2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Uncertainty Quantification for Stochastic Subspace Identification on Multi-Setup Measurements

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Abstract— **In Operational Modal Analysis, the modal parameters (natural frequencies, damping ratios and mode shapes), obtained from Stochastic System Identification of structures, are subject to statistical uncertainty from ambient vibration measurements. It is hence necessary to evaluate the uncertainty bounds of these obtained results. To obtain vibration measurements at many coordinates of a structure with only a few sensors, it is common practice to use multiple sensor setups for the measurements. Recently, a multi-setup subspace identification algorithm has been proposed that merges the data from different setups first to obtain one set of global modal parameters. This paper proposes an algorithm that efficiently estimates the uncertainty on modal parameters obtained from this multi-setup subspace identification.**

I. INTRODUCTION

The estimation of modal parameters can easily be carried out by using Stochastic System Identification methods on sensor measurements, where the vibrating characteristics (frequencies, damping ratios, mode shapes) are identified of mechanical or civil structures subject to uncontrolled, unmeasured and nonstationary excitation [1]. In [2], it was proved that the Instrumental Variable method and what was called the Balanced Realization method for linear eigenstructure identification are consistent in a nonstationary context. From that on, the family of subspace algorithms has been extensively studied and expanded rapidly [3], [4], [5], [6].

To obtain vibration measurements at many coordinates of a structure with only a few sensors, it is common practice to use multiple sensor setups for the measurements. For these multi-setup measurements, some of the sensors, the so-called reference sensors, stay fixed throughout all the setups, while the other sensors are moved from setup to setup. By fusing in some way the corresponding data, this allows to perform modal identification as if there was a very large number of sensors, even in the range of a few hundreds or thousands. In [7], [8] a method was proposed to merge the data from all the setups, before doing the global system identification on it. This method was designed for covariance-driven subspace identification and generalized in [9], [10] to a large range of subspace methods.

All identified modal parameters are afflicted with statistical uncertainty due to many reasons, such as finite number of data samples, undefined measurement noises, nonstationary excitation, nonlinear structure or model order reduction. Then the system identification algorithms do not yield the exact system matrices. To quantify the statistical uncertainty of the obtained modal parameters, the statistical uncertainty in the data can be evaluated and propagated to the system matrices and, thus, to the modal parameters. Such an algorithm was proposed in [11] for covariance-driven subspace identification. It has been shown how uncertainty bounds of modal parameters can be determined from the covariances of the system matrices, which are obtained from some covariance of the data. Many extensions are possible to this algorithm depending on the identification procedure of interest. Recently, uncertainty bounds have been derived for the Eigensystem-Realization-Algorithm, a class of subspace methods [12].

The current paper will expand on this and focus on the uncertainty computation for the modal parameters obtained from multi-setup stochastic subspace identification. Applying uncertainty computation as in [11] would yield to matrix computations for very large matrices in the multi-setup setting. An efficient uncertainty quantification algorithm is derived here, whose memory requirement does not increase with the number of setups or the total number of sensors.

In Section II, the generic stochastic subspace identification algorithm is introduced and the multi-setup subspace identification explained. Then uncertainty bounds for the multi-setup algorithm are derived in Section III.

II. STOCHASTIC SUBSPACE IDENTIFICATION (SSI)

A. The General Stochastic Subspace Identification Algorithm

The discrete time model in state-space form is:

$$\begin{cases} X_{k+1} &= AX_k + V_{k+1} \\ Y_k &= CX_k \end{cases} \tag{1}$$

with the state $X \in \mathbb{R}^n$, the output $Y \in \mathbb{R}^r$, the state transition matrix $A \in \mathbb{R}^{n\times n}$ and the observation matrix $C \in \mathbb{R}^{r\times n}$. The state noise V is unmeasured and assumed to be Gaussian, zero-mean, white.

Let r be the number of sensors, p and q be chosen parameters with $(p+1)r \geq qr \geq n$. From the output data, a matrix $\mathcal{H}$ is built according to a chosen SSI algorithm, see e.g. [6] for an overview. The matrix $\mathcal{H}$ will be called "subspace matrix" in the following, and the SSI algorithm is chosen such that the corresponding subspace matrix enjoys the factorization property

$$\mathcal{H} = \mathcal{O}\ \mathcal{Z} \tag{2}$$

This work was supported by the European project FP7-NMP CP-IP 213968-2 IRIS.

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into the matrix of observability

$$\mathcal{O} \stackrel{\text{def}}{=} \mathcal{O}(C,A) \stackrel{\text{def}}{=} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^p \end{bmatrix} \tag{3}$$

and a matrix $\mathcal{Z}$ depending on the selected SSI algorithm. Note that with (2) we restrict ourselves to SSI algorithms without a left weighting.

Example 1: Let $N+p+q$ be the number of available samples, and define the data matrices

$$\mathcal{Y}^+ = \frac{1}{\sqrt{N}} \begin{bmatrix} Y_{q+1} & Y_{q+2} & \vdots & Y_{N+q} \\ Y_{q+2} & Y_{q+3} & \vdots & Y_{N+q+1} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{q+p+1} & Y_{q+p+2} & \vdots & Y_{N+p+q} \end{bmatrix},$$

$$\mathcal{Y}^- = \frac{1}{\sqrt{N}} \begin{bmatrix} Y_q & Y_{q+1} & \vdots & Y_{N+q-1} \\ Y_{q-1} & Y_q & \vdots & Y_{N+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ Y_1 & Y_2 & \vdots & Y_N \end{bmatrix}.$$

For the *covariance-driven* SSI [2], [5], a block Hankel matrix containing correlations of the data is built, which is asymptotically equivalent to the subspace matrix $\mathcal{H}^{\text{cov}} \stackrel{\text{def}}{=} \mathcal{Y}^+\mathcal{Y}^{-T}$. It enjoys the factorization property (2), where $\mathcal{Z}$ is the controllability matrix.

