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# Optimal Control of Linear Systems with Stochastic Parameters for Variance Suppression

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*Abstract*— **In this paper, we consider an optimal control problem for a linear discrete time system with stochastic parameters in the infinite time horizon case. This paper focuses on optimal control for systems with stochastic parameters whereas the traditional stochastic optimal control theory mainly considers systems with deterministic parameters with stochastic noises. This paper extends the authors' former result on the same subject in the finite time horizon case to the infinite time horizon case. The main result is to provide a feedback controller suppressing the variation of the state and to prove stochastic stability of the corresponding feedback system by taking care of both the average and the variance of the state transient. Furthermore, a numerical simulations demonstrate the effectiveness of the proposed method.**

## I. Introduction

Stochastic and statistical methods are used in several problems in control systems theory. In particular, system identification and estimation methods rely on them. Recently, it is reported that Bayesian inference and the related estimation tools in machine learning [1], [2] are applied to system identification of state-space models [3], [4], [5], [6]. Consequently, statistical information of the system parameters become available for control. There also exists a result reporting that the quality of the manufactured products are estimated by those methods in process control [7]. So called randomized approach is also proposed to apply statistical tools to controller design problems [8], [9].

On the other hand, optimal control is an important and established control method. LQG (Linear Quadratic Gaussian) method can take care of optimal control problems with stochastic disturbances [10], [11]. Stochastic control theory has been developed by many authors based on it which mainly focuses on systems with deterministic parameters and stochastic noises. In stochastic control theory, MCV (Minimum Cost Variance) control [12], [13] and RS (Risk Sensitive) control [14] are proposed which can suppress the variance of the cost function with respect to the stochastic disturbance. There is a paper comparing those method as well [15].

Those existing results focus on the systems with deterministic parameters with stochastic disturbances. In order to utilize the Bayesian estimation results for state space models, controller design methods for state space systems with stochastic parameters are needed. For this problem, De Koning proposed a controller design method for state space systems with stochastic system parameters [16], [17], [18] which employs standard LQG type cost function. The results in the paper, however, cannot suppress the variance of the state transient of the resulting control system caused by the variation of the system parameters.

The authors proposed a generalized version of [16] to take the variation of the state into account by adopting a novel cost function including the covariance of the states [19]. But the result in [19] only considers an optimal control problem in the finite time horizon case. The present paper extends the authors' former result to the infinite time horizon case. The optimal state feedback controller suppressing the variation of the transient is derived. Furthermore, stability of the corresponding feedback system is proved using the analysis tools for stochastic systems provided by De Koning [16]. Furthermore, numerical simulations demonstrate the effectiveness of the proposed method. The proposed method can provide a new framework to stochastic control which can be used together with Bayesian inference of the system parameters.

## II. Preliminaries

This section gives notations and some preliminary results according to [1], [16].

- The symbol $\mathbb{N}$ denotes the space of natural numbers. $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. $\mathbb{M}^{mn}$ and $\mathbb{M}^n$ denote the space of $m \times n$ real valued rectangular matrices and that of $n \times n$ real valued square matrices, respectively. $\mathbb{S}^n$ denotes the space of $n \times n$ real valued symmetric matrices.
- The symbol $\mathrm{vec}(\cdot)$ denotes a function satisfying
$$\mathrm{vec}(A) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^{mn}, \quad a_i := \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} \in \mathbb{R}^m.$$
where $a_{ij}$ is the $(i,j)$ element of the matrix $A \in \mathbb{M}^{mn}$.
- The expectation of a time-varying stochastic parameter $a_t$ is denoted without the time parameter $t$ as $\mathrm{E}[a]$ instead of $\mathrm{E}[a_t]$, if its statistics is time invariant.

We use several transformations for symmetric matrices. Let us define a *monotonic* transformation.

*Definition 1:* [16] A transformation for symmetric matrices $\mathcal{A} : \mathbb{S}^n \to \mathbb{S}^n$ is said to be monotonic if it satisfies

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$\mathcal{A}X \preceq \mathcal{A}Y$, for any symmetric matrices $X, Y \in \mathbb{S}^n$ satisfying $0 \preceq X \preceq Y$.

The following lemma holds for monotonic transformations.

*Lemma 1:* [16] Consider the following equation.

$$X = \mathcal{A}X + B, \qquad X, B \in \mathbb{S}^n \tag{1}$$

Suppose that the transformation $\mathcal{A}$ is linear, monotonic and stable and that $B \succeq 0$. Then there exists a solution $X \succeq 0$ satisfying (1).

Further, the next lemma holds.

*Lemma 2:* Suppose that the transformation $\mathcal{A}_i : \mathbb{S}^n \to \mathbb{S}^n$ is monotonic and $X \succeq 0 \Rightarrow \mathcal{A}_i X \succeq 0$ holds for any integer $i$. Then a composition of the transformations $\mathcal{A}_i$'s

$$\mathcal{A}^{i:j} := \begin{cases} \mathcal{A}_j \mathcal{A}_{j-1} \cdots \mathcal{A}_i & (i \leq j) \\ \mathrm{id} & (i > j) \end{cases} \tag{2}$$

is monotonic and $X \succeq 0 \Rightarrow \mathcal{A}^{i:j} X \succeq 0$ holds for any integers $i$ and $j$.

