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State space representation of SISO periodic behaviors

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Abstract— **The aim of this paper is to obtain periodic state space representations for periodic input–output behavioral systems using a lifting technique which allows to associate a time–invariant behavior to a periodic one. Our approach differs from the classical one since we do not start from a transfer function description but rather from linear difference equations with periodically time-varying coefficients.**

Index Terms— **Mathematical systems theory, linear systems, discrete-time systems, behavior.**

I. Introduction

The problem of finding state space representations for periodic systems has deserved great attention, see, e.g., [1]–[8]. The available contributions consider exclusively the realization of transfer functions or impulse responses.

In this paper we are concerned with a similar problem, in the context of the behavioral approach. This means that our object of interest is the behavior of a system, a set that consists of all the signal trajectories that are admissible according to the system laws. In particular, such trajectories do not necessarily correspond to zero initial conditions, and therefore the behavior may contain more trajectories than the ones that can be generated by the system transfer function or impulse response. In this sense, our realization problem is aimed at a broader class of systems.

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Our starting point is the description of a periodic behavior by means of linear difference equations with periodically time–varying coefficients. Based on this, we obtain a description for the set of corresponding lifted trajectories, which is a time–invariant behavior, described in terms of a linear constant coefficient difference equations. Using behavioral techniques, it is possible to construct minimal state space representations for the lifted behavior that can be used (either directly or after a suitable transformation) to obtain a minimal periodic state space representation of constant dimension for the original periodic behavior.

II. Preliminaries

In the behavioral framework, see [9], [10], a dynamical system Σ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T} \subseteq \mathbb{R}$ is the time set, $\mathbb{W}$ is the signal space, and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}} := \{w\colon \mathbb{T} \to \mathbb{W}\}$ is the system behavior. The behavior is a set that consists of all the signal evolutions that are compatible with the system laws. In this paper we shall be concerned with the discrete–time case, that is, $\mathbb{T} = \mathbb{Z}$, and assume that the signal space is $\mathbb{W} = \mathbb{R}^2$. In general, in the behavioral approach all the system variables are treated in an equal footing, and no a priori distinction is made between inputs and outputs. This distinction can be made a posteriori, if at all appropriate. Here we shall assume that this input–output partition has been made and $w = (u, y)$ consists of the input component u and the output component y.

For $\tau \in \mathbb{Z}$, define the τ–shift as $\sigma^\tau\colon (\mathbb{R}^q)^{\mathbb{Z}} \to (\mathbb{R}^q)^{\mathbb{Z}}$, by

$$(\sigma^\tau w)(k) := w(k+\tau).$$

While the behavior $\mathfrak{B}$ of a time–invariant system over $\mathbb{Z}$ is characterized by its invariance under the time shifts, which amounts to $\sigma\mathfrak{B} = \mathfrak{B}$, [9], [10], P-periodic behaviors are required to be invariant only with respect to shifts that are powers of σ^P.

Definition 2.1: [11] A system Σ is said to be P–periodic, with $P \in \mathbb{N}$, if its behavior $\mathfrak{B}$ satisfies $\sigma^P\mathfrak{B} = \mathfrak{B}$. ◇

We consider single input–single output P–periodic behaviors $\mathfrak{B}$ described by difference equations with periodically time–varying coefficients, i.e.

$$\big(p_t\big(\sigma,\sigma^{-1}\big)\,y\big)(t+Pk)=\big(q_t\big(\sigma,\sigma^{-1}\big)\,u\big)(t+Pk)\ ,\qquad t=0,\ldots,P-1,\ k\in\mathbb{Z}\ , \tag{1}$$

where, for each time instant $t = 0,\ldots,P-1$, $p_t\big(\xi,\xi^{-1}\big)\,,q_t\big(\xi,\xi^{-1}\big)\in\mathbb{R}\big[\xi,\xi^{-1}\big]$ are Laurent polynomials in the indeterminate ξ. Note that (1) can also be written as

$$\big(p\big(\sigma,\sigma^{-1}\big)\,y\big)(Pk)=\big(q\big(\sigma,\sigma^{-1}\big)\,u\big)(Pk)\,,\quad k\in\mathbb{Z}, \tag{2}$$

where

$$p\big(\xi,\xi^{-1}\big):=\begin{bmatrix} p_0\big(\xi,\xi^{-1}\big)\\ \xi p_1\big(\xi,\xi^{-1}\big)\\ \vdots\\ \xi^{P-1}p_{P-1}\big(\xi,\xi^{-1}\big)\end{bmatrix},$$

and similarly for $q\left(\xi,\xi^{-1}\right)$. From now on, such systems will simply be called SISO P–periodic behaviors.

