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Stabilization of nonlinear uncertain systems based on interval observers

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***Abstract*—The problem of output stabilization of a class of nonlinear systems subject to parametric and signal uncertainties is studied. First, an interval observer is designed estimating the set of admissible values for the state. Next, it is proposed to design a control algorithm for the interval observer providing convergence to zero of the interval variables, that implies a similar convergence of the state for the original nonlinear system. An application of the proposed technique shows that a robust stabilization can be performed for linear time-varying and Linear-Parameter-Varying (LPV) systems without assumption that the vector of scheduling parameters is available for measurements. Efficiency of the proposed approach is demonstrated on two examples of computer simulation.**

***Index Terms*—interval estimation, nonlinear stabilization, LPV systems**

I. INTRODUCTION

THE problem of nonlinear system stabilization has been an area of active research during the last two decades [1], [2], [3]. Most of the proposed approaches appeal to particular structural characteristics of a considered nonlinear system. Frequently a partial similarity to linear systems is used to take advantage of the well established solutions for observer or control design. For example, the class of Lipschitz nonlinear systems forms a subclass of nonlinear ones, which can be estimated and regulated applying linear control approaches [4], [5]. This class of nonlinear systems is considered in the paper under assumption that the model contains uncertain time-varying parameters. The proposed methodology ensures the system output stabilization for all parameter values belonging to a given interval.

An important framework which has been largely investigated, to solve the problems of estimation and control for generic nonlinear systems, is based on LPV transformations [6], [7]. There exist several approaches to equivalently represent a nonlinear system in a LPV form [8], [9], [10]. It is worth to note that such a procedure is not based on approximate linearization. It is global and transforms the nonlinear system by introducing extended parametric uncertainties to the LPV setting. There are several methods for estimation of LPV systems, one of them is based on interval state observer design [11], [12], [13], that provides two variables evaluating the lower and upper bounds for state values of LPV systems in real time. Control of LPV systems is more challenging and has been intensively studied [14], [15], [16], [17]. Classically, the vector of scheduling parameters is assumed to be measured. However, in some cases, this assumption may become hard to satisfy because some relevant physical parameters, that can be served as scheduling parameters, are not measured or their measurement is not judged reliable. For example, to generate a LPV model for a nonlinear aircraft model, usually mass and center of gravity are used as scheduling parameters. Although these parameters are measured and available on-board (for example mass estimation based on fuel consumption), their measurements are relatively crude and they should be considered to be an interval, rather than a single point measurement.

In the present work, the assumption on measurements of the vector of scheduling parameters is dropped. The proposed dynamic output feedback approach is based on an interval state observer design for a given class of nonlinear uncertain systems. To the best of our knowledge, an interval observer has not been used for control before. The control is designed to stabilize the interval observer ensuring convergence to a vicinity of zero for the bounding variables. Since the computed interval observer bounds of the state vector have to be valid for any control, the plant state vector also converges to the origin. An advantage of the proposed approach is that it allows us to stabilize rather wide spectrum of nonlinear uncertain systems with partial measurements. Additionally, the proposed method may deal with systems with unstable or/and unobservable modes.

The paper is organized as follows. The preliminaries are given in Section 2. The problem statement and the system equations are given in Section 3. The interval observer design is presented in Section 4. Section 5 is devoted to the control design. Finally, two simulation examples are given in Section 6 to demonstrate efficiency of the developed techniques.

II. PRELIMINARIES

In this work for any two vectors x_1, x_2 or matrices A_1, A_2 the relations $x_1 \leq x_2$, $x_1 \geq x_2$, $A_1 \leq A_2$, $A_1 \geq A_2$ are understood elementwise. A square and symmetric positive definite matrix P is defined by $P \succ 0$. The symbol $|\cdot|$ is used to denote vector or corresponding induced matrix norms. The symbol I denotes the identity matrix, $\lambda_{min}(A)$ and $\lambda_{max}(A)$ are respectively stated for the minimal and maximal eigenvalues of the matrix A. The sequence of integers $1, ..., n$ is denoted as $\overline{1, n}$.

Recall that a square matrix S with dimension $n \times n$ is called Metzler if all its off-diagonal elements are nonnegative: $S_{i,j} \geq 0$, $1 \leq i \neq j \leq n$. For such Metzler matrix S, the system

$$\dot{z} = Sz + r(t),\ z \in \mathbb{R}^n,\ r : \mathbb{R}_+ \to \mathbb{R}^n_+$$

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is monotone (cooperative) [18] and has nonnegative solutions, i.e. if $z(0) \geq 0$ then $z(t) \geq 0$ for all $t \geq 0$.

III. Problem statement

Consider the nonlinear system

$$\dot{x} = Ax + B(\theta(t))u + f(x, y, \theta(t)),\; y = Cx, \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ are the state, the output and the control respectively, $\theta \in \Theta \subset \mathbb{R}^q$ is the vector of uncertain possibly time-varying parameters, the set Θ is assumed to be given, $t \geq 0$; the constant matrices A and C and the nonlinear functions $B : \mathbb{R}^q \to \mathbb{R}^{n\times m}$ and $f : \mathbb{R}^{n+p+q} \to \mathbb{R}^n$ are known, the function f ensures uniqueness and existence of the system solutions at least locally. Without any loss of generality assume that $f(0, 0, \theta) = 0$ for any $\theta \in \Theta$.