For the *data-driven* SSI with the Unweighted Principal Component (UPC) algorithm [4], [5], the matrix $\tilde{\mathcal{H}}^{\text{dat}} \stackrel{\text{def}}{=} \mathcal{Y}^+\mathcal{Y}^{-T}(\mathcal{Y}^-\mathcal{Y}^{-T})^{-1}\mathcal{Y}^-$ enjoys the factorization property (2) where $\mathcal{Z}$ is the Kalman filter state matrix. In practice, the respective subspace matrix $\mathcal{H}^{\text{dat}}$ is obtained from an RQ decomposition of the data, such that $\tilde{\mathcal{H}}^{\text{dat}} = \mathcal{H}^{\text{dat}}Q$ with an orthogonal matrix Q. See the mentioned references for details on the implementations. □

The eigenstructure of the system (1) is retrieved from a given matrix $\mathcal{H}$ with the general subspace algorithm stated in the following, with the condition that factorization property (2) holds for the selected subspace algorithm.

The observability matrix $\mathcal{O}$ is obtained from a thin Singular Value Decomposition (SVD) of the matrix $\mathcal{H}$ and its truncation at the desired model order n:

$$\begin{aligned} \mathcal{H} &= U\Sigma V^T \\ &= \begin{bmatrix} U_1 & U_0 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_0^T \end{bmatrix}, \end{aligned} \tag{4}$$

$$\mathcal{O} = U_1\Sigma_1^{1/2}. \tag{5}$$

with the matrices

$$U_1 = \begin{bmatrix} u_1 \dots u_n \end{bmatrix}, V_1 = \begin{bmatrix} v_1 \dots v_n \end{bmatrix}, \Sigma_1 = \text{diag}\{\sigma_1, \dots, \sigma_n\}$$

containing the first n left and right singular vectors, and singular values. The observation matrix C is then found in the first block-row of the observability matrix $\mathcal{O}$. The state transition matrix A is obtained from the shifting invariance property of $\mathcal{O}$, namely as the least squares solution of

$$\mathcal{O}^{\uparrow}A = \mathcal{O}^{\downarrow}, \text{ where } \mathcal{O}^{\uparrow} \stackrel{\text{def}}{=} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix}, \ \mathcal{O}^{\downarrow} \stackrel{\text{def}}{=} \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^p \end{bmatrix}. \tag{6}$$

Definition 2: In general, let $^{\uparrow}$ respectively $^{\downarrow}$ be operators, which remove the last respectively the first block row of an observability matrix as in (6). The size of the removed block row is the size of the observation matrix present in the observability matrix. □

The eigenstructure $(\lambda, \varphi_\lambda)$ results from

$$\det(A - \lambda I) = 0, \quad A\phi_\lambda = \lambda\phi_\lambda, \quad \varphi_\lambda = C\phi_\lambda, \tag{7}$$

where λ ranges over the set of eigenvalues of A. From λ, the natural frequency and damping ratio are obtained, and φ_λ is the corresponding mode shape.

There are many papers on the used subspace identification techniques. A complete description can be found in [2], [4], [5], [6], and the related references. A proof of non-stationary consistency of these subspace methods can be found in [6].

B. Multi-Setup Stochastic Subspace Identification

The problem of stochastic subspace identification using nonsimultaneously recorded data from multiple sensor setups was addressed in [7], [8] and generalized in [9], [10]. Instead of a single record for the output (Y_k) of the system (1), N_s records

$$\underbrace{\begin{pmatrix} Y_k^{(1,\text{ref})} \\ Y_k^{(1,\text{mov})} \end{pmatrix}}_{\text{Record 1}} \quad \underbrace{\begin{pmatrix} Y_k^{(2,\text{ref})} \\ Y_k^{(2,\text{mov})} \end{pmatrix}}_{\text{Record 2}} \quad \cdots \quad \underbrace{\begin{pmatrix} Y_k^{(N_s,\text{ref})} \\ Y_k^{(N_s,\text{mov})} \end{pmatrix}}_{\text{Record } N_s} \tag{8}$$

are now available collected successively. Each record j contains data $Y_k^{(j,\text{ref})}$ of dimension $r^{(\text{ref})}$ from a fixed *reference* sensor pool, and data $Y_k^{(j,\text{mov})}$ of dimension $r^{(j)}$ from a *moving* sensor pool. To each record $j = 1, \dots, N_s$ corresponds a state-space realization in the form [7], [8]

$$\begin{cases} X_{k+1}^{(j)} &= A\ X_k^{(j)} + V_{k+1}^{(j)} \\ Y_k^{(j,\text{ref})} &= C^{(\text{ref})}\ X_k^{(j)} \qquad \text{(reference pool)} \\ Y_k^{(j,\text{mov})} &= C^{(j,\text{mov})}\ X_k^{(j)} \qquad \text{(sensor pool n}^o j\text{)} \end{cases} \tag{9}$$

with a single state transition matrix A, since the same system is being observed. The observation matrix $C^{(\text{ref})}$ with respect to the reference sensors is independent of the measurement setup as the reference sensors are fixed throughout the measurements, while the observation matrices $C^{(j,\text{mov})}$ correspond to the moving sensor pool of each setup j.