By factoring p and q as, see [12],

$$p\left(\xi,\xi^{-1}\right) = P^L\big(\xi^P,\xi^{-P}\big)\,\Omega_P(\xi) \tag{3}$$
$$q\left(\xi,\xi^{-1}\right) = Q^L\big(\xi^P,\xi^{-P}\big)\,\Omega_P(\xi) \tag{4}$$

where

$$\Omega_P(\xi):=\Big[1\quad \xi\quad \cdots\quad \xi^{P-1}\Big]^T,$$

we write down relation (2) as

$$\begin{aligned}&\left(P^L\left(\sigma^P,\sigma^{-P}\right)\Omega_P\left(\sigma\right)y\right)\left(Pk\right)\\ &\qquad=\left(Q^L\left(\sigma^P,\sigma^{-P}\right)\Omega_P\left(\sigma\right)u\right)\left(Pk\right),k\in\mathbb{Z}\ .\end{aligned} \tag{5}$$

Define the lifted input and output trajectories

$$u^L\left(k\right):=\left(Lu\right)\left(k\right):=\begin{bmatrix}u\left(Pk\right)\\ \vdots\\ u\left(Pk+P-1\right)\end{bmatrix}$$
$$y^L\left(k\right):=\left(Ly\right)\left(k\right):=\begin{bmatrix}y\left(Pk\right)\\ \vdots\\ y\left(Pk+P-1\right)\end{bmatrix}$$

see [11]–[14], and note that $L\left(\sigma^P v\right)=\sigma\left(Lv\right)$. Then (5) can be written as

$$\left(P^L\big(\sigma,\sigma^{-1}\big)\,y^L\right)(k) = \left(Q^L\big(\sigma,\sigma^{-1}\big)\,u^L\right)(k)\,,\ k\in\mathbb{Z}. \tag{6}$$

The behavior $\mathfrak{B}^L$, defined by $L(\mathfrak{B}):=\{(Lu,Ly)\,,(u,y)\in\mathfrak{B}\}$, called the *lifted behavior* associated with $\mathfrak{B}$, is time–invariant, and equals the set of trajectories

$$\Big\{\big(u^L,y^L\big)\in\big(\mathbb{R}^P\big)^{\mathbb{Z}}\times\big(\mathbb{R}^P\big)^{\mathbb{Z}}\mid \text{(6) holds}\Big\},$$

that is,

$$\mathfrak{B}^L=\ker\Big[P^L\big(\sigma,\sigma^{-1}\big)\quad -Q^L\big(\sigma,\sigma^{-1}\big)\Big]\ .$$

Given a P–periodic input–output behavior $\mathfrak{B}$, we say that a P–periodic state space system $\Sigma(\cdot) = (A(\cdot)\,,B(\cdot)\,,C(\cdot)\,,D(\cdot))$

$$\begin{cases}(\sigma x)\,(k)=A\,(k)\,x\,(k)+B\,(k)\,u\,(k)\\ \quad y\,(k)=C\,(k)\,x\,(k)+D\,(k)\,u\,(k)\end{cases}\qquad k\in\mathbb{Z}, \tag{7}$$

where $A(\cdot)\,,B(\cdot)\,,C(\cdot)\,,D(\cdot)$ are periodic functions with period P, is a *(periodic) state space representation* of $\mathfrak{B}$ if

$$\mathfrak{B}=\{(u,y)\mid \exists x \text{ such that } (u,x,y) \text{ satisfies (7)}\}\ .$$

The definition of a (time–invariant) state space representation $\Sigma=(A,B,C,D)$ for a time invariant behavior is analogous, see [15]. A state space representation of a behavior will be called minimal if the dimension of the state vector is the smallest among all the representations of the same behavior.