The goal of the paper is to design a dynamic output feedback ensuring the system (1) (practical) stabilization at the origin. The forthcoming investigation is based on the following properties of the system (1):

(A1) There are functions $\underline{f}, \overline{f} : \mathbb{R}^{2n+p} \to \mathbb{R}^n$ and matrices B_{min}, B_{max} such that the relations

$$\begin{aligned} \underline{f}(\underline{x}, \overline{x}, y) &\leq f(x, y, \theta) \leq \overline{f}(\underline{x}, \overline{x}, y), \\ B_{min} &\leq B(\theta) \leq B_{max} \end{aligned}$$

are satisfied provided that $\underline{x} \leq x \leq \overline{x}$ and $\theta \in \Theta$.

The functions $\underline{f}$, $\overline{f}$ can be computed under assumption that $\underline{\theta} \leq \theta \leq \overline{\theta}$ for all $\theta \in \Theta$ and some $\underline{\theta}, \overline{\theta} \in \mathbb{R}^q$. Note that under assumption (A1) for any $u \in \mathbb{R}^m$ the inequalities

$$\underline{B}(u)u \leq B(\theta)u \leq \overline{B}(u)u$$

are satisfied for

$$\underline{B}^{(i)}(u) = \begin{cases} B^{(i)}_{min} \; if \; u_i \geq 0; \\ B^{(i)}_{max} \; if \; u_i < 0, \end{cases}$$

$$\overline{B}^{(i)}(u) = \begin{cases} B^{(i)}_{max} \; if \; u_i \geq 0; \\ B^{(i)}_{min} \; if \; u_i < 0, \end{cases}$$

where the upper index i for a matrix $B^{(i)}$ denotes the i-th column of the matrix, and the lower index i for a vector u_i returns the i-th element of the vector.

In this paper the vector θ may play a role of the vector of scheduling parameters in the LPV representation:

$$\dot{x} = A(\theta(t))x + B(\theta(t))u \tag{2}$$

or simply a parameter vector of (1). The system (2) can be presented in the form (1) with $f(x, y, \theta) = [A(\theta) - A]x$. The control design will be presented for more generic nonlinear system (1), the system (2) will be used as a special case. Note that assumption (A1) is satisfied for the LPV system (2) if, for example, there exists a matrix $\Delta A \geq 0$ such that

$$A - \Delta A \leq A(\theta) \leq A + \Delta A,$$

for any $\theta \in \Theta$. In this case $\underline{f}(\underline{x}, \overline{x}, y) = -\Delta A(\overline{x} - \underline{x})$, $\overline{f}(\underline{x}, \overline{x}, y) = \Delta A(\overline{x} - \underline{x})$.

IV. Interval state observer

In this section based on assumption (A1) we are going to design an interval observer for (1) using the results from [11], [13]. To simplify the presentation we introduce the following auxiliary assumption (weaker than assumptions in [11], [13]).

(A2) There exists a matrix L such that the matrix $A - LC$ is Metzler.

Note that we do not require any stability properties for the matrix $A-LC$. In this case, an interval observer for the system (1) can be written as follows [11], [13]:

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{B}(u)u + \underline{f}(\underline{x}, \overline{x}, y) + L(y - C\underline{x}), \\ \dot{\overline{x}} &= A\overline{x} + \overline{B}(u)u + \overline{f}(\underline{x}, \overline{x}, y) + L(y - C\overline{x}). \end{aligned} \tag{3}$$

In this paragraph we implicitly assume that the control u does not violate the conditions of solutions existence into the system (3), and that system (3) solutions are defined for all $t \geq 0$ with a such control.

Theorem 1. *Let assumptions (A1),(A2) and the constraint* $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$ *be satisfied, then for any control* u *the solutions of the system (1), (3) satisfy:*

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t),\; \forall t \geq 0.$$

Proof: Introducing the estimation errors $\underline{e} = x - \underline{x}$ and $\overline{e} = \overline{x} - x$ from (1) and (3) we get that

$$\dot{\underline{e}} = (A - LC)\underline{e} + \underline{d}(t),\; \dot{\overline{e}} = (A - LC)\overline{e} + \overline{d}(t),$$

$$\begin{aligned} \underline{d} &= [B(\theta) - \underline{B}(u)]u + [f(x, y, \theta) - \underline{f}(\underline{x}, \overline{x}, y)], \\ \overline{d} &= [\overline{B}(u) - B(\theta)]u + [\overline{f}(\underline{x}, \overline{x}, y) - f(x, y, \theta)]. \end{aligned}$$

By (A1) we have $\underline{e}(0) \geq 0$, $\overline{e}(0) \geq 0$ and $\underline{d}(0) \geq 0$, $\overline{d}(0) \geq 0$. From assumption (A2) the dynamics of the estimation errors is cooperative. Therefore, $\underline{e}(t) \geq 0$, $\overline{e}(t) \geq 0$ for all $t \geq 0$, that is necessary to prove. ■

The theorem does not claim that variables $\underline{x}(t)$ and $\overline{x}(t)$ are bounded, it establishes the order relations only (for any control u). Due to nonlinear nature of the plant and coupling among the systems (1), (3), even Hurwitz property of the matrix $A - LC$ (usual assumption for interval observers [11], [13]) does not ensure boundedness of the solutions $\underline{x}(t)$ and $\overline{x}(t)$. The solution $x(t)$ also can be unbounded. To ensure the overall boundedness of solutions for the systems (1), (3) we have to design a stabilizing control algorithm.