*Proof:* Lemma is proved by induction. In the case $j - i \leq 0$, the claim holds obviously. Suppose that it holds when $j - i = k$ for a non-negative integer $k$. Then $\mathcal{A}^{i:j}$ is monotonic. Since $\mathcal{A}_{j+1}$ is also monotonic, we have $X \succeq 0 \Rightarrow \mathcal{A}^{i:j} X \succeq 0 \Rightarrow \mathcal{A}_{j+1}\mathcal{A}^{i:j} X \succeq \mathcal{A}_{j+1} 0 \Rightarrow \mathcal{A}^{i:j+1} X \succeq 0$. Further, the monotonicity of $\mathcal{A}^{i:j}$ and $\mathcal{A}_{j+1}$ implies $0 \preceq X \preceq Y \Rightarrow \mathcal{A}^{i:j} X \preceq \mathcal{A}^{i:j} Y \Rightarrow \mathcal{A}_{j+1}\mathcal{A}^{i:j} X \preceq \mathcal{A}_{j+1}\mathcal{A}^{i:j} Y \Rightarrow 0 \preceq \mathcal{A}^{i:j+1} X \preceq \mathcal{A}^{i:j+1} Y$. Therefore, $\mathcal{A}^{i:j+1}$ is monotonic. This implies that the claim holds for the case $j - i = k + 1$. Hence the claim holds for any integers $i$ and $j$, which proves the lemma. ■

Next, let us consider a discrete time system

$$x_{t+1} = A_t x_t + B_t u_t \tag{3}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the control input, $A_t \in \mathbb{M}^n$ and $B_t \in \mathbb{M}^{nm}$ are the system matrices. Suppose that the system matrices $A_t$ and $B_t$ consist of stochastic variables with time invariant statistics. Assume also that the initial state $x_0$ is deterministic. Apply a state feedback

$$u_t = -L x_t \tag{4}$$

to the system (3) with a feedback gain $L \in \mathbb{M}^{mn}$. Then the resulting feedback system is described by

$$x_{t+1} = \Psi_{L,t} x_t \tag{5}$$

with a new system matrix $\Psi_{L,t} := A_t - B_t L \in \mathbb{M}^n$. Stability of this feedback system is defined as follows.

*Definition 2:* [16] The feedback system (5) is said to be m-stable if $\lim_{t\to\infty} \mathrm{E}[x_t | x_0] = 0$ holds for $\forall x_0 \in \mathbb{R}^n$. It is said to be ms-stable if $\lim_{t\to\infty} \mathrm{E}[\| x_t \|^2 | x_0] = 0$ holds for $\forall x_0 \in \mathbb{R}^n$.

In order to describe the behavior of the state $x_t$ with its variance, let us define a transformation $\mathcal{A}_L : \mathbb{S}^n \to \mathbb{S}^n$ describing the expectation of a quadratic function along the feedback system (5) as follows.

$$\mathcal{A}_L X := \mathrm{E}[\Psi_L^{\mathrm{T}} X \Psi_L], \qquad X \in \mathbb{S}^n \tag{6}$$

The transformation $\mathcal{A}_L$ thus defined satisfies the following lemma.

*Lemma 3:* [16] (a) $\mathrm{E}[x_i^{\mathrm{T}} X x_i | x_0] = x_0^{\mathrm{T}} \mathcal{A}_L^i X x_0$ holds for any natural number $i \in \mathbb{N}$ and any symmetric matrix $X \in \mathbb{S}^n$.
(b) The transformation $\mathcal{A}_L^i$ is linear and monotonic for any natural number $i \in \mathbb{N}$.
(c) The feedback system (5) is m-stable if and only if $\mathrm{E}[\Psi_L]$ is Hurwitz (asymptotically stable). Furthermore, it is ms-stable if and only if the transformation $\mathcal{A}_L$ is stable.

Stabilizability of the system (3) with $(A_t, B_t)$ is defined as follows.

*Definition 3:* [16] The pair $(A_t, B_t)$ is said to be m-stabilizable if there exists a feedback gain $L \in \mathbb{M}^{mn}$ in such a way that the feedback system (5) is m-stable. It is said to be ms-stable if if there exists $L$ such that (5) is ms-stable.

Lemma 3 (c) and Definition 3 imply the following lemma.

*Lemma 4:* [16] The pair $(A_t, B_t)$ is m-stabilizable if and only if there exists a feedback gain $L$ in such a way that $\mathrm{E}[\Psi_L]$ is asymptotically stable. Furthermore, it is ms-stabilizable if and only if there exists $L$ such that $\mathcal{A}_L$ is stable.

## III. Optimal control for variance suppression

This section is devoted to optimal control for variance suppression.

### A. The finite time horizon case

In the authors' former result [19], optimal control for variance suppression in the finite time case is presented. This subsection briefly reviews this result.