In [10] an algorithm is derived that constructs a global observability matrix $\mathcal{O}^{(\text{all})} = \mathcal{O}(C^{(\text{all})}, A)$ (cf. (3)) from all the records (8), where the global observation matrix containing information of all sensor positions is defined as

$$C^{(\text{all})} \stackrel{\text{def}}{=} \begin{bmatrix} C^{(\text{ref})T} & C^{(1,\text{mov})T} & \dots & C^{(N_s,\text{mov})T} \end{bmatrix}^T.$$

From $\mathcal{O}^{(\text{all})}$, the eigenstructure of system (9) can then be identified in one run using the steps (6)-(7) of the general subspace identification algorithm from Section II-A.

The global observability matrix $\mathcal{O}^{(\text{all})}$ is constructed as follows [10]:

(a) For each setup j, the subspace matrix $\mathcal{H}^{(j)}$ is built according to the chosen subspace algorithm using data $Y_k^{(j,\text{ref})}$ and $Y_k^{(j,\text{mov})}$, such that $\mathcal{H}^{(j)}$ fulfills factorization property (2) with observability matrix $\mathcal{O}(\begin{bmatrix}C^{(\text{ref})T} & C^{(j,\text{mov})T}\end{bmatrix}^T, A)$.

(b) Obtain $\mathcal{O}^{(j)}$ from an SVD of $\mathcal{H}^{(j)}$ and truncation at the desired model order n as in (4)-(5).

(c) Separate $\mathcal{O}^{(j)}$ into $\mathcal{O}^{(j,\text{ref})}$ and $\mathcal{O}^{(j,\text{mov})}$ by choosing the appropriate block rows, where $\mathcal{O}^{(j,\text{ref})} = \mathcal{O}(C^{(\text{ref})}, A)T_j$ and $\mathcal{O}^{(j,\text{mov})} = \mathcal{O}(C^{(j,\text{mov})}, A)T_j$ with an unknown change of basis matrix T_j. Note that

$$\mathcal{O}^{(j)} = P_j \begin{bmatrix}\mathcal{O}^{(j,\text{ref})} \\ \mathcal{O}^{(j,\text{mov})}\end{bmatrix},$$

where P_j is an appropriate permutation matrix.

(d) Set $\mathcal{O}^{(\text{ref})} \stackrel{\text{def}}{=} \mathcal{O}^{(1,\text{ref})}$ and compute the observability matrix parts

$$\overline{\mathcal{O}}^{(j,\text{mov})} \stackrel{\text{def}}{=} \mathcal{O}^{(j,\text{mov})}\mathcal{O}^{(j,\text{ref})\dagger}\mathcal{O}^{(\text{ref})},$$

which are in the same modal basis.

(e) Interleave the block rows of the matrices $\mathcal{O}^{(\text{ref})}$ and $\overline{\mathcal{O}}^{(j,\text{mov})}$, $j = 1, \ldots, N_s$, to obtain the global observability matrix

$$\mathcal{O}^{(\text{all})} = P \begin{bmatrix}\mathcal{O}^{(\text{ref})} \\ \overline{\mathcal{O}}^{(1,\text{mov})} \\ \vdots \\ \overline{\mathcal{O}}^{(N_s,\text{mov})}\end{bmatrix}, \tag{10}$$

where P is an appropriate permutation matrix.

Remark 3: Due to factorization property (2) and the restriction to subspace algorithms without a left weighting, step (d) of the merging algorithm is equivalent to

$$\overline{\mathcal{O}}^{(j,\text{mov})} \stackrel{\text{def}}{=} \mathcal{H}^{(j,\text{mov})}\mathcal{H}^{(j,\text{ref})\dagger}\mathcal{O}^{(\text{ref})}, \tag{11}$$

where $\mathcal{H}^{(j,\text{mov})}$ and $\mathcal{H}^{(j,\text{ref})}$ are defined by

$$\mathcal{H}^{(j)} = P_j \begin{bmatrix}\mathcal{H}^{(j,\text{ref})} \\ \mathcal{H}^{(j,\text{mov})}\end{bmatrix}, \tag{12}$$

analogously to $\mathcal{O}^{(j,\text{mov})}$ and $\mathcal{O}^{(j,\text{ref})}$. □

From $\mathcal{O}^{(\text{all})}$, the global observation matrix $C^{(\text{all})}$ is recovered as the first block row. The state transition matrix A is the least squares solution of $\mathcal{O}^{(\text{all})\uparrow}A = \mathcal{O}^{(\text{all})\downarrow}$. With (10) and using Definition 2, this least squares solution can be expressed as

$$\begin{aligned} A &= \left(\mathcal{O}^{(\text{all})\uparrow T}\mathcal{O}^{(\text{all})\uparrow}\right)^{-1}\mathcal{O}^{(\text{all})\uparrow T}\mathcal{O}^{(\text{all})\downarrow} \\ &= \left(\mathcal{O}^{(\text{ref})\uparrow T}\mathcal{O}^{(\text{ref})\uparrow} + \sum_{j=1}^{N_s}\overline{\mathcal{O}}^{(j,\text{mov})\uparrow T}\overline{\mathcal{O}}^{(j,\text{mov})\uparrow}\right)^{-1} \\ &\quad \cdot \left(\mathcal{O}^{(\text{ref})\uparrow T}\mathcal{O}^{(\text{ref})\downarrow} + \sum_{j=1}^{N_s}\overline{\mathcal{O}}^{(j,\text{mov})\uparrow T}\overline{\mathcal{O}}^{(j,\text{mov})\downarrow}\right). \end{aligned} \tag{13}$$