Notice that according to this definition, a state space representation should describe the whole system behavior and not only its transfer function (or impulse response). This issue is particularly relevant in the case of non controllable behaviors, [9], [10].

III. PERIODIC STATE SPACE REPRESENTATIONS

In this section we investigate the construction of periodic state space representations for SISO periodic behaviors. This differs from the realization problems previously considered by other authors, see, e.g., [7], [16], since we are not interested in merely realizing the transfer function or the impulse response. Instead, we start from a periodic behavior $\mathfrak{B}$ described by a linear difference equation with periodically time–varying coefficients as (1), and exploit the connections $\mathfrak{B}$ and its (time–invariant) lifted version $\mathfrak{B}^L$ in order to construct periodic state space representations. More concretely, we show how to obtain periodic state space representations for $\mathfrak{B}$ from state space representations of the time–invariant behavior $\mathfrak{B}^L$, which can be obtained by standard algorithms, see [17, pp. 72–77].

For this purpose we start by studying the relationship between periodic state space representations of a given periodic behavior $\mathfrak{B}$ and time–invariant state space representations of the lifted behavior $\mathfrak{B}^L$. For the sake of simplicity we consider only the case of $P=2$. General P follows along the same lines.

Let us start by assuming that a n–dimensional state space representation $\Sigma(\cdot) = (A(\cdot)\,,B(\cdot)\,,C(\cdot)\,,D(\cdot))$ of $\mathfrak{B}$ is given, i.e.,

$$\begin{cases}(\sigma x)\,(k)=A\,(k)\,x\,(k)+B\,(k)\,u\,(k)\\ \quad y\,(k)=C\,(k)\,x\,(k)+D\,(k)\,u\,(k)\end{cases}\qquad k\in\mathbb{Z},$$

where $A(\cdot)\in\mathbb{R}^{n\times n}$, $B(\cdot)\in\mathbb{R}^{n\times 1}$, $C(\cdot)\in\mathbb{R}^{1\times n}$ and $D(\cdot)\in\mathbb{R}$ are periodic functions with period 2. Letting

$$\begin{aligned} z(k) &= x(2k) \\ u^L(k) &= \begin{bmatrix} u(2k) \\ u(2k+1) \end{bmatrix} \\ y^L(k) &= \begin{bmatrix} y(2k) \\ y(2k+1) \end{bmatrix} \end{aligned}$$

we obtain the following time–invariant n–dimensional state space representation $\Sigma^L=(F,G,H,J)$ for $\mathfrak{B}^L$:

$$\begin{cases} z(k+1) = Fz(k) + Gu^L(k) \\ y^L(k) = Hz(k) + Ju^L(k) \ , \end{cases} \tag{8}$$

with

$$F = A(1)A(0) \qquad G = \begin{bmatrix} A(1)B(0) & B(1) \end{bmatrix}$$

$$H = \begin{bmatrix} C(0) \\ C(1)A(0) \end{bmatrix} \quad J = \begin{bmatrix} D(0) & 0 \\ C(1)B(0) & D(1) \end{bmatrix} .$$

This representation is also obtained in [7], [14], [18] in a different context.

We shall say that the representation $\Sigma^L = (F,G,H,J)$ of $\mathfrak{B}^L$ is *induced by* the representation $\Sigma(\cdot) = (A(\cdot),B(\cdot),C(\cdot),D(\cdot))$ of $\mathfrak{B}$, or equivalently, that $\Sigma(\cdot)$ induces Σ^L. Moreover, we shall call a representation Σ^L of $\mathfrak{B}^L$ *induced* whenever it is induced by some periodic representation Σ of $\mathfrak{B}$.