Remark 2. Before we proceed with the control design, it is worth to note that the assumption (A2) can be relaxed. Actually the existence of a nonsingular matrix T is needed such that the matrix $S = T^{-1}(A - LC)T$ is Metzler. The problem of design of the matrix T is studied in [19]. Indeed, introducing new variables $x = Tz$, the system (1) can be presented as follows:

$$\begin{aligned} \dot{z} = T^{-1}ATz + T^{-1}B(\theta(t))u + T^{-1}f(Tz, y, \theta(t)), \\ y = CTz. \end{aligned} \tag{4}$$

Obviously, if the assumption (A1) is satisfied for the system (1), then a similar property holds for the system (4) with the functions $T^{-1}B(\theta(t))$ and $T^{-1}f(Tz,y,\theta(t))$. Indeed, in this case under assumption (A1) we have:

$$\overline{T}_i\underline{\varphi}(\underline{z},\overline{z},y)-\underline{T}_i\overline{\varphi}(\underline{z},\overline{z},y)\leq T^{-1}f(Tz,y,\theta(t))\leq$$
$$\leq\overline{T}_i\overline{\varphi}(\underline{z},\overline{z},y)-\underline{T}_i\underline{\varphi}(\underline{z},\overline{z},y),$$
$$[\overline{T}_i\underline{B}(u)-\underline{T}_i\overline{B}(u)]u\leq T^{-1}B(\theta)u\leq$$
$$\leq[\overline{T}_i\overline{B}(u)-\underline{T}_i\underline{B}(u)]u,$$

where

$$\underline{\varphi}(\underline{z},\overline{z},y)=\underline{f}(\overline{T}\underline{z}-\underline{T}\overline{z},\overline{T}\overline{z}-\underline{T}z,y),$$
$$\overline{\varphi}(\underline{z},\overline{z},y)=\overline{f}(\overline{T}\underline{z}-\underline{T}\overline{z},\overline{T}\overline{z}-\underline{T}\underline{z},y)$$

and $\overline{T}=max\{0,T\}$, $\underline{T}=\overline{T}-T$, $\overline{T}_i=max\{0,T^{-1}\}$, $\underline{T}_i=\overline{T}_i-T^{-1}$. Then, the interval observer (3) can be written for the system (4) as follows:

$$\begin{aligned}\dot{\underline{z}}&=S\underline{z}+\underline{B}^z(u)u+\underline{f}^z(\underline{z},\overline{z},y)+T^{-1}Ly,\\ \dot{\overline{z}}&=S\overline{z}+\overline{B}^z(u)u+\overline{f}^z(\underline{z},\overline{z},y)+T^{-1}Ly,\end{aligned}\tag{5}$$

where $\underline{B}^z(u)=\overline{T}_i\underline{B}(u)-\underline{T}_i\overline{B}(u)$, $\overline{B}^z(u)=\overline{T}_i\overline{B}(u)-\underline{T}_i\underline{B}(u)$, $\underline{f}^z(\underline{z},\overline{z},y)=\overline{T}_i\underline{\varphi}(\underline{z},\overline{z},y)-\underline{T}_i\overline{\varphi}(\underline{z},\overline{z},y)$ and $\overline{f}^z(\underline{z},\overline{z},y)=\overline{T}_i\overline{\varphi}(\underline{z},\overline{z},y)-\underline{T}_i\underline{\varphi}(\underline{z},\overline{z},y)$. In the new coordinates, the observer (5) is similar to (3). By introducing the estimation errors $\underline{e}=z-\underline{z}$ and $\overline{e}=\overline{z}-z$ it is possible to show that their dynamics are cooperative and

$$\underline{z}(t)\leq z(t)\leq\overline{z}(t),\ \forall t\geq 0$$

provided that $\underline{z}(0)\leq z(0)\leq\overline{z}(0)$. Therefore, for brevity of presentation all results below are stated for the case of assumption (A2), taking in mind the possibility of its relaxation.