III. Uncertainty Bounds of Modal Parameters in Stochastic Subspace Identification

A. Covariances of Modal Parameters from Single-Setup SSI

Consider the Stochastic Subspace Identification from Section II-A, where the modal parameters (natural frequencies f_λ, damping ratios d_λ and mode shapes φ_λ) are obtained from output-only data of one measurement setup. The statistical uncertainty of the obtained modal parameters at a chosen system order can be computed from the uncertainty of the system matrices, which depends on the uncertainty of the corresponding subspace matrix $\mathcal{H}$. For any function $y = f(\mathcal{H})$ and a small perturbation $\Delta\mathcal{H}$, the uncertainty on $\mathcal{H}$ is propagated by $\Delta y \approx J_f \Delta\mathcal{H}$, where J_f is the sensitivity of function f. The uncertainty of $\mathcal{H}$ can be evaluated by cutting the sensor data into blocks on which instances of the subspace matrix are computed. Thus, this offers a possibility to compute the uncertainty bounds of the modal parameters at a certain system order without repeating the system identification. In [11], this algorithm was described in detail for the covariance-driven SSI. The uncertainty ΔA and ΔC of the system matrices A and C are connected to the uncertainty of the subspace matrix through a Jacobian matrix $J_{A,C}$ by

$$\begin{bmatrix}\operatorname{vec}\Delta A \\ \operatorname{vec}\Delta C\end{bmatrix} = J_{A,C}\ \operatorname{vec}\Delta\mathcal{H}, \tag{14}$$

where vec is the vectorization operator. Hence, the covariance of the vectorized system matrices can be expressed as

$$\operatorname{cov}_{A,C} \stackrel{\text{def}}{=} \operatorname{cov}\left(\begin{bmatrix}\operatorname{vec} A \\ \operatorname{vec} C\end{bmatrix}\right) = J_{A,C}\operatorname{cov}_{\mathcal{H}} J_{A,C}^T, \tag{15}$$

where $\operatorname{cov}_{\mathcal{H}} \stackrel{\text{def}}{=} \operatorname{cov}(\operatorname{vec}\mathcal{H})$ is the covariance of the vectorized subspace matrix. Note that $\operatorname{cov}_{\mathcal{H}}$ depends on the selected subspace method. For covariance-driven SSI, it is stated in [11] and for data-driven SSI with the UPC algorithm, it is derived in [13].

As the modal parameters are functions of the system matrices A and C, their uncertainty yields

$$\Delta f_\lambda = J_{f_\lambda}\begin{bmatrix}\operatorname{vec}\Delta A \\ \operatorname{vec}\Delta C\end{bmatrix}, \quad \Delta d_\lambda = J_{d_\lambda}\begin{bmatrix}\operatorname{vec}\Delta A \\ \operatorname{vec}\Delta C\end{bmatrix},$$
$$\Delta\varphi_\lambda = J_{\varphi_\lambda}\begin{bmatrix}\operatorname{vec}\Delta A \\ \operatorname{vec}\Delta C\end{bmatrix},$$

where J_{f_λ}, J_{d_λ} and J_{φ_λ} are the respective Jacobians [11] that are computed for each mode λ. Finally, the covariances of the modal parameters are obtained as

$$\begin{aligned} \operatorname{cov}(f_\lambda) = J_{f_\lambda}\operatorname{cov}_{A,C}J_{f_\lambda}^T,\ \operatorname{cov}(d_\lambda) = J_{d_\lambda}\operatorname{cov}_{A,C}J_{d_\lambda}^T, \\ \operatorname{cov}(\varphi_\lambda) = J_{\varphi_\lambda}\operatorname{cov}_{A,C}J_{\varphi_\lambda}^T. \end{aligned} \tag{16}$$

B. Covariances of Modal Parameters from Multi-Setup SSI

As the system matrices A and C are obtained differently for multi-setup measurements in Section II-B than for a single measurement in Section II-A, their covariance computation has to be adapted. Equations (14) and (15) do not hold anymore and $\operatorname{cov}_{A,C}$ needs to be derived for multi-setup

measurements. Then, the covariance of the modal parameters is obtained from (16).

For evaluating the uncertainties of system matrices A and C from the multi-setup SSI, the uncertainties of $\mathcal{O}^{(\text{ref})}$ and $\overline{\mathcal{O}}^{(j,\text{mov})}$ are required, as A and C depend on these matrices.

Lemma 4: The uncertainties of $\mathcal{O}^{(\text{ref})}$ and $\overline{\mathcal{O}}^{(j,\text{mov})}$, $j = 1,\dots,N_s$, with respect to small perturbations in $\mathcal{H}^{(j)}$ are

$$\operatorname{vec}\Delta\mathcal{O}^{(\text{ref})} = J_{\mathcal{O}^{(\text{ref})},\mathcal{H}^{(\text{ref})}} \operatorname{vec}\Delta\mathcal{H}^{(1)}, \tag{17}$$

$$\operatorname{vec}\Delta\overline{\mathcal{O}}^{(j,\text{mov})} = J_{\overline{\mathcal{O}}^{j},\mathcal{H}^{(\text{ref})}} \operatorname{vec}\Delta\mathcal{H}^{(1)} + J_{\overline{\mathcal{O}}^{j},\mathcal{H}^{j}} \operatorname{vec}\Delta\mathcal{H}^{(j)},$$

where the Jacobians $J_{\mathcal{O}^{(\text{ref})},\mathcal{H}^{(\text{ref})}}$, $J_{\overline{\mathcal{O}}^{j},\mathcal{H}^{(\text{ref})}}$ and $J_{\overline{\mathcal{O}}^{j},\mathcal{H}^{j}}$ are defined in Equations (18), (23) and (24), respectively.

Proof: See Appendix. ∎

From the uncertainties of $\mathcal{O}^{(\text{ref})}$ and $\overline{\mathcal{O}}^{(j,\text{mov})}$, $j = 1,\dots,N_s$, the uncertainties of A and C are derived in the following lemma, using (13) and (10).