Note that, for the n–dimensional induced representation (8), the $(n+1)$–square matrix

$$\mathcal{M} := \begin{bmatrix} F & G_1 \\ H_2 & J_{21} \end{bmatrix} ,$$

with G_1, H_2, and J_{21} defined by

$$\begin{bmatrix} G_1 & G_2 \end{bmatrix} := G \qquad \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} := H$$

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} := J ,$$

can be factored as

$$\begin{aligned} \mathcal{M} &= \begin{bmatrix} A(1)A(0) & A(1)B(0) \\ C(1)A(0) & C(1)B(0) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} A(1) \\ C(1) \end{bmatrix}}_{n} \begin{bmatrix} A(0) & B(0) \end{bmatrix} \Big\} n \ . \end{aligned}$$

Therefore $\operatorname{rank}\mathcal{M} \leqslant n$.

Conversely, let now $\Sigma^L = (F,G,H,J)$ be a state space representation of $\mathfrak{B}^L$, with

$$F \in \mathbb{R}^{n\times n} \qquad G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \in \mathbb{R}^{n\times 2}$$

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \in \mathbb{R}^{2\times n} \quad J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2} .$$

Define

$$\mathcal{M} := \begin{bmatrix} F & G_1 \\ H_2 & J_{21} \end{bmatrix} . \tag{9}$$

Assume that $\operatorname{rank}\mathcal{M} \leqslant n$ and decompose this matrix as

$$\mathcal{M} = \underbrace{\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}}_{n} \begin{bmatrix} D_1 & D_2 \end{bmatrix} \Big\} n \ . \tag{10}$$

Then it can be shown that $\Sigma(\cdot)=(A(\cdot),B(\cdot),C(\cdot),D(\cdot))$, with

$$A(0) = D_1 \qquad B(0) = D_2 \tag{11}$$

$$A(1) = E_1 \qquad B(1) = G_2 \tag{12}$$

$$C(0) = H_1 \qquad D(0) = J_{11} \tag{13}$$

$$C(1) = E_2 \qquad D(1) = J_{22} \ , \tag{14}$$

is a periodic state space representation of $\mathfrak{B}$ of dimension n.

These considerations lead to the following result.

Proposition 3.1: Let $\mathfrak{B}$ be a SISO 2–periodic behavior and $\mathfrak{B}^L$ the lifted behavior associated to $\mathfrak{B}$. Then a n–dimensional state space representation $\Sigma^L=(F,G,H,J)$ of $\mathfrak{B}^L$, with

$$F \in \mathbb{R}^{n\times n} \qquad G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \in \mathbb{R}^{n\times 2}$$

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \in \mathbb{R}^{2\times n} \quad J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2} ,$$

is induced if and only if

$$\operatorname{rank}\begin{bmatrix} F & G_1 \\ H_2 & J_{21} \end{bmatrix} \leqslant n .$$

Moreover, in this case, $\Sigma(\cdot) = (A(\cdot),B(\cdot),C(\cdot),D(\cdot))$ defined as in (11)–(14) is a periodic n–dimensional state space representation of $\mathfrak{B}$ that induces Σ^L. ◇

As the next example shows, not every state space representation of $\mathfrak{B}^L$ is an induced one.

Example 3.2: Consider the 2–periodic input–output behavior $\mathfrak{B}$ described by

$$\left(\begin{bmatrix} \sigma^2 - 1 \\ \sigma \end{bmatrix} y\right)(2k) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} u\right)(2k) .$$

Its associated lifted behavior $\mathfrak{B}^L$, defined according to (6), is given by

$$\left(\begin{bmatrix} \sigma-1 & 0 \\ 0 & 1 \end{bmatrix} y^L\right)(k) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} u^L\right)(k) ,$$

for which a minimal state space representation, of dimension 1, is

$$\begin{cases} \sigma z(k) = z(k) \\ y^L(k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} z(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} u^L(k) . \end{cases}$$