Remark 3. Another interval observer is proposed in [12], for the system (1) under assumption (A1) it can be written as follows

$$\begin{aligned}\dot{\underline{x}}&=A^1\underline{x}+A^2\overline{x}+\underline{B}(u)u+\underline{f}(\underline{x},\overline{x},y),\\ \dot{\overline{x}}&=A^1\overline{x}+A^2\underline{x}+\overline{B}(u)u+\overline{f}(\underline{x},\overline{x},y),\end{aligned}\tag{6}$$

where $A^1_{i,i}=A_{i,i}$, $A^1_{i,j}=\max\{0,A_{i,j}\}$ for $i=\overline{1,n}$, $j=\overline{1,n}$, $j\neq i$ and $A^2=A-A^1$. The interval observer (6) has not the gain L, thus it is independent on assumption (A2). Note that, if $\underline{f}$ and $\overline{f}$ are independent on y, then the observer is an autonomous system. This observer under the constraint $\underline{x}(0)\leq x(0)\leq\overline{x}(0)$ for any control u ensures that $\underline{x}(t)\leq x(t)\leq\overline{x}(t)$, $\forall t\geq 0$. The proposed below control approach also can be applied to (6). This possible extension is omitted for brevity of presentation.

V. Control design

The main idea of this work is to solve the stabilization problem for the completely known system (3) instead of (1). Under conditions of Theorem 1, if both $\underline{x}(t)$ and $\overline{x}(t)$ converge to zero, then the state $x(t)$ also has to converge to zero, and boundedness of $x(t)$ follows by the same property of $\underline{x}(t)$ and $\overline{x}(t)$. In this case the signal $y(t)$ is treated in the system (3) as a state dependent disturbance with upper bound

$$|y(t)|\leq|C|(|\underline{x}(t)|+|\overline{x}(t)|),\ t\geq 0.$$

Therefore, it is required to stabilize the system (3) uniformly (or robustly) with respect to the input y. Applying the same arguments in the case of Remark 2, the system (1) stabilization follows by the interval observer stabilization in coordinates $\underline{z}$,$\overline{z}$ (the matrix T is nonsingular). In the case of Remark 3, stabilization of the system (1) follows by the system (6) stabilization.

The advantages of such reduction are that the system (3) is completely known and the state vector $\underline{x}(t)$, $\overline{x}(t)$ is available. However, the dimension of (3) is two times bigger than the corresponding dimension of the system (1) while the control vector u preserves its size. Another difficulty is that the system (3) has variable structure if the matrix functions $\underline{B}$ and $\overline{B}$ depend on u or if the matrix function B in (1) depends on θ (if this is not the case and $B(\theta)=B$, then $\underline{B}(u)=\overline{B}(u)=B$).

The nonlinearity of the system (3) inherited from (1) represents an obstacle, since there is no common approach to robustly stabilize a generic nonlinear system. From another side, there exist several techniques obtained for special classes of nonlinear systems [2], [3]. In the present work some of them are chosen and applied here to stabilize (3) and (1). For this purpose some restrictions on the functions $\underline{f}$ and $\overline{f}$ are introduced.

A. Lipschitz case

(A3) Let there exist constants $\overline{a}_i\geq 0$, $\underline{a}_i\geq 0$, $i=1,2,3$ such that for any $\overline{x},\underline{x}\in\mathbb{R}^n$, $y\in\mathbb{R}^p$ the inequalities

$$|\overline{f}(\underline{x},\overline{x},y)|\leq\overline{a}_1|\overline{x}|+\overline{a}_2|\underline{x}|+\overline{a}_3|y|,$$
$$|\underline{f}(\underline{x},\overline{x},y)|\leq\underline{a}_1|\overline{x}|+\underline{a}_2|\underline{x}|+\underline{a}_3|y|$$

are satisfied.

The introduced assumption does not restrict behavior of the nonlinearity f in the plant equations (1). In addition, if stabilization is required for a predefined set of initial conditions, then for rather wide spectrum of nonlinear functions f, the majorant nonlinearities $\underline{f}$ and $\overline{f}$ can be chosen locally to satisfy the assumption (A3). Indeed, roughly speaking this assumption says that the nonlinearities $\underline{f}$ and $\overline{f}$ are globally Lipschitz. The locally Lipschitz property is required if stabilization into a predefined set is needed.

For the system (2) the assumption (A3) is naturally satisfied for $\overline{a}_1=\overline{a}_2=\underline{a}_1=\underline{a}_2=|\Delta A|$ and $\overline{a}_3=\underline{a}_3=0$.

Due to Lipschitz property of the system (3) under assumption (A3), the control is chosen in the conventional state linear feedback form:

$$u=\underline{K}\underline{x}+\overline{K}\overline{x},\tag{7}$$

where $\underline{K}$ and $\overline{K}$ are two feedback gains to be designed. Substitution of the control (7) into the equations (3) gives:

$$\begin{aligned}\dot{\underline{x}}&=[A-LC+\underline{B}(u)\underline{K}]\underline{x}+\underline{B}(u)\overline{K}\overline{x}+\underline{f}(\underline{x},\overline{x},y)+Ly,\\ \dot{\overline{x}}&=[A-LC+\overline{B}(u)\overline{K}]\overline{x}+\overline{B}(u)\underline{K}\underline{x}+\overline{f}(\underline{x},\overline{x},y)+Ly.\end{aligned}\tag{8}$$

The linear part of the system depends on the sign of the control (7) and is defined by the following 2^m matrices:

$$G_k=\begin{bmatrix}A-LC+\underline{B}_k\underline{K} & \underline{B}_k\overline{K}\\ \overline{B}_k\underline{K} & A-LC+\overline{B}_k\overline{K}\end{bmatrix},k=1,...,2^m,$$

where $\overline{B}_k=B_{max}+B_{min}-\underline{B}_k$ and $\underline{B}_k^{(i)}$ can be composed by $B_{min}^{(i)}$ or $B_{max}^{(i)}$ for each $i=1,...m$ (there are 2^m variants of such compositions). Thus the system (8) is nonlinear and it has multiple-mode linear part.