Lemma 5: The uncertainties of A and C with respect to small perturbations in $\mathcal{O}^{(\text{ref})}$ and $\overline{\mathcal{O}}^{(j,\text{mov})}$ write as

$$\operatorname{vec}\Delta A = J_{A,\mathcal{O}^{(\text{ref})}} \operatorname{vec}\Delta\mathcal{O}^{(\text{ref})} + \sum_{j=1}^{N_s} J_{A,\overline{\mathcal{O}}^{j}} \operatorname{vec}\Delta\overline{\mathcal{O}}^{(j,\text{mov})},$$

$$\operatorname{vec}\Delta C = J_{C,\mathcal{O}^{(\text{ref})}} \operatorname{vec}\Delta\mathcal{O}^{(\text{ref})} + \sum_{j=1}^{N_s} J_{C,\overline{\mathcal{O}}^{j}} \operatorname{vec}\Delta\overline{\mathcal{O}}^{(j,\text{mov})},$$

where the Jacobians $J_{A,\mathcal{O}^{(\text{ref})}}$, $J_{A,\overline{\mathcal{O}}^{j}}$, $J_{C,\mathcal{O}^{(\text{ref})}}$ and $J_{C,\overline{\mathcal{O}}^{j}}$ are defined in Equations (28), (29) and (30), respectively.

Proof: See Appendix. ∎

From Lemma 4 and 5 the computation of the covariances of the system matrices follows finally in the following proposition.

Proposition 6: The covariance of the system matrices obtained from multi-setup SSI writes as

$$\operatorname{cov}_{A,C} = \sum_{j=1}^{N_s} J_{AC,j} \operatorname{cov}_{\mathcal{H}^{(j)}} J_{AC,j}^{T},$$

where

$$J_{AC,1} = \begin{bmatrix} J_{A,\mathcal{O}^{(\text{ref})}} \\ J_{C,\mathcal{O}^{(\text{ref})}} \end{bmatrix} J_{\mathcal{O}^{(\text{ref})},\mathcal{H}^{(\text{ref})}} + \sum_{j=1}^{N_s} \begin{bmatrix} J_{A,\overline{\mathcal{O}}^{j}} \\ J_{C,\overline{\mathcal{O}}^{j}} \end{bmatrix} J_{\overline{\mathcal{O}}^{j},\mathcal{H}^{(\text{ref})}},$$

$$J_{AC,j} = \begin{bmatrix} J_{A,\overline{\mathcal{O}}^{j}} \\ J_{C,\overline{\mathcal{O}}^{j}} \end{bmatrix} J_{\overline{\mathcal{O}}^{j},\mathcal{H}^{j}}, \qquad j \geq 2,$$

and $\operatorname{cov}_{\mathcal{H}^{(j)}} = \operatorname{cov}(\operatorname{vec}\mathcal{H}^{(j)})$, $j = 1,\dots,N_s$, are the covariances of the local subspace matrices according to the selected SSI algorithm.

Proof: Plugging the results of Lemma 4 into Lemma 5, the uncertainties of the system matrices can be expressed by the uncertainties of the local subspace matrices and it holds

$$\begin{bmatrix} \operatorname{vec}\Delta A \\ \operatorname{vec}\Delta C \end{bmatrix} = \sum_{j=1}^{N_s} J_{AC,j} \operatorname{vec}\Delta\mathcal{H}^{(j)}.$$

As the data records from different measurement setups are collected at different times, we can assume that they are uncorrelated. Hence, the local subspace matrices $\mathcal{H}^{(j)}$ are statistically independent and it holds $\operatorname{cov}(\mathcal{H}^{(j_1)},\mathcal{H}^{(j_2)}) = 0$ for $j_1 \neq j_2$. Thus, the assertion follows. ∎

Note that in Proposition 6, the size of the involved covariance matrices is reduced considerably by assuming statistical independence of the data from different setups, as only the matrices $\operatorname{cov}_{\mathcal{H}^{(j)}}$ are needed.

Using Proposition 6, the covariance and hence the uncertainty bounds of the modal parameters can be computed as stated in (16).

IV. Numerical Results

The paper presents multi-setup system identification results and their uncertainty bounds on a multilayer E-glass reinforced composite panel [14], which is similar to the load carrying laminate in a wind turbine blade. The nominal dimension are $20 \times 320 \times 320$ mm. Vibration measurements of the composite panel were taken in three setups with 14 moving sensors and one setup with 7 moving sensors, while one reference sensor stayed fixed throughout all the measurements.

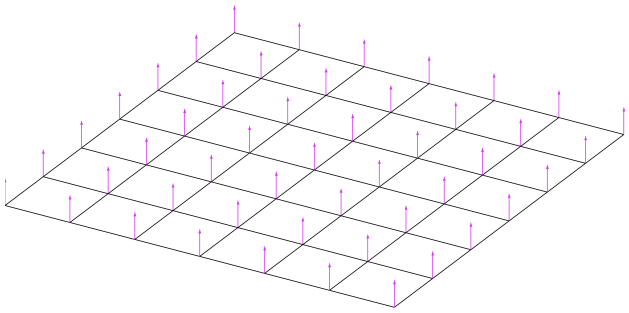

Fig. 1. Schematic view of the investigated composite panel.

For the construction of the local subspace matrices for multi-setup system identification in Section II-B, the parameters $p+1 = q = 40$ and the model order $n = 40$ were used. The covariance-driven subspace identification algorithm was used. A summary of the obtained natural frequencies and damping ratios from the multi-setup identification is given in Table I, together with their uncertainty bounds obtained from the algorithm described in Section III-B. Note that in Table I relative standard deviations (standard deviation of the value divided by this value) are presented, which are obtained from the square root of the estimated covariance from (16).

