and a constant height $2h_{\rm b}$, while its length is L. The piezoelectric layer is bonded on top of the base layer. Let $h_{\rm p}$ denote the height of the piezoelectric layer and let the width of this layer be $2d_{\rm p}$. Each side of the piezoelectric layer in the z_1z_2 plane is covered by an electrode to which a homogeneous voltage distribution is applied. The voltage distribution will generate an electrical field between the electrodes. Hence, due to the piezoelectric properties, the material will deform. This electrical field can be controlled and thus we can also control the shape of the piezoelectric beam.

The spatial discretization scheme proposed in [3] works as follows. First one divides the beam, which is described in the interval $Z = [0, L]$, into n subintervals. On each of these n subintervals, e.g., $Z_{ab} = [a, b]$ with $0 \le a < b \le L$, we spatially discretize the dynamics while considering the following steps, for more details see [9]:

- approximate the efforts and flows on Z_{ab},
- define the boundary ports over which the elements exchange energy with neighboring elements,
- discretize the interconnection structure,
- formulate the finite dimensional interconnection structure,
- discretize the energy function.

All these steps combined yield then a finite dimensional approximation for the infinite dimensional dynamics of our piezoelectric composite on the interval Z_{ab}. These n finite dimensional pH models are then interconnected in a physical way via the boundary ports which are defined during the spatial discretization scheme. The interconnected model then approximates the dynamics of the total piezoelectric beam on the interval Z.

The finite dimensional system which describes the dynamics on the interval Z_{ab} is then given by

$$\begin{aligned} \dot{x} &= J\nabla_x H + B_{int}u_{int} + B_{ext}u_{ext} \\ y_{int} &= B_{int}^{\top}\nabla_x H + D_{int}u_{int} \\ y_{ext} &= B_{ext}^{\top}\nabla_x H + D_{ext}u_{ext} \end{aligned} \tag{3}$$

where $x = [p_u, p_w, p_\phi, u', w', \phi, \phi', E]$ represents the state on the interval Z_{ab} . The p_i, $i \in \{u, w, \phi\}$ are the momenta in the u, w, and ϕ direction. The states u', w', ϕ, and ϕ' are the strain parameters which we also denote as $\varepsilon = [u', w', \phi, \phi']$ — here the prime operator stands for $x' = \frac{\partial x}{\partial z_1}$. The last component of the state E is the electrical field which is generated between the two electrodes. The energy function H can be stated as

$$H = \frac{1}{2}p^{\top}M^{-1}p + \frac{1}{2}\varepsilon^{\top}\mathbf{C}(\varepsilon)\varepsilon + \frac{1}{2}\epsilon^{e}E^2 \tag{4}$$

where $\mathbf{M}$ is the mass matrix of the beam, $\mathbf{C}$ is a nonlinear smooth positive definite matrix which relates the stresses and the strains in the system and ϵ^e is the permittivity of the piezoelectric material.

The matrix J is given by

$$J = \begin{bmatrix} 0 & K \\ -K^{\top} & 0 \end{bmatrix}$$

with

$$K = \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 & 0 & c_1(x) \\ 0 & \frac{1}{\alpha} & 0 & 0 & c_2(x) \\ 0 & 0 & -\frac{1}{\bar{\alpha}} & \frac{1}{\alpha} & c_3(x) \end{bmatrix}$$

where α and $\bar{\alpha}$ are given non zero constants and the functions $c_i(x)$ depend smoothly on x.

The internal input matrix B_{int} is used to connect the local finite dimensional approximation on the interval Z_{ab} in a physical way with neighboring elements via the mechanical input-output pair $(u_{\rm int}, y_{\rm int})$, which consists of forces and velocities at the boundaries of the finite element. From now on we neglect the terms related to the internal input matrix and the related input-output pair because these components cannot be used as control inputs and therefore do not influence the stabilization properties of the original model.

The external input matrix B_{ext} is used to inject energy into the system via the input-output pair (u_{ext}, y_{ext}) and is defined as

$$B_{ext} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\epsilon^e A_{\rm e}} \end{bmatrix}^{\top}$$

where A_e is the surface of the electrode and the input u_{ext} is the spatially uniform voltage distribution to be applied to the electrode of the piezoelectric beam. The output y_{ext} is the related uniform current distribution which can be measured at the electrode. The input-output pair (u_{ext}, y_{ext}) will be the only port which can be used to control the dynamics of the system. In the next section we will show that the finite dimensional model on the interval Z_{ab} cannot be stabilized by means of active control.