Theorem 4. *Let assumptions (A1)-(A3) hold,* $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$ *and for the chosen matrices* $\underline{K}$, $\overline{K}$ *there exist matrices* $P^T = P \succ 0$ *and* $Q^T = Q \succ 0$ *such that the Riccati equation*

$$G_k^T P + PG_k + I + \alpha^2 P^2 + Q = 0$$

is verified for all $k = 1, ..., 2^m$, *where* $\alpha = max_{s=1,2}\{\overline{a}_s + \underline{a}_s\} + 2|C|(\overline{a}_3 + \underline{a}_3 + 2|L|)$. *Then, solutions of the system (1), (3), (7) are bounded and asymptotically converge to the origin.*

Proof: Since all conditions of Theorem 1 are satisfied, we have $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$ and $|y| \leq 2|C||\xi|$, where $\xi = [\underline{x}^T \; \overline{x}^T]^T$ is the state vector of the system (8). Consider for the system (8) the Lyapunov function $V = \xi^T P \xi$:

$$\dot{V} = \xi^T (G_k^T P + PG_k)\xi + 2\xi^T P[F(\xi, y) + \Lambda y],$$

where $F(\xi, y) = [\underline{f}(\underline{x}, \overline{x}, y)^T \; \overline{f}(\underline{x}, \overline{x}, y)^T]^T$ and $\Lambda = [L^T \; L^T]^T$. Owing the previous definitions

$$\begin{aligned} |F(\xi, y)| &\leq (\overline{a}_1 + \underline{a}_1)|\overline{x}| + (\overline{a}_2 + \underline{a}_2)|\underline{x}| + (\overline{a}_3 + \underline{a}_3)|y| \\ &\leq max_{s=1,2}\{\overline{a}_s + \underline{a}_s\}|\xi| + (\overline{a}_3 + \underline{a}_3)|y|, \\ |\Lambda| &\leq 2|L|, \end{aligned}$$

and completing squares, we obtain ($|F(\xi, y) + \Lambda y| \leq \alpha|\xi|$):

$$\begin{aligned} \dot{V} &\leq \xi^T (G_k^T P + PG_k + I)\xi + \\ &\quad +[F(\xi, y) + \Lambda y]^T P^2 [F(\xi, y) + \Lambda y] \\ &\leq \xi^T (G_k^T P + PG_k + I + \alpha^2 P^2)\xi \leq -\xi^T Q \xi. \end{aligned}$$

Therefore, the variable ξ is bounded and asymptotically converges to zero (the system (8) is asymptotically stable). Since all conditions of Theorem 1 are satisfied we have that $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$, that implies boundedness and convergence to zero for $x(t)$. ■

The proposed theorem establishes stability conditions for the interval observer-based control (7) and it provides a constructive technique for the matrices $\underline{K}$ and $\overline{K}$ choice resolving the Riccati equations. Note that in this case, to ensure the system stability, we do not require a stability property of the matrix $A - LC$, it could be unstable, but Metzler. That allows us considering systems with nonobservable or non-detectable pair (A, C). Moreover, the matrix G_k stability can be ensured by the gains L, $\underline{K}$, $\overline{K}$ choice. The pairs of matrices (A, C), $(A, \underline{B}_k)$ and $(A, \overline{B}_k)$ can be unobservable and uncontrollable (separately the gains L or $\underline{K}$, $\overline{K}$ can not ensure the matrix A stability), but their combined application $A - LC + \underline{B}_k \underline{K}$, $A - LC + \overline{B}_k \overline{K}$ may solve the problem.

B. Known control gain

In this section we consider more simple situation assuming that $B(\theta) = B$, then $\underline{B}(u) = \overline{B}(u) = B$. The result of Theorem 1 is valid in this case, and the assumption (A3) can be relaxed as follows.

(A4) Assume there exist constants $\rho \geq 0$, $\phi \geq 0$, $\gamma \geq 0$, $\sigma \geq 0$ and $\nu \geq 0$ such that for any $\overline{x}, \underline{x} \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ the inequalities

$$|\delta f(\underline{x}, \overline{x}, y)| \leq \rho|\overline{x} - \underline{x}| + \phi,$$
$$|f_a(\underline{x}, \overline{x}, y)| \leq 0.5\gamma|\overline{x} + \underline{x}| + \sigma|y| + \nu,$$

are satisfied, where

$$\delta f(\underline{x}, \overline{x}, y) = \overline{f}(\underline{x}, \overline{x}, y) - \underline{f}(\underline{x}, \overline{x}, y),$$
$$f_a(\underline{x}, \overline{x}, y) = 0.5[\overline{f}(\underline{x}, \overline{x}, y) + \underline{f}(\underline{x}, \overline{x}, y)].$$

Again, the introduced assumption does not restrict behavior of the nonlinearity f in the plant equations (1). It imposes mainly the same restriction, that the difference δf and the average f_a are globally Lipschitz. The locally Lipschitz property is required if stabilization into a predefined set is considered. The assumption (A4) is less restrictive as (A3) due to nonzero constants ϕ and ν appearance.