In this case the matrix $\mathcal{M}$ defined in (9) is given by

$$\mathcal{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \tag{15}$$

which is clearly of full rank, and hence not decomposable as in (10). $\diamond$

However, as we shall see, it is always possible to construct an induced state space representation of $\mathfrak{B}^L$ starting from a non–induced one. In fact, let $\Sigma^L = (F, G, H, J)$ be a n–dimensional representation of $\mathfrak{B}^L$,

$$\begin{cases} \sigma x(k) = Fx(k) + Gu^L(k) \\ y^L(k) = Hx(k) + Ju^L(k) , \end{cases}$$

which is not induced by a periodic state space representation of $\mathfrak{B}$. This means that the $(n+1) \times (n+1)$ matrix

$$\mathcal{M} = \begin{bmatrix} F & G_1 \\ H_2 & J_{21} \end{bmatrix}$$

has rank $n+1$. Augment this matrix by adding a zero row to F and G_1 and a zero column to F and H_2, yielding

$$\widetilde{\mathcal{M}} = \left[\begin{array}{ccc} & 0 & \\ F & \vdots & G_1 \\ & 0 & \\ 0 \cdots 0 & 0 & 0 \cdots 0 \\ & 0 & \\ H_2 & \vdots & J_{21} \\ & 0 & \end{array}\right] . \tag{16}$$

Note that this corresponds to adding a superfluous (zero) state x^s to the original representation, in order to obtain a higher dimensional one, of the form:

$$\begin{cases} \sigma \overline{x}^{\mathrm{e}}(k) = \left[\begin{array}{c|c} F & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 0 \end{array}\right] \overline{x}^{\mathrm{e}}(k) + \left[\begin{array}{c} G \\ \hline 0 \end{array}\right] u^L(k) \\ y^L(k) = \left[\begin{array}{c|c} H & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \overline{x}^{\mathrm{e}}(k) + Ju^L(k) . \end{cases}$$

Clearly $\widetilde{\mathcal{M}}$ is an $(n+2) \times (n+2)$ matrix with rank $n+1$ that can be decomposed as

$$\widetilde{\mathcal{M}} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \begin{bmatrix} D_1 & D_2 \end{bmatrix} ,$$

with

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} := \left[\begin{array}{cc} F & G_1 \\ 0 & 0 \\ \hline H_2 & J_{21} \end{array}\right]$$

$$\begin{bmatrix} D_1 & D_2 \end{bmatrix} := \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & 0 & I \end{array}\right] .$$

This yields a $(n+1)$–dimensional state space representation $\Sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ of $\mathfrak{B}$, given by:

$$A(0) = D_1 \qquad B(0) = D_2 \tag{17}$$

$$A(1) = E_1 \qquad B(1) = \begin{bmatrix} G_2 \\ 0 \end{bmatrix} \tag{18}$$

$$C(0) = \begin{bmatrix} H_1 & 0 \end{bmatrix} \qquad D(0) = J_{11} \tag{19}$$

$$C(1) = E_2 \qquad D(1) = J_{22} , \tag{20}$$

cf. (11)–(14).

Thus, it is always possible to construct a periodic state space representation Σ for a SISO 2–periodic behavior $\mathfrak{B}$ starting from a state space representation Σ^L of $\mathfrak{B}^L$. Moreover $\dim(\Sigma) = \dim(\Sigma^L)$ or $\dim(\Sigma) = \dim(\Sigma^L) + 1$.

Example 3.3: Recall Example 3.2. Adding a zero row and a zero column to matrix $\mathcal{M}$ given in (15), as illustrated previously, we obtain a new 3×3 matrix $\widetilde{\mathcal{M}}$ given by

$$\widetilde{\mathcal{M}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

which can be decomposed as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] .$$

This factorization allows us to obtain a 2–dimensional state space reapresentation $\Sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ of $\mathfrak{B}$, given by:

$$A(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C(0) = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D(0) = 0$$

$$C(1) = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad D(1) = 0 .$$

$\diamond$

Since every time–invariant input–output behavior has a state space representation [17], we conclude that every periodic input–output behavior also has a periodic state space representation.

Notice that every state space representation Σ of $\mathfrak{B}$ induces a representation Σ^L of $\mathfrak{B}^L$ with the same dimension. Moreover, as a consequence of the following lemma, either the minimal state space representations of $\mathfrak{B}$ are all induced or none of them is.