A. Stabilizability of the model

In this section we check whether the spatial discretized model (3) is stabilizable via the external electrical port. We first state Proposition 4.2.14 from [7].

Theorem 1: Consider the generalized Hamiltonian system (1) with equilibrium x_0. A necessary condition for asymptotic stabilizability around x_0 is that for every $\varepsilon > 0$

$$\bigcup_{\|x-x_0\|<\varepsilon} [Im\, J(x) + Im\, B(x)] = \mathbb{R}^n.$$

For the proof we refer the interested reader to [7]. Now we can state the following theorem.

Theorem 1: The system (3) does not fulfill the necessary condition for stabilizability of Proposition 1 if one considers the electrical input as the only control input.

Proof: The rank of the external input matrix B_{ext} is 1. The rank of the interconnection matrix J is 6, so, the interconnection matrix has a six-dimensional image. But to fulfill the necessary condition of Proposition 1 we need that $Im(\mathbf{J}) + Im(\mathbf{B}_{ext}) = \mathbb{R}^8$, which obviously is not the case. Hence, the system is not stabilizable. ■

The result is not surprising since ϕ and ϕ' are not independent to each other. The reason why we have chosen a model which depends on ϕ and ϕ' is that it simplifies the spatial discretization, since it avoids a nonlinear interconnection structure. To overcome the non-stabilizability problem, we now first study another choice of coordinates. In particular, one can define an infinite dimensional coordinate change between the two different models, one with 4 strain states and one with 3 strain states, and use the coordinate change to derive a finite dimensional coordinate change as explained next.

B. Coordinate transformation

Due to the fact that the derivation of the coordinate transformation is rather extensive we present here just the general idea and refer the interested reader to [11]. The infinite dimensional model we used is a valid pH model. However, due to the strong relation between the states ϕ and $\phi' = \frac{\partial \phi}{\partial z_1}$ it turns out that the system is not asymptotically stabilizable. One way to overcome this problem is to define a system with only 3 nonlinear strain states and find an infinite dimensional coordinate transformation between the 3 nonlinear strain state model and (3). We are able to derive such an infinite dimensional coordinate change because the dynamics of the model are independent to the choice of states — 3 strain states or 4 strain states. This coordinate change we use then to construct a finite dimensional coordinate projection which transforms the system (3) into a system with states which are independent to each other.

The strains of an infinite dimensional nonlinear Timoshenko beam are given by

$$\begin{aligned} \varepsilon_{11} &= u_0' + \frac{1}{2}(w')^2 - z\phi' \\ \varepsilon_{13} &= \frac{1}{2}(w - \phi'). \end{aligned} \tag{5}$$

Note that the strain determines the energy function. But one needs to parametrize the strain to be able to define the states of the dynamical system. Moreover, since the strain (5) is nonlinear there are several ways to parametrize the strain, and with this choose the states of the infinite dimensional system. For the system (3) we have chosen a linear strain parametrization

$$\varepsilon := [u_0', w', \phi, \phi']^\top \tag{6}$$

which results in a linear infinite dimensional interconnection structure, but yields a finite dimensional system that cannot be stabilized. Another way to parametrize the strains is to define the following nonlinear strain states

$$\varepsilon := \left[\varepsilon_{11}^0, \varepsilon_{11}^1, \varepsilon_{13}\right]^\top \tag{7}$$

where

$$\varepsilon_{11}^0 := \left(u_0' + \frac{1}{2}(w')^2\right), \quad \varepsilon_{11}^1 := \phi', \quad \varepsilon_{13} := \frac{1}{2}(w - \phi').$$

But, a system with this choice of nonlinear strain states will have a non constant infinite dimensional interconnection structure. Hence, it is impossible to apply a structure preserving spatial discretization scheme. However, one can use this parametrization to define a finite dimensional coordinate change which then renders (3) stabilizable. Although one can choose different strain states, this has no impact on the actual dynamics of the beam because we have not changed the energy function in at all.