Uncertainty bounds of the frequencies are much smaller than those of damping ratios. This is coherent with statistical theory, since the lower bound of the covariance given by Fisher information matrix is smaller for the frequencies than for the damping ratios [15].

V. Conclusions

In this paper, a memory efficient algorithm for the uncertainty quantification of modal parameters, which are obtained from the multi-setup subspace identification algorithm presented in [9], [10], has been derived. It has been shown that

TABLE I

IDENTIFIED FREQUENCIES f AND DAMPING RATIOS d WITH THEIR RELATIVE STANDARD DEVIATIONS.

Mode	f (Hz)	$\sigma_f/f \cdot 100$ (%)	d (%)	$\sigma_d/d \cdot 100$ (%)
1	358.1	0.40	2.12	9.0
2	551.9	0.15	2.56	6.8
3	787.5	0.36	3.64	16
4	923.4	0.21	2.42	8.5
5	1096	0.09	2.20	4.6
6	1262	0.86	3.50	20
7	1508	0.11	2.45	3.6
8	1855	0.43	2.74	27
9	1928	0.45	2.67	31

the uncertainty on modal parameters is a weighted sum of the uncertainty of all local subspace matrices for each setup and then can be computed efficiently and iteratively. The method was successfully applied and tested on the ambient vibration data of a composite panel.

ACKNOWLEDGMENTS

We thank M. Luczak and B. Peeters from LMS International for providing us with the data of the composite panel.

APPENDIX

PROOFS OF SECTION III-B

Before actually proofing Lemma 4 and Lemma 5, results from [16] are presented on the uncertainty propagation to singular values and vectors.

Definition 7: For $a, b \in \mathbb{N}$ define the permutation

$$P_{a,b} = \sum_{k=1}^{a}\sum_{l=1}^{b} E_{k,l}^{a,b} \otimes E_{l,k}^{b,a},$$

where $E_{k,l}^{a,b}$ is a matrix of size $a \times b$ that is equal to 1 at position (k,l) and zero elsewhere. For any matrix $X \in \mathbb{R}^{a,b}$ it has the property [16]

$$\operatorname{vec} X^T = P_{a,b}\operatorname{vec} X.$$

Lemma 8 ([16]): Let σ_i, u_i and v_i be the ith singular value, left and right singular vector of some matrix $X \in \mathbb{R}^{a,b}$ and ΔX a small perturbation on X. Then,

$$\Delta\sigma_i = (v_i \otimes u_i)^T \operatorname{vec}\Delta X, \quad \begin{bmatrix}\Delta u_i \\ \Delta v_i\end{bmatrix} = B_i^{\dagger} C_i \operatorname{vec}\Delta X,$$

where

$$B_i \stackrel{\text{def}}{=} \begin{bmatrix}\sigma_i I_a & -X \\ -X^T & \sigma_i I_b\end{bmatrix}, \; C_i \stackrel{\text{def}}{=} \begin{bmatrix} v_i^T \otimes (I_a - u_i u_i^T) \\ (u_i^T \otimes (I_b - v_i v_i^T))P_{a,b}\end{bmatrix},$$

$\otimes$ denotes the Kronecker product, I_n is identity matrix of size $n \times n$ and $P_{a,b}$ is defined in Definition 7.

Proof of Lemma 4

Using Lemma 8 and following the lines of [11], the uncertainty of $\mathcal{O}^{(\text{ref})}$ is obtained as follows. Let $\mathcal{O}^{(\text{ref})}$ be computed from $\mathcal{H}^{(1)}$ as described in steps (a)-(d) in Section II-B, i.e.

$$\mathcal{H}^{(1)} = \begin{bmatrix}U_1 & U_0\end{bmatrix}\begin{bmatrix}\Sigma_1 & 0 \\ 0 & \Sigma_0\end{bmatrix}\begin{bmatrix}V_1^T \\ V_0^T\end{bmatrix}, \; \mathcal{O}^{(\text{ref})} = S_0 U_1 \Sigma_1^{1/2},$$

where S_0 is an appropriate selection matrix. Then, for a small perturbation $\Delta\mathcal{H}^{(1)}$ equation (17) holds, where

$$J_{\mathcal{O}^{(\text{ref})},\mathcal{H}^{(\text{ref})}} = \mathcal{B} + \mathcal{C} \tag{18}$$

and [11]

$$\mathcal{B} \stackrel{\text{def}}{=} \left(I_n \otimes \left(\frac{1}{2} S_0 U_1 \Sigma_1^{-1/2}\right)\right) S_1 \begin{bmatrix}(v_1 \otimes u_1)^T \\ \vdots \\ (v_n \otimes u_n)^T\end{bmatrix},$$

$$\mathcal{C} \stackrel{\text{def}}{=} (\Sigma_1^{1/2} \otimes S_0 \begin{bmatrix}I_{n_1} & 0_{n_1,n_2}\end{bmatrix}) \begin{bmatrix}B_1^{\dagger} C_1 \\ \vdots \\ B_n^{\dagger} C_n\end{bmatrix}$$

and in which $S_1 = \sum_{k=1}^{n} E_{(k-1)n+k,k}^{nn,n}$ is a selection matrix and n_1 and n_2 are the number of rows and columns of $\mathcal{H}^{(1)}$.