Lemma 3.4: Let $\mathfrak{B}^L$ be the lifted behavior associated to a SISO 2–periodic behavior $\mathfrak{B}$. If $\mathfrak{B}^L$ has one minimal state space representation which is induced, then all its minimal representations are induced. $\diamond$

On the other hand, if the minimal state space representations of Σ^L of $\mathfrak{B}^L$ (with dimension, say, $n_{\mathfrak{B}^L}$) are not induced, then there exists an induced representation of dimension $n_{\mathfrak{B}^L}+1$. This implies that

$$n_{\mathfrak{B}^L} \leqslant n_{\mathfrak{B}} \leqslant n_{\mathfrak{B}^L}+1 \ ,$$

where $n_{\mathfrak{B}}$ denotes the minimal dimension of the state space representations of $\mathfrak{B}$.

The previous considerations can be summarized as follows.

Theorem 3.5: Let $\mathfrak{B}$ be a SISO 2–periodic behavior and let $\mathfrak{B}^L$ be the corresponding lifted behavior. Then:

(i) $\mathfrak{B}$ has a 2–periodic state space representation $\Sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$;

(ii) The dimensions $n_{\mathfrak{B}}$ and $n_{\mathfrak{B}^L}$ of the minimal state space representations of $\mathfrak{B}$ and of $\mathfrak{B}^L$, respectively, are such that:
$$n_{\mathfrak{B}^L} \leqslant n_{\mathfrak{B}} \leqslant n_{\mathfrak{B}^L}+1;$$

(iii) A minimal periodic state space representation of $\mathfrak{B}$ can be obtained by Algorithm 3.6.

$\diamond$

Algorithm 3.6:

- **Input**: A SISO 2–periodic behavior $\mathfrak{B}$;
- **Output**: A minimal 2–periodic state space representation of $\mathfrak{B}$.

Step 1. Construct the lifted behavior $\mathfrak{B}^L$;

Step 2. Compute a minimal representation $\Sigma^L = (F, G, H, J)$ of $\mathfrak{B}^L$ and its dimension $n_{\mathfrak{B}^L}$;

Step 3. Construct the matrix $\mathcal{M}$ as in (9);

Step 4. If $\operatorname{rank} \mathcal{M} \leqslant n_{\mathfrak{B}^L}$:

4.1. decompose $\mathcal{M}$ as in (10), with $n = n_{\mathfrak{B}^L}$, to obtain E_1, E_2, D_1 and D_2;

4.2. define $\Sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ as in (11)–(14);

go to Step 6.

Else: continue;

Step 5. Construct the $(n_{\mathfrak{B}^L}+2) \times (n_{\mathfrak{B}^L}+2)$ matrix $\widetilde{\mathcal{M}}$ (of rank $n_{\mathfrak{B}^L}+1$) as in (16):

5.1. Decompose $\widetilde{\mathcal{M}}$ as
$$\widetilde{\mathcal{M}} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \begin{bmatrix} D_1 & D_2 \end{bmatrix} ,$$
where $\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$ has full column rank and $\begin{bmatrix} D_1 & D_2 \end{bmatrix}$ has full row rank;

5.2. Define $\Sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$ as in (17)–(20);

Step 6. Output: "Minimal state space representation of $\mathfrak{B}$": $\Sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$;

Step 7. End. $\diamond$

IV. Conclusion

In this paper we have investigated the problem of obtaining state space representations for periodic input–output behaviors by exploiting the lifting technique. This classical approach to periodic systems has been previously used in a different context for the realization of periodic transfer functions or of impulse responses. In this paper we have approached the realization problem from the behavioral point of view and aimed at obtaining a state space representation for the whole system behavior. As pointed out earlier, this differs from the classical transfer function or impulse response realization problem for the case of noncontrollable behaviors. We have shown that every SISO behavior described by a linear difference equation with 2–periodic time–varying coefficients can be represented in state space form, and have presented an algorithm to obtain a corresponding minimal state space representation from a minimal representation of the lifted behavior. The generalization of these results to the case of MIMO P–periodic systems with $P > 2$ can be achieved along the same lines will be reported elsewhere. Apart from the realization problem itself, many other relevant questions concerning state space representations of periodic behaviors, such as, for instance, the characterization of minimality, are currently under investigation.

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