Consider a linear discrete-time system with stochastic parameters

$$x_{t+1} = A_t x_t + B_t u_t + G_t \epsilon_t \tag{7}$$

where $\epsilon_t \in \mathbb{R}^n$ is a stochastic external noise. Matrices $A_t \in \mathbb{M}^n$, $B_t \in \mathbb{M}^{nm}$, and $G_t \in \mathbb{M}^n$ are stochastic parameters defined by time invariant statistics. For this system, let us consider the following cost function.

$$J_N(U_N, x_0) = \mathrm{E}\left[ \sum_{t=0}^{N-1} \left\{ x_t^{\mathrm{T}} Q x_t + u_t^{\mathrm{T}} R u_t + \mathrm{tr}(S \,\mathrm{cov}[x_{t+1}|x_t]) \right\} + x_N^{\mathrm{T}} F x_N \middle| x_0 \right] \tag{8}$$

Here the matrices $Q \succ 0 \in \mathbb{S}^n$, $R \succ 0 \in \mathbb{S}^m$, $F \succeq 0 \in \mathbb{S}^n$, and $S \succeq 0 \in \mathbb{S}^n$ are design parameters and $U_N = \{u_t, 0 \leq t \leq N-1\}$ denotes the collection of the input. The third term in the right hand side of Equation (8) reduces to

$$\begin{aligned} &\mathrm{tr}\,(S \,\mathrm{cov}[x_{t+1}|x_t]) \\ &= \mathrm{E}\left[x_{t+1}^{\mathrm{T}} S x_{t+1} | x_t\right] - \mathrm{E}[x_{t+1}|x_t]^{\mathrm{T}} S\, \mathrm{E}[x_{t+1}|x_t] \end{aligned}$$

which is the weighted sum of the covariance of the states. The coefficient matrix $S$ can be used to select the weights between the variation and the average of the state.

For this system, the optimal control problem for variance suppression is defined as follows.

*Definition 4:* Consider the system (7) and the cost function (8). For a given initial condition $x_0 \in \mathbb{R}^n$, find an input sequence $U_N^* = \{u_t^*, 0 \le t \le N-1\}$ minimizing the cost function $J_N(U_N, x_0)$ and the corresponding minimum value of the cost function $J_N^*(x_0)$. We call this control problem as a *finite time optimal control for variance suppression*.

Before solving the variance suppression problem, let us consider the relationship between the value of the cost function at the time $t$ with the state $x_t$ and that at the time $t-1$ with the state $x_{t-1}$. Suppose that a state feedback $u_t = -Lx_t$ is employed, then the term to evaluate the variance of the state $\mathrm{E}[\mathrm{tr}(S\,\mathrm{cov}[x_{t+1}|x_t])|x_0]$ in the cost function can be calculated as $\mathrm{E}[\mathrm{tr}(S\,\mathrm{cov}[x_{t+1}|x_t])|x_0]$

$$\begin{aligned}
&\mathrm{E}\left[\mathrm{tr}(S\,\mathrm{cov}[x_{t+1}|x_t])|x_0\right] \\
&= \mathrm{E}\left[\mathrm{E}[x_{t+1}^{\mathrm{T}} S x_{t+1}|x_t] - \mathrm{E}[x_{t+1}|x_t]^{\mathrm{T}} S\,\mathrm{E}[x_{t+1}|x_t]|x_0\right] \\
&= \mathrm{E}\left[x_t^{\mathrm{T}}\left(\mathrm{E}[\Psi_{L_t}^{\mathrm{T}} S \Psi_{L_t}] - \mathrm{E}[\Psi_{L_t}^{\mathrm{T}}] S\,\mathrm{E}\left[\Psi_{L_t}\right]\right) x_t|x_0\right] \\
&= \mathrm{E}\left[x_t^{\mathrm{T}}\left(\mathcal{A}_{L_t} S - \mathrm{E}[\Psi_{L_t}^{\mathrm{T}}] S\,\mathrm{E}\left[\Psi_{L_t}\right]\right) x_t|x_0\right] \\
&= \mathrm{E}\left[x_{t-1}^{\mathrm{T}} \mathcal{A}_{L_t}\left(\mathcal{A}_{L_t} S - \mathrm{E}[\Psi_{L_t}^{\mathrm{T}}] S\,\mathrm{E}[\Psi_{L_t}]\right) x_{t-1}|x_0\right] \\
&= \mathrm{E}\left[x_0^{\mathrm{T}}\left\{\mathcal{A}_L^{0:t} S - \mathcal{A}_L^{1:t}\left(\mathrm{E}[\Psi_{L_0}^{\mathrm{T}}] S\,\mathrm{E}[\Psi_{L_0}]\right)\right\} x_0|x_0\right] \\
&= x_0^{\mathrm{T}}\left\{\mathcal{A}_L^{0:t} S - \mathcal{A}_L^{1:t}\left(\mathrm{E}[\Psi_{L_0}^{\mathrm{T}}] S\,\mathrm{E}[\Psi_{L_0}]\right)\right\} x_0
\end{aligned} \tag{9}$$