Having now two different parametrizations for the strains one can define an *infinite* dimensional coordinate transformation between the two strain state definitions (6) and (7). This infinite dimensional coordinate transformation can then be spatially discretized using the same method that we have used to derive (3). We can now derive a finite dimensional transformation, $z = T(x)$, $z \in \mathbb{R}^7$ and transform the 4 strain state system (3) into

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & \tilde{K} \\ -\tilde{K}^\top & 0 \end{bmatrix} \nabla_z H + \tilde{B}_{int} u_{int} + \tilde{B}_{ext} u_{ext} \\ y_{int} &= \tilde{B}_{int}^\top \nabla_z H + D_{int} u_{int} \\ y_{ext} &= \tilde{B}_{ext}^\top \nabla_z H + D_{ext} u_{ext} \end{aligned} \tag{8}$$

where

$$\begin{aligned} \tilde{K} &= \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 & -\frac{1}{\alpha} c_1(z) \\ \frac{1}{\alpha} c_4(z) & \frac{1}{\alpha} & 0 & -\frac{1}{\alpha} c_2(z) \\ 0 & -\frac{1}{2\bar{\alpha}} & -\frac{1}{\alpha} & -\frac{1}{\alpha} c_3(z) \end{bmatrix} \\ \tilde{\mathbf{B}}_{ext} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\epsilon^e A_e} \end{bmatrix}^\top. \end{aligned} \tag{9}$$

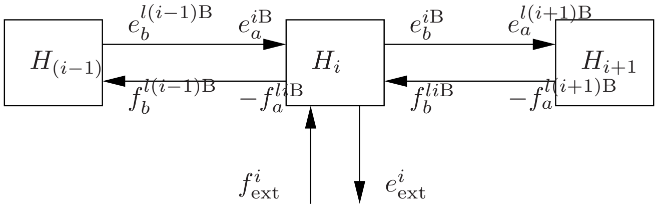

Fig. 2. Interconnection of the ith system with the neighboring systems

The mechanical related part in interconnection matrix has changed entirely but because the coordinate change is working on the mechanical domain the external input matrix has only changed in dimension. Furthermore, note that although the internal input matrix has also changed we skip its definition because the internal inputs are not used for control. Now we can prove that (8) fulfills the necessary conditions for stabilization.

Theorem 2: The system (8) fulfills the necessary condition for stabilizability of Proposition 1 if one considers the electrical input as the only control input.

Proof: The rank of the electrical input matrix $\tilde{\mathbf{B}}_{ext}$ has not changed (so, it is still 1). The rank of the interconnection matrix $\tilde{\mathbf{J}}$ is still 6. Hence, the interconnection matrix has a six-dimensional image. We can also check by calculating the span of the two matrices that indeed $Im(\tilde{\mathbf{J}}) + Im(\tilde{\mathbf{B}}_{ext}) = \mathbb{R}^7$. Hence, the system now satisfies the necessary condition for stabilizability via the electrical input. ■

Note that the result is as expected since we have chosen the coordinate in such a way to remove the dependency between the states and render the system stabilizable.

C. Interconnection of the subsystems to achieve global approximation

With the procedure presented in the past sections we can calculate n finite dimensional pH systems, also called finite elements, which describe the dynamics of the beam locally (on the interval $[a_i, b_i]$ where it holds that $a_i = b_{i-1}$). In order to achieve a global approximation for the dynamics of our beam we have to interconnect the system in a simple manner.

In (3) we have introduced the internal inputs and outputs of a local system, forces and velocities, respectively. The interconnection with the neighboring finite elements is done in such away that the forces and velocities at the boundaries of the neighboring elements coincide. So, the ith system is interconnected to the $(i-1)$st and the $(i+1)$st system. This gives us an interconnection of the ith system with the neighboring systems, as illustrated in Figure 2.

D. Stabilizability of the interconnected model

We have already showed that the model for one element (8) fulfills the condition of Proposition 1. But this does not mean that the fully interconnected system also fulfills this property. We state the following theorem.

Theorem 3: Any interconnection of the systems (8) by using the procedure described in Section III-C is not stabilizable under the assumption that only homogeneous electrical input can be used to stabilize the system.