The control is chosen in the same form (7) with $u = \underline{K}\underline{x} + \overline{K}\overline{x} = K_d e_d + K_a e_a$, $K_d = 0.5[\overline{K} - \underline{K}]$, $K_a = \overline{K} + \underline{K}$, where $e_d = \overline{x} - \underline{x}$ is the estimated interval length (difference between the upper and the lower estimates) and $e_a = 0.5[\overline{x} + \underline{x}]$ is the interval average. The dynamics of these errors have the form:

$$\dot{e}_d = [A - LC + B_d K_d]e_d + B_d K_a e_a + \delta f(\underline{x}, \overline{x}, y),$$
$$\dot{e}_a = [A - LC + B_a K_a]e_a + B_a K_d e_d + f_a(\underline{x}, \overline{x}, y) + Ly,$$

where $B_d = \overline{B}(u) - \underline{B}(u) = 0$, $B_a = 0.5[\overline{B}(u) + \underline{B}(u)] = B$. In this case the dynamics of e_d does not depend on the control and stability of this variable has to be ensured by a choice of the matrix L (the pair (A, C) is observable for instance). The choice $K_d = 0$ explicitly decouples dynamics of the errors e_d and e_a:

$$\begin{aligned} \dot{e}_d &= [A - LC]e_d + \delta f(\underline{x}, \overline{x}, y), \\ \dot{e}_a &= [A - LC + B_a K_a]e_a + f_a(\underline{x}, \overline{x}, y) + Ly. \end{aligned} \tag{9}$$

In this case the matrices L and K_a are responsible for stability of different variables, e_d and e_a respectively.

Theorem 5. *Let assumptions (A1), (A2),(A4) hold,* $\underline{x}(0) \leq x(0) \leq \overline{x}(0)$ *and for the chosen matrix* K_a, *there exist matrices* $P_d^T = P_d \succ 0$, $P_a^T = P_a \succ 0$ *and* $Q_d^T = Q_d \succ 0$, $Q_a^T = Q_a \succ 0$ *such that the Riccati equations*

$$\begin{aligned} (A - LC)^T P_d + P_d(A - LC) + I + 2\rho^2 P_d^2 + Q_d = 0, \\ [A - LC + B_a K_a]^T P_a + P_a[A - LC + B_a K_a] + \\ +I + \alpha^2 P_a^2 + Q_a = 0 \end{aligned}$$

are verified for $\alpha = \gamma + \beta$, $\beta = 2|C|(\sigma + |L|)$. *Then, the solutions of the system (1), (3), (7) are bounded and admit the estimates:*

$$|e_d(t)| \leq \kappa_d(|e_d(0)|e^{-0.5\eta_d t} + \varrho_d \phi),$$
$$|e_a(t)| \leq \kappa_a(|e_a(0)|e^{-0.5\eta_a t} + \varrho_a \sqrt{\pi[|e_d(0)|, \phi, \nu]})$$

with

$$\pi[x, \phi, \nu] = \frac{\beta^2 \kappa_d^2 x^2}{1 - \eta_d \eta_a^{-1}}(e^{-\eta_d t} - e^{-\eta_a t}) + $$
$$+ \frac{2\alpha\beta\kappa_d x}{1 - 0.5\eta_d \eta_a^{-1}}(e^{-0.5\eta_d t} - e^{-\eta_a t}) + (\beta\kappa_d \varrho_d \phi + \nu)^2,$$

where $\eta_d = \frac{\lambda_{min}(Q_d)}{\lambda_{max}(P_d)}$, $\kappa_d = \sqrt{\frac{\lambda_{max}(P_d)}{\lambda_{min}(P_d)}}$, $\varrho_d = \sqrt{\frac{2\lambda_{max}(P_d)}{\eta_d}}$, $\eta_a = \frac{\lambda_{min}(Q_a)}{\lambda_{max}(P_a)}$, $\varrho_a = \sqrt{\frac{2\lambda_{max}(P_a)}{\eta_a}}$ *and* $\kappa_a = \sqrt{\frac{\lambda_{max}(P_a)}{\lambda_{min}(P_a)}}$.