Here we use the notation $\mathcal{A}_L^{i:j} = \mathcal{A}_{L_j}\mathcal{A}_{L_{j-1}}\cdots\mathcal{A}_{L_i}$ as in Equation (2). Now the function $\mathcal{B}_L : \mathbb{S}^n \to \mathbb{S}^n$ is defined by

$$\begin{aligned}
&\mathcal{B}_L X := \mathcal{A}_L X + Q + L^{\mathrm{T}} R L + \mathcal{A}_L S - \mathrm{E}[\Psi_L^{\mathrm{T}}] S\,\mathrm{E}[\Psi_L], \\
&X \in \mathbb{S}^n.
\end{aligned} \tag{10}$$

The function $\beta : \mathbb{S}^n \to \mathbb{R}$ is defined by

$$\beta(X) := \mathrm{E}[\epsilon^{\mathrm{T}} G^{\mathrm{T}} X G \epsilon]. \tag{11}$$

Then the value of the cost function (8) becomes

$$\begin{aligned}
&J_N(U_N, x_0) \\
&\quad= \mathrm{E}\left[\sum_{t=0}^{N-1}\left\{x_t^{\mathrm{T}}(Q + L_t^{\mathrm{T}} R L_t) x_t + \mathrm{tr}(S\,\mathrm{cov}[x_{t+1}|x_t])\right\} + x_N^{\mathrm{T}} F x_N \;\middle|\; x_0\right] \\
&\quad= x_0^{\mathrm{T}}\left\{\sum_{t=0}^{N-1} \mathcal{A}_L^{0:t}\left(Q + L_0^{\mathrm{T}} R L_0 + \mathcal{A}_{L_0} S - \mathrm{E}[\Psi_{L_0}^{\mathrm{T}}] S\,\mathrm{E}[\Psi_{L_0}]\right) + \mathcal{A}_L^{0:N} F\right\} x_0 \\
&\qquad + \beta\left(\sum_{t=0}^{N-1}\left\{\mathcal{A}_L^{1:t} F + \sum_{j=0}^{t-1} \mathcal{A}_L^{1:j}\left(Q + L_0^{\mathrm{T}} R L_0 + \mathcal{A}_{L_0} S - \mathrm{E}[\Psi_{L_0}^{\mathrm{T}}] S\,\mathrm{E}[\Psi_{L_0}]\right)\right\} + NS\right) \\
&\quad= x_0^{\mathrm{T}} \mathcal{B}_L^{0:N-1} F x_0 + \beta\left(NS + \sum_{t=0}^{N-1} \mathcal{B}_L^{1:t} F\right)
\end{aligned} \tag{12}$$

We can prove a property of the transformation $\mathcal{B}_L^{i:j}$ as follows.

*Lemma 5:* The transformation $\mathcal{B}_L^{i:j}$ is monotonic and $X \succeq 0 \Rightarrow \mathcal{B}_L^{i:j} X \succeq 0$ holds for any non-negative integers $i$ and $j$.

*Proof:* Proof is omitted for the limitation of space. ■

Using this lemma, we can prove the main theorem which provides the solution to the optimal control for variance suppression.

*Theorem 1:* [19] The optimal control law to minimize the cost function (8) and the minimum value of the cost function $J_N^*(x_0)$ are given as follows.

$$u_t^* = -L_{\mathcal{B}_*^{N-t-1} F} x_t, \quad t = 0, \ldots, N-1 \tag{13}$$

$$J_N^*(x_0) = x_0^{\mathrm{T}} \mathcal{B}_*^N F x_0 + \beta\left(NS + \sum_{t=0}^{N-1} \mathcal{B}_*^t F\right), \quad \forall x_0 \tag{14}$$

Here the function $\mathcal{B}_* : \mathbb{S}^n \to \mathbb{S}^n$ is defined by

$$\mathcal{B}_* X := \mathcal{B}_{L_X} X, \quad X \in \mathbb{S}^n. \tag{15}$$

The function $\mathcal{B}_{L_X}$ is defined in Equation (10) and the gain matrix $L_X$ is defined by

$$\begin{aligned}
&L_X := (\mathrm{E}[B^{\mathrm{T}} X B] + \Sigma_{BB} + R)^{-1}(\mathrm{E}[B^{\mathrm{T}} X A] + \Sigma_{BA}), \\
&X \in \mathbb{S}^n
\end{aligned} \tag{16}$$

where $\Sigma_{XY} := \mathrm{E}[X^{\mathrm{T}} S Y] - \mathrm{E}[X]^{\mathrm{T}} S\,\mathrm{E}[Y]$.