Proof: An interconnection, as described in Section III-C, of two finite elements S_1 and S_2 given by (8) will lead to the following pH system

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} J_1 & -B_{1,r}B_{2,l}^\top \\ B_{2,l}B_{1,r}^\top & J_2 \end{bmatrix} \begin{bmatrix} \nabla_{x_1} H \\ \nabla_{x_2} H \end{bmatrix} \\
&\quad + B_{int}u_{int} + \begin{bmatrix} B_{ext}^1 \\ B_{ext}^2 \end{bmatrix} u_{ext} \\
y_{int} &= B_{int}^\top \begin{bmatrix} \nabla_{x_1} H \\ \nabla_{x_2} H \end{bmatrix} + D_{int}u_{int} \\
y_{ext} &= \begin{bmatrix} B_{ext}^{1^\top} & B_{ext}^{2^\top} \end{bmatrix} \begin{bmatrix} \nabla_{x_1} H \\ \nabla_{x_2} H \end{bmatrix} + D_{ext}u_{ext}
\end{aligned}$$

where J_i is the interconnection matrix of the system S_i, while $B_{i,l}$, and $B_{i,r}$ are the internal input matrices of the system S_i related to the the left and right side respectively. We define that $B_{int}^i = [B_{i,l}, B_{i,r}]$. The matrix B_{ext}^i is the external input matrix of the system S_i. As already discussed the only control input we can use is the electrical one u_{ext} and is assumed to be equal for both finite elements since we consider a homogeneous spatial voltage field. The rank of the electrical input matrix $\left[B_1^{e\top}, B_2^{e\top}\right]^\top$ is clearly 1. So, the image of this matrix is one-dimensional. The rank of the interconnection matrix is given by the sum of the rank of J_1 and J_2 because it clearly holds that $\text{rank}(B_{1,r}B_{2,l}^\top) \leq \text{rank}J_i$. Moreover, since the systems S_i ($i \in \{1,2\}$) have the same interconnection structure, it is clear that the rank of the total interconnection matrix is $2 \cdot \text{rank}(J_1)$. It is easy to see that the rank of J_1 is 6. So, we obtain that the image of the interconnection matrix is 12 dimensional. But $Im(J) + Im(B_{ext})$ can never be equal to $\mathbb{R}^{14}$. Hence, the system is not stabilizable via an electrical input. Moreover, note that the proof can easily be extended to the interconnection of n finite elements. ■

The result is physically explainable since we have simplified the model by modeling a quasi static electrical field. This means that the model only represents the stored electrical energy and completely neglects the magnetic energy. But since we neglect the magnetic energy we also neglect the necessary energy exchange between the two energies (magnetic and electrical) and thus a crucial part of the dynamics so that we cannot change the electrical field. This makes it impossible to control the electrical field. Consequently, one is not able to control the deformation of the piezoelectric material. One can circumvent this problem by deriving a model where the magnetic energy part of the electromagnetic domain is also modeled. This model has then also the coupling between the electrical and magnetic energy included, which enables us to control the dynamics of the electrical field such that we are able to change the dynamics of the mechanical domain due to the piezoelectric coupling.

So, we can summarize that neglecting energies in the field of pH modeling can yield systems which are non stabilizable, because one neglects crucial dynamics.

IV. 1-D Timoshenko beam with dynamic electrical field

In this section we investigate the stability of a discretized model of a 1-D Timoshenko beam with a dynamic electrical field, see [11]. The reason for modeling a dynamic electrical field is that a model with a quasi static electrical field is non stabilizable, see Section III-D. The derivation of the model will be done using the same procedure as in [9] which we have used to derive (3). Then one applies the same coordinate change which was used to derive [9] in order to get independent strain states. We skip the details and just state the result of the spatial discretization and the coordinate transformation — for more details we refer the interested reader to [9]. The dynamics of a piezoelectric beam with dynamical electromagnetic field on the interval Z_{ab} can be described as