For the second part of Lemma 4, choose any $j \in \{1, \ldots, N_s\}$ and let the dimensions of $\mathcal{H}^{(j,\text{ref})}$ and $\mathcal{H}^{(j,\text{mov})}$ be $n_1 \times n_3$ and $n_2 \times n_3$, respectively. From (11) follows

$$\begin{aligned}\Delta\overline{\mathcal{O}}^{(j,\text{mov})} &= \Delta(\mathcal{H}^{(j,\text{mov})})\mathcal{H}^{(j,\text{ref})\dagger}\mathcal{O}^{(\text{ref})} \\ &+ \mathcal{H}^{(j,\text{mov})}\Delta(\mathcal{H}^{(j,\text{ref})\dagger})\mathcal{O}^{(\text{ref})} \\ &+ \mathcal{H}^{(j,\text{mov})}\mathcal{H}^{(j,\text{ref})\dagger}\Delta(\mathcal{O}^{(\text{ref})})\end{aligned}$$

and in vectorized form

$$\begin{aligned}\operatorname{vec}\Delta\overline{\mathcal{O}}^{(j,\text{mov})} &= (\mathcal{H}^{(j,\text{ref})\dagger}\mathcal{O}^{(\text{ref})} \otimes I_{n_2})^T \operatorname{vec}\Delta\mathcal{H}^{(j,\text{mov})} \\ &+ (\mathcal{O}^{(\text{ref})T} \otimes \mathcal{H}^{(j,\text{mov})})\operatorname{vec}\Delta\mathcal{H}^{(j,\text{ref})\dagger} \\ &+ (\mathcal{H}^{(j,\text{mov})}\mathcal{H}^{(j,\text{ref})\dagger} \otimes I_n)\operatorname{vec}\Delta\mathcal{O}^{(\text{ref})}. \end{aligned} \tag{19}$$

Let derive the required uncertainties in this equation, starting with $\Delta\mathcal{H}^{(j,\text{ref})\dagger}$. The pseudoinverse is defined with the SVD decomposition

$$\mathcal{H}^{(j,\text{ref})} = \begin{bmatrix}U_1 & U_0\end{bmatrix}\begin{bmatrix}\Sigma_1 & 0 \\ 0 & \Sigma_0\end{bmatrix}\begin{bmatrix}V_1^T \\ V_0^T\end{bmatrix}$$

by $\mathcal{H}^{(j,\text{ref})\dagger} = V_1\Sigma_1^{-1}U_1^T$, where Σ_1 is of size $n \times n$. Hence,

$$\begin{aligned}\Delta\mathcal{H}^{(j,\text{ref})\dagger} &= \Delta(V_1)\Sigma_1^{-1}U_1^T - V_1\Sigma_1^{-1}\Delta(\Sigma_1)\Sigma_1^{-1}U_1^T \\ &+ V_1\Sigma_1^{-1}\Delta(U_1^T).\end{aligned}$$

With Lemma 8 follows

$$\Delta\operatorname{vec}\mathcal{H}^{(j,\text{ref})\dagger} = J_{\mathcal{H}^{(j,\text{ref})\dagger}}\Delta\operatorname{vec}\mathcal{H}^{(j,\text{ref})}, \tag{20}$$

where

$$\begin{aligned}J_{\mathcal{H}^{(j,\text{ref})\dagger}} &= \mathcal{B} + (U_1\Sigma_1^{-1} \otimes \begin{bmatrix}0_{n_3,n_1} & I_{n_3}\end{bmatrix})\mathcal{C} \\ &+ (I_{n_1} \otimes V_1\Sigma_1^{-1})P_{n_1,n}(I_n \otimes \begin{bmatrix}I_{n_1} & 0_{n_1,n_3}\end{bmatrix})\mathcal{C}\end{aligned}$$

and

$$\mathcal{B} \stackrel{\text{def}}{=} \sum_{i=1}^{n}\sigma_i^{-2}(u_i v_i^T \otimes v_i u_i^T), \quad \mathcal{C} \stackrel{\text{def}}{=} \begin{bmatrix}B_1^{\dagger}C_1 \\ \vdots \\ B_n^{\dagger}C_n\end{bmatrix}.$$

From (12) follows

$$\operatorname{vec}\Delta\mathcal{H}^{(j,\text{ref})} = S^{(j,\text{ref})}\operatorname{vec}\Delta\mathcal{H}^{(j)}, \tag{21}$$

$$\operatorname{vec}\Delta\mathcal{H}^{(j,\text{mov})} = S^{(j,\text{mov})}\operatorname{vec}\Delta\mathcal{H}^{(j)}, \tag{22}$$

where $S^{(j,\text{ref})}$ and $S^{(j,\text{mov})}$ are appropriate selection matrices. Plugging (17), (20), (21) and (22) into (19), yields the assertion with

$$J_{\overline{\mathcal{O}}^j,\mathcal{H}^{(\text{ref})}} = (\mathcal{H}^{(j,\text{mov})}\mathcal{H}^{(j,\text{ref})\dagger} \otimes I_n) J_{\mathcal{O}^{(\text{ref})}}, \tag{23}$$

$$\begin{aligned} J_{\overline{\mathcal{O}}^j,\mathcal{H}^j} = {} & (\mathcal{H}^{(j,\text{ref})\dagger}\mathcal{O}^{(\text{ref})} \otimes I_{n_2})^T S^{(j,\text{mov})} \\ & + (\mathcal{O}^{(\text{ref})T} \otimes \mathcal{H}^{(j,\text{mov})}) J_{\mathcal{H}^{(j,\text{ref})\dagger}} S^{(j,\text{ref})} \end{aligned} \tag{24}$$