Let us denote the value of the cost function at the time $t$ by $V_t = x_0^{\mathrm{T}} \Pi_t x_0$. Then Equation (15) reduces to the following recursive equation similar to the Riccati equation.

$$\begin{aligned}
\Pi_{t-1} = Q + \Sigma_{AA} + \mathrm{E}[A^{\mathrm{T}} \Pi_t A] - \left(\mathrm{E}[A^{\mathrm{T}} \Pi_t B] + \Sigma_{AB}\right) \\
\left(\mathrm{E}[B^{\mathrm{T}} \Pi_t B] + \Sigma_{BB} + R\right)^{-1}\left(\mathrm{E}[B^{\mathrm{T}} \Pi_t A] + \Sigma_{BA}\right)
\end{aligned} \tag{17}$$

Here $\Pi_N = F$ and $\mathcal{B}_*^k F = \Pi_{N-k}$, namely,

$$u_t^* = -L_{\Pi_t} x_t.$$

Furthermore, Equation (17) reduces to a Riccati equation for the conventional LQG problem if the parameters $A_t$ and $B_t$ are deterministic. This implies that the proposed method is a natural generalization of the conventional LQG method. Note that the computation of $\Pi_t$ using Equation (17) needs the 2nd order statistics of the stochastic variables $A_t$ and $B_t$.

### *B. The infinite time horizon case*

This subsection derives the optimal control law for the infinite time horizon which is the main result of the present paper. Let us consider the system (7) with $G_t \equiv 0$, that is, the system (3) without the noise $\epsilon_t$ is considered.

For this system, the following cost function with the infinite time horizon is adopted

$$J_\infty(U_\infty, x_0) = \mathrm{E}\left[\sum_{t=0}^{\infty}\left\{x_t^{\mathrm{T}} Q x_t + u_t^{\mathrm{T}} R u_t + \mathrm{tr}(S\,\mathrm{cov}[x_{t+1}|x_t])\right\} \;\middle|\; x_0\right] \tag{18}$$

where $U_\infty = \{u_0, u_1, \ldots\}$.

The optimal control problem considered here is formalized as follows.

*Definition 5:* Consider the system (3) and the cost function (18). For a given initial condition $x_0$, find the control input $U_\infty^* = \{u_t^*\}$ minimizing the cost function $J_\infty(U_\infty, x_0)$. The corresponding minimum value is denoted by $J_\infty^*(x_0)$. This problem is called an *infinite time optimal control problem for variance suppression*.

Before stating the main result, we need to clarify some properties of the optimal control for variance suppression in the finite time horizon case.

The transformation $\mathcal{B}_* : \mathbb{S}^n \to \mathbb{S}^n$ in Equation (15) is to convert the matrix $\Pi_t$ characterizing the cost $V_t(x_t) = x_t^{\mathrm{T}}\Pi_t x_t$ to another $\Pi_{t-1}$ corresponding the cost $V_{t-1}(x_{t-1}) = x_{t-1}^{\mathrm{T}}\Pi_{t-1}x_{t-1}$. The difference between $\mathcal{B}_*$ and $\mathcal{B}_L$ defined in Equation (10) is explained as follows. Since $\mathcal{B}_L$ corresponds to the constant feedback gain $L$, the resulting cost function is $J(x_0) = x_0^{\mathrm{T}}\mathcal{B}_L^N F x_0$. On the other hand, since $\mathcal{B}_* := \mathcal{B}_{L_X}$ corresponds to the time-varying feedback gain $L_X$, the resulting cost function satisfies $J_N^*(x_0) \neq x_0^{\mathrm{T}}\mathcal{B}_{L_X}^N F x_0$ whereas we can still use the expression $J_N^*(x_0) = x_0^{\mathrm{T}}\mathcal{B}_*^N F x_0$. The following lemma on the transformation $\mathcal{B}_*$ is useful in proving the main theorem.

*Lemma 6:* (a) An inequality $0 \preceq \mathcal{B}_*^N F \preceq \mathcal{B}_L^{0:N-1} F$ holds for any natural number $N \in \mathbb{N}$, any matrix $L \in \mathbb{M}^{n\times m}$, and any symmetric matrix $F \succeq 0$, $F \in \mathbb{S}^n$.

(b) The transformation $\mathcal{B}_*^N$ is monotonic for any natural number $N \in \mathbb{N}$.

*Proof:* (a) Due to the optimality of the solution, we have

$$0 \le J_N^*(x_0) \le J_N(U_N, x_0)$$
$$0 \le x_0^{\mathrm{T}}\mathcal{B}_*^N F x_0 \le x_0^{\mathrm{T}}\mathcal{B}_L^{0:N-1} F x_0, \quad \forall x_0$$

Hence the claim of the lemma holds.

(b) The part (a) implies that $\mathcal{B}_* X \succeq 0$ holds for any $X \succeq 0$. Lemma 5 suggests that $\mathcal{B}_* X = \mathcal{B}_{L_X} X \preceq \mathcal{B}_{L_Y} Y = \mathcal{B}_* Y$ holds for any $X$ and $Y$ satisfying $0 \preceq X \preceq Y$. Hence $\mathcal{B}_*$ is monotonic. It follows from Lemma 2 (a) that $\mathcal{B}_*^N$ is monotonic for any natural number $N \in \mathbb{N}$. ■

Now we are ready to prove the main result.