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & J_m & 0 & 0 \\ -J_m^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & J_e \\ 0 & 0 & -J_e^\top & 0 \end{bmatrix} \nabla_{\mathbf{x}} H \qquad (10) \\ &\quad + \mathbf{B}_{int}\mathbf{u}_{int} + \mathbf{B}_{ext}\mathbf{u}_{ext} \\ \mathbf{y}_{int} &= \mathbf{B}_{int}^\top \nabla_{\mathbf{x}} H + D_{int}\mathbf{u}_{int} \\ \mathbf{y}_{ext} &= \mathbf{B}_{ext}^\top \nabla_{\mathbf{x}} H + D_{ext}\mathbf{u}_{ext} \end{aligned}$$

with

$$\begin{aligned} \tilde{J}_m &= \begin{bmatrix} \frac{1}{\alpha_b^{u_0'}} & 0 & 0 \\ \frac{1}{\alpha_b^{u_0'}} c_4(\mathbf{x}) & \frac{1}{2\alpha_b^{w'}} & 0 \\ 0 & -\frac{1}{2\bar{\alpha}} & -\frac{1}{\alpha_b^{\phi'}} \end{bmatrix}, \; J_e = \frac{1}{\alpha} (11) \\ \mathbf{B}_{ext} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\alpha} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\alpha} \end{bmatrix} \end{aligned}$$

where the α**s** in the interconnection matrix are nonzero constants and $c_4(\mathbf{x})$ is a smooth function. The mechanical states of the systems are the same as for the (8), while the electrical states are now expressing the charge and flux distribution on the electrode. Hence, instead of one electrical state, the electrical field, we now have two electrical states. Again we do not define the internal input matrix and the related power port since these ports cannot be used for control. The external inputs are now the current and the voltage applied at a specific point of the electrodes — the electrical part now behaves similar to a transmission line. Next we have to interconnect the finite elements in order to achieve an approximation of the beam on the whole spatial domain. This will be done in the same way described in Section III-C. Then we can state the following theorem.

Theorem 4: The interconnection of systems (10) fulfills the necessary condition of Proposition 1.

Proof: An interconnection, as described in Section III-C, of two systems S_1 and S_2 given by (10) will lead to the following pH system:

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} &= \begin{bmatrix} J_1 & -B_{1,r}B_{2,l}^\top \\ B_{2,l}B_{1,r}^\top & J_2 \end{bmatrix} \begin{bmatrix} \nabla_{\mathbf{x}_1} H \\ \nabla_{\mathbf{x}_2} H \end{bmatrix} \\ &\quad + \mathbf{B}_{int}\mathbf{u}_{int} + \begin{bmatrix} B_{ext}^1 \\ B_{ext}^2 \end{bmatrix} \mathbf{u}_{ext} \\ \mathbf{y}_{int} &= \mathbf{B}_{int}^\top \begin{bmatrix} \nabla_{\mathbf{x}_1} H \\ \nabla_{\mathbf{x}_2} H \end{bmatrix} + D_{int}\mathbf{u}_{int} \\ \mathbf{y}_{exy} &= \begin{bmatrix} B_{ext}^{1\top} & B_{ext}^{2\top} \end{bmatrix} \begin{bmatrix} \nabla_{\mathbf{x}_1} H \\ \nabla_{\mathbf{x}_2} H \end{bmatrix} + D_{ext}\mathbf{u}_{ext} \end{aligned}$$

where J_i is the interconnection matrix of the system S_i, while $B_{i,l}$ and $B_{i,r}$ are the internal input matrices of the system S_i related to the the left and right side respectively. We have that $\mathbf{B}_{int}^i = [B_{i,l}, B_{i,r}]$. The matrix $\mathbf{B}_{ext}^i$ is the external input matrix of the system S_i. We have that the only control input that we can use is the electrical input $\mathbf{u}_{ext}$. The rank of the interconnection matrix is given by the sum of the rank of J_1 and J_2 because it clearly holds that $\text{rank}(B_{1,r}B_{2,l}^\top) \leq \text{rank}J_i$. The systems S_i $(i \in \{1,2\})$ have the same structure. So, it is clear that the rank of the total interconnection matrix is $2 \cdot \text{rank}(J_1)$. It is easy to see that the rank of J_1 is 8, which means that the image of the interconnection matrix is 16-dimensional. So, $Im(J) + Im(B) = \mathbb{R}^{16}$ no matter what the rank of the electrical input matrix is. Hence, the system fulfills the necessary condition for being stabilizable via an electrical input. By induction the statement can now be proved straight forwardly for the interconnection of n systems. ■