Proof: All conditions of Theorem 1 are satisfied and we have $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$ for all $t \geq 0$ and $|y| \leq 2|C|(|e_a| + |e_d|)$. Since the dynamics of variables e_d and e_a are decoupled in (9) it is possible to analyze stability of these variables separately. Consider the Lyapunov function $V_d = e_d^T P_d e_d$, then

$$\dot{V}_d = e_d^T([A-LC]^T P_d + P_d[A-LC])e_d + 2e_d^T P_d \delta f(\underline{x}, \overline{x}, y).$$

Taking in mind that $|\delta f(\underline{x}, \overline{x}, y)| \le \rho|e_d| + \phi$ we obtain:

$$\begin{aligned}
\dot{V}_d &\le e_d^T([A-LC]^T P_d + P_d[A-LC] + I)e_d + \\
&\quad + \delta f(\underline{x}, \overline{x}, y)^T P_d^2 \delta f(\underline{x}, \overline{x}, y) \\
&\le e_d^T([A-LC]^T P_d + P_d[A-LC] + I + \\
&\quad + 2\rho^2 P_d^2)e_d + 2|P_d|^2\phi^2 \\
&\le -e_d^T Q_d e_d + 2|P_d|^2\phi^2.
\end{aligned}$$

Therefore, the variable e_d is practically stable and it admits the required estimate.

Next, consider the Lyapunov function $V_a = e_a^T P_a e_a$, then:

$$\begin{aligned}
\dot{V}_a &= e_a^T([A-LC+B_aK_a]^T P_a + P_a[A-LC+ \\
&\quad + B_aK_a])e_a + 2e_a^T P_a(f_a(\underline{x}, \overline{x}, y) + Ly).
\end{aligned}$$

By assumption (A4), we have $|f_a(\underline{x}, \overline{x}, y) + Ly| \le \gamma|e_a| + (\sigma + |L|)|y| + \nu$, then

$$\begin{aligned}
\dot{V}_a &\le e_a^T([A-LC+B_aK_a]^T P_a + P_a[A-LC+B_aK_a] + \\
&\quad + I + \alpha^2 P_a^2)e_a + 2|P_a|^2(\beta|e_d| + \nu)^2 \le \\
&\le -e_a^T Q_a e_a + 2|P_a|^2(\beta|e_d| + \nu)^2.
\end{aligned}$$

Using the proven boundedness of the error e_d, the required estimate is satisfied. ■

Comparing with Theorem 4, Theorem 5 presents more simple way of treatment for the case $B(\theta) = B$. For the case $\phi = \nu = 0$ the boundedness and asymptotic convergence to zero are substantiated in Theorem 5.

VI. EXAMPLES

In this section we illustrate the proposed approach on examples of time-varying uncertain systems.

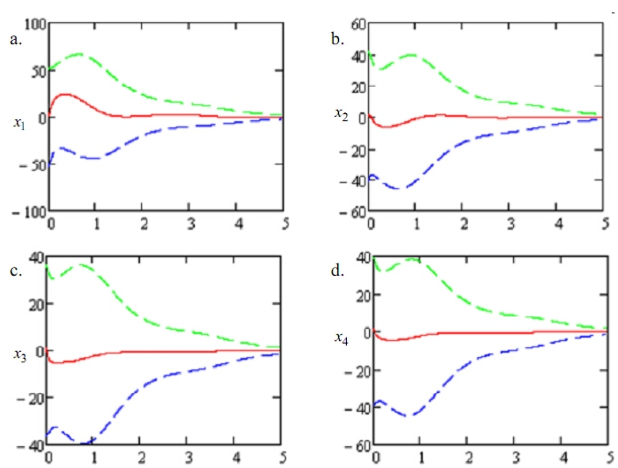

Fig. 1. The results of LPV system stabilization.

A. LPV system

Consider the uncertain system:

$$\dot{x} = A(\theta)x + Bu, \ y = Cx,$$

$$A - \Delta A \le A(\theta) \le A + \Delta A, \ \theta \in \mathbb{R},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 35.6 & 50.7 & 45.6 & 75.6 \\ -1.8 & -25.5 & -3.8 & -6.3 \\ -18.1 & -20 & -38.9 & -31 \\ -5.8 & -7 & -6.6 & -30.5 \end{bmatrix}$$

$$\Delta A = \frac{1}{4}\begin{bmatrix} 1 & 1 & 5 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 1 & 2 & 2 \\ 1 & 4 & 1 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ -2 & 1 \\ -2 & 2 \end{bmatrix},$$

where the matrix ΔA defines the admissible time-varying deviations from the nominal value A. The matrix A is unstable and it does not exist a matrix L such that $A - LC$ is Metzler (assumption (A2) is never satisfied). However for

$$L = \begin{bmatrix} 5 & 2 & 3 & 1 \end{bmatrix}^T,$$

$$T = \begin{bmatrix} -22.179 & 8 & 3 & 19 \\ 8 & -21.179 & 7 & 5 \\ 3 & 7 & -20.179 & 6 \\ 9 & 5 & 6 & -19.179 \end{bmatrix}$$

the matrix $T^{-1}(A - LC)T$ is Hurwitz and Metzler as it is required in Remark 2 and Theorem 4. In addition, all other conditions of Theorem 4 are satisfied for the system (4) and the interval observer (5) with

$$\underline{K} = \begin{bmatrix} -2.8 & -5.6 & 1.2 & -1.8 \\ -8.4 & -9.8 & -7 & -13.2 \end{bmatrix},$$

$$\overline{K} = \begin{bmatrix} -9.6 & -13.2 & -0.9 & -12.6 \\ -15.6 & -18.9 & -14.4 & -23.7 \end{bmatrix}.$$

For simulation we choose $A[\theta(t)] = A(t) = A + V(t)$, where

$$V(t) = \\ = \frac{1}{4}\begin{bmatrix} sin(t) & cos(0.5t) & 5sin(2t) & 2cos(t) \\ sin(0.5t) & 2cos(2t) & cos(t) & 3sin(0.5t) \\ 2cos(2t) & sin(t) & 2cos(0.5t) & 2sin(2t) \\ cos(t) & 4sin(0.5t) & cos(2t) & 2sin(0.5t) \end{bmatrix}.$$

The results of simulation are presented in Fig. 1. On plots Fig. 1,a – Fig. 1,d the state coordinates are shown (solid line) with the corresponding bounding variables from the interval observer (dashed lines).