*Theorem 2:* Suppose that the system $(A_t, B_t)$ is ms-stabilizable. Then $\Pi := \lim_{N\to\infty}\mathcal{B}_*^N 0$ exists and $\Pi$ is the minimum positive semi-definite solution to

$$\begin{aligned}\Pi &= \mathcal{B}_*\Pi \\ &= \mathcal{A}_{L_\Pi}\Pi + Q + L_\Pi^{\mathrm{T}} R L_\Pi + \mathcal{A}_{L_\Pi} S - \mathrm{E}[\Psi_{L_\Pi}^{\mathrm{T}}] S\, \mathrm{E}[\Psi_{L_\Pi}] \\ &= Q + \Sigma_{AA} + \mathrm{E}[A^{\mathrm{T}}\Pi A] - (\mathrm{E}[A^{\mathrm{T}}\Pi B] + \Sigma_{AB}) \\ &\quad (\mathrm{E}[B^{\mathrm{T}}\Pi B] + \Sigma_{BB} + R)^{-1}(\mathrm{E}[B^{\mathrm{T}}\Pi A] + \Sigma_{BA}).\end{aligned} \tag{19}$$

Furthermore, the optimal feedback input is given by

$$\begin{aligned}u_t^* &= -L_\Pi x_t \\ &= -(\mathrm{E}[B^{\mathrm{T}}\Pi B] + R + \Sigma_{BB})^{-1}(\mathrm{E}[B^{\mathrm{T}}\Pi A] + \Sigma_{BA})x_t\end{aligned} \tag{20}$$

for which the cost function (18) takes its minimum value $J_\infty^*(x_0) = x_0^{\mathrm{T}}\Pi x_0$.

*Proof:* First of all, we prove that the function $J_N^*(x_0)$ is monotonically non-decreasing with respect to the terminal time $N$. Select $N_1$ and $N_2$ satisfying $0 < N_1 < N_2$, then the following equations hold for any initial state $x_0$.

$$J_{N_1}^*(x_0) = x_0^{\mathrm{T}}\mathcal{B}_*^{N_1} 0 x_0$$
$$J_{N_2}^*(x_0) = x_0^{\mathrm{T}}\mathcal{B}_*^{N_2} 0 x_0.$$

The definition (15) of the transformation $\mathcal{B}_*$ implies that $\mathcal{B}_* 0 \succeq 0$. Furthermore, it follows from Lemma 6 (b) that $\mathcal{B}_*^N$ is monotonic hence

$$\begin{aligned}\mathcal{B}_*^{N_2} 0 = \mathcal{B}_*^{N_2-1}\mathcal{B}_* 0 \succeq \mathcal{B}_*^{N_2-1} 0 \succeq \cdots \\ \succeq \mathcal{B}_*^{N_1+1} 0 = \mathcal{B}_*^{N_1}\mathcal{B}_* 0 \succeq \mathcal{B}_*^{N_1} 0.\end{aligned}$$

This means $J_{N_1}^*(x_0) \le J_{N_2}^*(x_0)$ and, consequently, the cost function $J_N^*(x_0)$ is monotonically non-decreasing with respect to the terminal time $N$. Therefore, there exists a lower bond of $J_N^*(x_0)$.

Next we prove the existence of $\Pi = \lim_{N\to\infty}\mathcal{B}_* X$. Since the system $(A_t, B_t)$ is ms-stabilizable, it follows from Lemma 5 that there exists a feedback gain $L$ such that $\mathcal{A}_L$ is stable. Hence Lemma 1 implies that the equation

$$\Pi = \mathcal{B}_L\Pi (= \mathcal{A}_L\Pi + Q + L^{\mathrm{T}}RL + \mathcal{A}_L S - \mathrm{E}[\Psi_L^{\mathrm{T}}] S\, \mathrm{E}[\Psi_L])$$

has a solution $\Pi \succeq 0$. Further, by induction, we have $\mathcal{B}_L^N\Pi = \Pi$. Lemma 6 suggests

$$\Pi = \mathcal{B}_L^N\Pi \succeq \mathcal{B}_*^N\Pi \succeq \mathcal{B}_*^N 0. \tag{21}$$

So the sequence $\{\mathcal{B}_*^N 0\}$ has an upper bound and is monotonically non-decreasing. Therefore, the limit $\Pi = \lim_{N\to\infty}\mathcal{B}_*^N \succeq 0$ exists. The equation $\mathcal{B}_*^{N+1}0 = \mathcal{B}_*\mathcal{B}_*^N 0$ implies

$$\Pi = \mathcal{B}_*\Pi \tag{22}$$

by taking the limit $N \to \infty$.

Next we prove that $\Pi = \lim_{N\to\infty}\mathcal{B}_*^N \succeq 0$ is the minimum solution to (22). Suppose that $\tilde{\Pi} \succeq 0$ is another solution of (22). Then Lemma 6 (b) implies that $\mathcal{B}_*^N$ is monotonic for any natural number $N \in \mathbb{N}$. Hence $\mathcal{B}_*^N 0 \preceq \mathcal{B}_*^N\tilde{\Pi} = \tilde{\Pi}$ holds. Taking the limit $N \to \infty$, we obtain $\Pi \preceq \tilde{\Pi}$ which means that $\Pi$ is the minimum solution.