A. Comparison between a piezoelectric composite with and without dynamical electrical field

The difference between the model of a piezoelectric composite with and without dynamical electrical field is that for the quasi static electrical field we neglect the magnetic energy of the beam. This is a standard assumption in the field of piezoelectricity, see [6]. The reason why this is done is that the effect of the magnetic field is extremely small. So, the magnetic field has hardly any influence on the dynamics apart from adding a very small delay in the electrical dynamics of the system. This assumption is true if one is only interested in the simulation of the dynamics of the piezoelectric material. However, if one decides to control the shape of the piezoelectric material in the pH framework neglecting the magnetic energy, this results into a model which cannot be stabilized. The reason for this is that the magnetic field represents crucial dynamics of the electromagnetic field. So, if one neglects the magnetic field, it is impossible to stabilize the electric field around a given equilibrium. Moreover, recall that the stabilization of the electromagnetic field around a given equilibrium is the main actuation force in our beam, due to the fact that we treat a piezoelectric material. As a consequence, we are then not able to stabilize the shape around a given equilibrium.

This result is a typical example which shows that sometimes neglecting small details during a modeling process can yield huge problems if one is interested in controlling the dynamics of a given system. Hence, one has to be careful not to neglect too many details if the goal is to obtain a stabilizable model.

V. Conclusion

In this paper we have first shown that the dependency of states in infinite dimensional pH modeling, normally done to avoid non constant interconnection matrices, yields, if one spatially discretizes the system with the method proposed in [3], to a finite dimensional system that is non stabilizable. We have also shown — by using an example of a piezoelectric composite — how a spatially discretized coordinate transformation can be utilized to change the system such that the resulting system then fulfills a necessary condition. Moreover, we have proven that neglecting on first sight unimportant parts of the system's energy can yield infinite dimensional pH systems that are not anymore stabilizable. After adapting the model, with the changed proposed in this paper, the resulting finite dimensional system of a piezoelectric composite fulfills a necessary condition for being stabilizable by means of active control.

VI. Acknowledgments

We would like to thank the MicroNed programme for the funding of the research.

References

[1] M. Dalsmo and A.J. van der Schaft. On the representation and integrability of mathematical structures in energy-conserving physical systems. *SIAM Journal On Control and Optimization*, 37:54–91, 1998.

[2] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx, editors. *Modeling and Control of Complex Physical Systems*. Springer-Verlag Berlin Heidelberg, 2009.

[3] G. Golo, V. Talasila, A.J. van der Schaft, and B.M. Maschke. Hamiltonian discretization of boundary control systems. *Automatica*, 40(5):757–771, 2004.

[4] A. Macchelli. *Port Hamiltonian systems. A unified approach for modeling and control finite and infinite dimensional physical systems*. PhD thesis, University of Bologna–DEIS, Bologna, Italy, 2003.

[5] B.M. Maschke and A.J. van der Schaft. Hamiltonian formulation of distributed parameter systems with boundary energy flow. *Journal of Geometry and Physics*, 42:166 – 194, 2002.

[6] The Institute of Electrical and Electronics Engineers, Inc. *An American National Standard IEEE Standard on Piezoelectricity*, IEEE Standards Board edition, 1987.

[7] A.J. van der Schaft. *L_2-Gain and Passivity Techniques in Nonlinear Control*. Springer London, 2nd edition edition, 2000.

[8] A.J. Van der Schaft and B.M. Maschke. The Hamiltonian formulation of energy conserving physical systems with external ports. *AEU. Archiv für Elektronik und Übertragungstechnik*, 49(5-6):362–371, 1995.

[9] T. Voß and J.M.A. Scherpen. Structure preserving spatial discretization of a 1D piezoelectric Timoshenko beam. *SIAM Journal on Multiscale Modeling and Simulation*, 9:129–154, 2011.

[10] T. Voß, J.M.A. Scherpen, and P.R. Onck. Modeling for control of an inflatable space reflector, the nonlinear 1-D case. In *47th IEEE Conference on Decision and Control*, pages 1777–1782, 2008.

[11] Thomas Voß. *Port-Hamiltonian Modeling and Control of Piezoelectric Beams and Plates: Application to Inflatable Space Structures*. PhD thesis, University of Groningen, 2010.