□

Proof of Lemma 5

From (13) follows $A = K^{-1}L$, where

$$K = \mathcal{O}^{(\text{ref})\uparrow T}\mathcal{O}^{(\text{ref})\uparrow} + \sum_{j=1}^{N_s} \overline{\mathcal{O}}^{(j,\text{mov})\uparrow T}\overline{\mathcal{O}}^{(j,\text{mov})\uparrow}, \tag{25}$$

$$L = \mathcal{O}^{(\text{ref})\uparrow T}\mathcal{O}^{(\text{ref})\downarrow} + \sum_{j=1}^{N_s} \overline{\mathcal{O}}^{(j,\text{mov})\uparrow T}\overline{\mathcal{O}}^{(j,\text{mov})\downarrow}. \tag{26}$$

Hence, $\Delta A = -K^{-1}\Delta K K^{-1}L + K^{-1}\Delta L = -K^{-1}\Delta K A + K^{-1}\Delta L$ and it follows

$$\text{vec}\,\Delta A = -(A^T \otimes K^{-1})\text{vec}\,\Delta K + (I \otimes K^{-1})\text{vec}\,\Delta L. \tag{27}$$

Let $\mathcal{O} \in \mathbb{R}^{(p+1)r \times n}$ be a placeholder for $\mathcal{O}^{(\text{ref})}$ or $\overline{\mathcal{O}}^{(j,\text{mov})}$, where $r = r^{(\text{ref})}$ or $r = r^{(j)}$, respectively. Let furthermore $S_\uparrow$ and $S_\downarrow$ be selection matrices, such that $\mathcal{O}^\uparrow = S_\uparrow\mathcal{O}$ and $\mathcal{O}^\downarrow = S_\downarrow\mathcal{O}$. Then,

$$\Delta(\mathcal{O}^{\uparrow T}\mathcal{O}^\uparrow) = \Delta(\mathcal{O}^T)S_\uparrow^T\mathcal{O}^\uparrow + \mathcal{O}^{\uparrow T}S_\uparrow\Delta(\mathcal{O}).$$

and after vectorization and using Definition 7

$$\begin{aligned} \text{vec}\,\Delta(\mathcal{O}^{\uparrow T}\mathcal{O}^\uparrow) = {} & \left((\mathcal{O}^{\uparrow T}S_\uparrow \otimes I_n)P_{pr,n} + (I_n \otimes \mathcal{O}^{\uparrow T}S_\uparrow)\right) \\ & \cdot\text{vec}\,\Delta\mathcal{O}. \end{aligned}$$

Analogously, it holds

$$\begin{aligned} \text{vec}\,\Delta(\mathcal{O}^{\uparrow T}\mathcal{O}^\downarrow) = {} & \left((\mathcal{O}^{\downarrow T}S_\uparrow \otimes I_n)P_{pr,n} + (I_n \otimes \mathcal{O}^{\uparrow T}S_\downarrow)\right) \\ & \cdot\text{vec}\,\Delta\mathcal{O}. \end{aligned}$$

Then, from (25), (26) and (27) the assertion follows for ΔA, where

$$\begin{aligned} J_{A,\mathcal{O}^{(\text{ref})}} = {} & -(A^T \otimes K^{-1})\Big((\mathcal{O}^{(\text{ref})\uparrow T}S_{(\text{ref})\uparrow} \otimes I_n)P_{pr^{(\text{ref})},n} \\ & + (I_n \otimes \mathcal{O}^{(\text{ref})\uparrow T}S_{(\text{ref})\uparrow})\Big) \\ & + (I_n \otimes K^{-1})\Big((\mathcal{O}^{(\text{ref})\downarrow T}S_{(\text{ref})\uparrow} \otimes I_n)P_{pr^{(\text{ref})},n} \\ & + (I_n \otimes \mathcal{O}^{(\text{ref})\uparrow T}S_{(\text{ref})\downarrow})\Big), \end{aligned} \tag{28}$$

$$\begin{aligned} J_{A,\overline{\mathcal{O}}^j} = {} & -(A^T \otimes K^{-1})\Big((\overline{\mathcal{O}}^{(j,\text{mov})\uparrow T}S_{j\uparrow} \otimes I_n)P_{pr^{(j)},n} \\ & + (I_n \otimes \overline{\mathcal{O}}^{(j,\text{mov})\uparrow T}S_{j\uparrow})\Big) \\ & + (I_n \otimes K^{-1})\Big((\overline{\mathcal{O}}^{(j,\text{mov})\downarrow T}S_{j\uparrow} \otimes I_n)P_{pr^{(j)},n} \\ & + (I_n \otimes \overline{\mathcal{O}}^{(j,\text{mov})\uparrow T}S_{j\downarrow})\Big). \end{aligned} \tag{29}$$

It remains the uncertainty of C. As C is the first block row of (10), its uncertainty can be written as

$$\Delta C = S_C \begin{bmatrix} \Delta\mathcal{O}^{(\text{ref})} \\ \Delta\overline{\mathcal{O}}^{(1,\text{mov})} \\ \vdots \\ \Delta\overline{\mathcal{O}}^{(N_s,\text{mov})} \end{bmatrix}$$

with an appropriate selection matrix S_C. Partition $S_C = \begin{bmatrix} S_{C,(\text{ref})} & S_{C,1} & \dots & S_{C,N_s} \end{bmatrix}$ such that

$$\Delta C = S_{C,(\text{ref})}\Delta\mathcal{O}^{(\text{ref})} + \sum_{j=1}^{N_s} S_{C,j}\Delta\overline{\mathcal{O}}^{(j,\text{mov})}.$$

Vectorizing this equation and setting

$$J_{C,\mathcal{O}^{(\text{ref})}} = I_n \otimes S_{C,(\text{ref})},\ J_{C,\overline{\mathcal{O}}^j} = I_n \otimes S_{C,j} \tag{30}$$

leads to the assertion. □

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