Finally, let us prove that the feedback (20) is the optimal input. Let $\hat{U}_\infty := \{\hat{u}_t\}$, $t \ge 0$ denote the sequence of the feedback input $\hat{u}_t = -L_\Pi x_t$. Let $J_N(U_N, x_0, X)$ denote the cost (8) where the terminal time is $N$ and the terminal cost $F = X$. Then we have

$$J_N(\hat{U}_\infty, x_0, 0) \le J_N(\hat{U}_\infty, x_0, \Pi) = x_0^{\mathrm{T}}\mathcal{B}_{L_\Pi}^N\Pi x_0 = x_0^{\mathrm{T}}\Pi x_0$$

for any $N \in \mathbb{N}$. Taking the limit $N \to \infty$,

$$J_\infty(\hat{U}_\infty, x_0) \le x_0^{\mathrm{T}}\Pi x_0. \tag{23}$$

On the other hand,

$$x_0^{\mathrm{T}}\mathcal{B}_*^N 0 x_0 = J_N^*(x_0, 0) \le J_\infty^*(x_0) \le J_\infty(\hat{U}_\infty, x_0) \tag{24}$$

also holds. Inequalities (23) and (24) suggest

$$x_0^{\mathrm{T}}\mathcal{B}_*^N 0 x_0 \le J_\infty^*(x_0) \le J_\infty(\hat{U}_\infty, x_0) \le x_0^{\mathrm{T}}\Pi x_0.$$

Again, taking the limit $N \to \infty$,

$$J^*_\infty(x_0) = J_\infty(\hat{U}_\infty, x_0) = x_0^{\mathrm{T}} \Pi x_0. \tag{25}$$

which implies $U^*_\infty = \hat{U}_\infty$. Therefore $u^*_t = \hat{u}_t = -L_\Pi x_t$ for $t \geq 0$. This completes the proof. ■

Equation (19) reduces to the algebraic Riccati equation for the (conventional) discrete time optimal control if the system parameters $A_t$ and $B_t$ are constant (deterministic). However, in the stochastic case, Equation (19) is not a Riccati equation anymore and it cannot be solved with the conventional technique in the linear control systems theory. For instance, although $\mathrm{E}[A^{\mathrm{T}}\Pi B]$ is linear in $\Pi$, there does not exist constant (deterministic) matrices $X$ and $Y$ satisfying $\mathrm{E}[A^{\mathrm{T}}\Pi B] = X\Pi Y$ in general because the system parameters $A_t$ and $B_t$ are stochastic variables. Therefore we need to employ nonlinear optimization to solve Equation (19). Here we use the following procedure to obtain the solution $\Pi$. First of all, define the cost function for nonlinear optimization $\Gamma(\Pi)$ as follows.

$$\begin{aligned}\Gamma(\Pi) :=& -\Pi + Q + \Sigma_{AA} + \mathrm{E}[A^{\mathrm{T}}\Pi A] - (\mathrm{E}[A^{\mathrm{T}}\Pi B] + \Sigma_{AB}) \\ & (\mathrm{E}[B^{\mathrm{T}}\Pi B] + \Sigma_{BB} + R)^{-1}(\mathrm{E}[B^{\mathrm{T}}\Pi A] + \Sigma_{BA})\end{aligned}$$

Since Equation $\Gamma(\Pi) = 0$ is equivalent to Equation (19), we compute $\Pi$ numerically by minimizing $\|\Gamma(\Pi)\|_F = \|\mathrm{vec}(\Gamma(\Pi))\|$ where $\|\cdot\|_F$ is the Frobenius norm. Here we select the initial value $\Pi = \Pi_0$ the solution to the conventional Riccati equation in the deterministic case

$$\begin{aligned}\Pi_0 =& Q + \Sigma_{AA} + \mathrm{E}[A]^{\mathrm{T}}\Pi_0\,\mathrm{E}[A] - (\mathrm{E}[A]^{\mathrm{T}}\Pi_0\,\mathrm{E}[B] + \Sigma_{AB}) \\ & (\mathrm{E}[B]^{\mathrm{T}}\Pi_0\,\mathrm{E}[B] + \Sigma_{BB} + R)^{-1}(\mathrm{E}[B]^{\mathrm{T}}\Pi_0\,\mathrm{E}[A] + \Sigma_{BA}).\end{aligned}$$

This procedure often gives us the true solution when the variances of the system parameters are small. When the nonlinear optimization does not give the global optimal (the solution to Equation (19)), then applying the nonlinear optimization procedure recursively by changing the variances of the system parameters from 0 to the true value would give us the global optimal.