B. Nonlinear system

Consider the time-varying nonlinear pendulum:

$$\dot{x}_1 = x_2; \ \dot{x}_2 = -\omega^2(t)sin(x_1) - \kappa(t)x_2 + b(t)u; \ y = x_1,$$

where $x_1 \in [-\pi, \pi]$ is the angle, $x_2 \in \mathbb{R}$ is the angular velocity. The parameters satisfy the inequalities

$$\omega_m \le \omega(t) \le \omega_M, \ \kappa_m \le \kappa(t) \le \kappa_M, \ b_m \le b(t) \le b_M$$

for some known $\omega_m, \omega_M, \kappa_m, \kappa_M, b_m, b_M$. Clearly, the system is in the form (1) for

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\kappa_a \end{bmatrix}, f[x_1, x_2, \theta(t)] = \begin{bmatrix} 0 \\ \theta_1(t)sin(x_1) + \theta_2(t)x_2 \end{bmatrix},$$

$$B[\theta(t)] = \begin{bmatrix} 0 \\ \theta_3(t) \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T,$$

$$\theta(t) = [-\omega^2(t)\ \Delta\kappa(t)\ b(t)]^T,$$

where $\kappa_a = 0.5(\kappa_m + \kappa_M)$, $\Delta\kappa(t) = \kappa(t) - \kappa_a$, $|\Delta\kappa(t)| \leq \delta\kappa = 0.5(\kappa_M - \kappa_m)$ (assumption (A1) is satisfied). It is required to stabilize this pendulum in the upper unstable equilibrium $(\pi, 0)$ starting from the origin (the stable equilibrium). Such problem is called the pendulum swinging up, a solution of this problem for uncertain and time-varying pendulum is rather complicated. For simulation the following values of parameters have been chosen:

$$\omega_m = 0.1,\ \omega_M = 1.2,\ \kappa_m = 0.4,\ \kappa_M = 1.5, \\ b_m = 0.5,\ b_M = 2.5$$

and

$$\omega(t) = 0.5[\omega_M + \omega_m + (\omega_M - \omega_m)cos(t)], \\ b(t) = 0.5[b_M + b_m + (b_M - b_m)cos(2t)], \\ \kappa(t) = \kappa_a + \delta\kappa sin(t).$$

For

$$L = [5\ 0]^T$$

the matrix $A - LC$ becomes Hurwitz and Metzler (assumption (A2) is verified). Clearly, assumption (A3) holds since the nonlinearity f is globally Lipschitz in this case. All conditions of Theorem 4 are satisfied for

$$\underline{K} = -[1\ 1],\ \overline{K} = -[5\ 1].$$

The results of simulation are presented in Fig. 2, where the state coordinates (solid lines) and their interval bounds (dashed lines) are plotted.

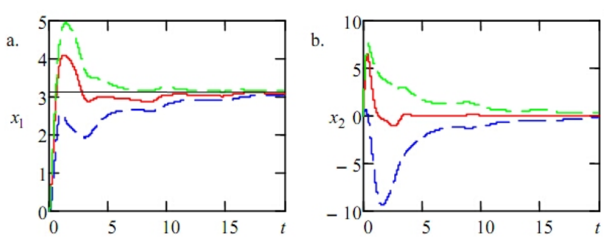

Fig. 2. The results of nonlinear pendulum stabilization.

VII. Conclusions

The problem of nonlinear uncertain systems stabilization has been addressed in this paper. An approach for output dynamic feedback design is proposed, where the conventional observer is replaced with an interval one. In this way, the problem of output stabilization of an uncertain nonlinear system is reduced to the problem of robust state feedback design for two completely known nonlinear systems. Applicability conditions of the interval observers are less conservative than for the conventional state observers. Thus the proposed approach extends the class of uncertain nonlinear systems having stabilizing control laws. Application of the proposed stabilizing method to LPV systems does not require measurements of the vector of scheduling parameters making the method very attractive for uncertain systems with partial measurements. The price to pay is that the dimension of the controller may become rather large (the interval observer has the dimension two times bigger than the plant model). The proposed approach is illustrated through simulation examples.

Finally, the work presented in this paper is focused on the case of Lipschitz nonlinearities. Other state feedback techniques for robust stabilization of nonlinear systems (e.g. backstepping/forwarding, passivation approach, feedback linearization) can be applied in a similar way. An appealing direction for future work is to relax conservatism of the proposed Lipschitz stability conditions.

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