## IV. Numerical Example

This section gives a numerical example to demonstrate the effectiveness of the proposed method in comparison to the conventional LQG method. Let us consider the plant described by Equation (3) with the state $x_t \in \mathbb{R}^2$ and the input $u_t \in \mathbb{R}$.

$$\mathrm{E}[A] = \begin{bmatrix} 1 & 0.1 \\ -0.01 & 0.99 \end{bmatrix}$$

$$\mathrm{E}[B] = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$$

$$\mathrm{cov}\,[\mathrm{vec}([A, B])] = \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4.0\times 10^{-6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0392 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.0\times 10^{-6} \end{array}\right]$$

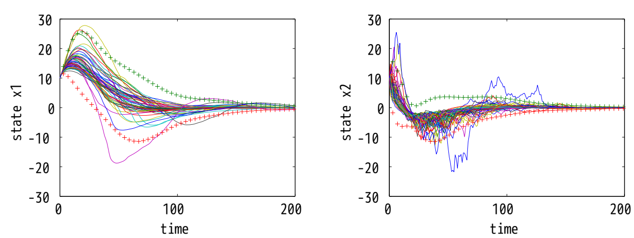
$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$R = 1$$

$$x_0 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

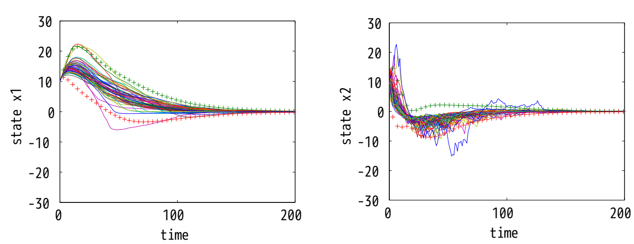
The covariance $\mathrm{cov}\,[\mathrm{vec}([A, B])]$ are selected in such a way that the parameters $A_{21}$, $A_{22}$ and $B_2$ have 20% standard deviations. Fig.1 shows the time responses of 100 random samples of the feedback system with the conventional LQG controller. Fig.2 shows the time responses by the proposed method with the design parameter $S = 1000I$. In those figures, the plus signs + denote the upper and lower bounds of the $1\sigma$ deviation from the average and the solid lines denote the sampled responses.

Figures show that both the conventional LQG controller and the proposed controller achieve both m-stability and ms-stability. The response with the proposed controller in Fig.2 achieves smaller variance than that with the LQG controller in Fig.1 which indicates the effect of the proposed method.




(i) State transition of $x^1$ (ii) State transition of $x^2$

Fig. 1. State transition (LQG)




(i) State transition of $x^1$ (ii) State transition of $x^2$

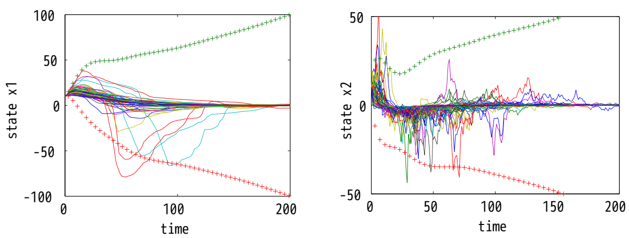
Fig. 2. State transition (proposed method)

Next, Figs.3–4 show the case in which the variances of the parameters are bigger than the case in Figs.1–2. The covariance of the system parameters $A$ and $B$ is selected as follows in such a way that the parameters $A_{21}$, $A_{22}$ and $B_2$

have 40% standard deviations.

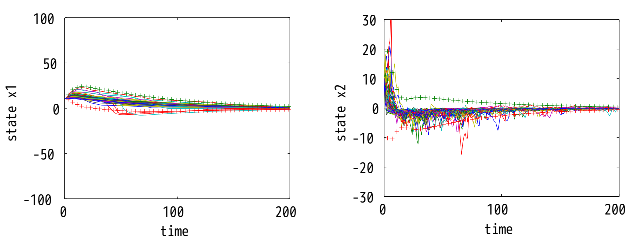
$$
\begin{aligned}
&\mathrm{cov}\,[\mathrm{vec}([A, B])] \\
&= \left[\begin{array}{cccc|cc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.6\times10^{-5} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.1568 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.6\times10^{-5}
\end{array}\right]
\end{aligned}
$$

Figs.3–4 show the time responses of 100 random samples by the LQG method and the proposed method with the design parameter $S = 1000I$, respectively. In those figures, the plus signs $+$ denote the upper and lower bounds of the $1\sigma$ deviation from the average and the solid lines denote the sampled responses as in Figs.1–2.




(i) State transition of $x^1$ (ii) State transition of $x^2$

Fig. 3. State transition (LQG)




(i) State transition of $x^1$ (ii) State transition of $x^2$

Fig. 4. State transition (proposed method)

Fig.3 shows the time responses of the state of the feedback system with the LQG controller. In the figure the states diverge, since the LQG controller does not take care of the variance of the system parameters. Fig.4 shows the time responses of the state with the proposed controller for the parameter $S = 1000I$. This result shows that the proposed method can stabilize the variance of the states in the sense of ms-stability, whereas the variance of the feedback system with the LOG controller becomes unstable (in the sense of ms-stability). Those results confirms the effectiveness of the proposed method.

## V. Conclusion

This paper generalizes the authors' former result on optimal control with variance suppression to the infinite time horizon case. We have derived an algebraic equation similar to the algebraic Riccati equation to characterize the solution to the optimal control problem and proved stochastic stability of the corresponding feedback system by taking care of the average and the variance of the state transient. Furthermore, some numerical simulations exhibit the effectiveness of the proposed method. We believe that the proposed method provides a new stochastic control framework to work with the Bayesian inference methods.

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