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Nonsmooth Approximate Maximum Principle in Optimal Control

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***Abstract*— In this paper we investigate stability of the Pontryagin maximum principle with respect to time discretization for optimal control problems with convex cost function and endpoint constraints. A generalization of the so-called approximate maximum principle from smooth to nonsmooth systems is obtained.**

Keywords. Optimal control, discrete approximations, approximate maximum principle.

I. Introduction

Consider the following Mayer-type optimal control problem with endpoint constraints:

$$(P)\begin{cases} \text{minimize } \varphi_0(x(t_1)) \\ \text{subject to} \\ \dot{x}(t) = f(t,x(t),u(t)) \text{ a.e. } t \in [t_0,t_1], \\ x(t_0) = x_0 \in I\!R^n, \\ u(t) \in U(t) \text{ a.e. } t \in [t_0,t_1], \\ \varphi_i(x(t_1)) \le 0, \quad i = 1,\dots,r_1, \\ \varphi_i(x(t_1)) = 0, \quad i = r_1+1,\dots,r_1+r_2, \end{cases}$$

over measurable controls $u(\cdot)$ and absolutely continuous trajectories $x(\cdot)$ on the fixed time interval $T := [t_0,t_1]$. The data for this problem comprise functions $f : I\!R \times I\!R^n \times I\!R^m \to I\!R^n$, $\varphi_i : I\!R^n \to I\!R$, $i = 0,\dots,r_1+r_2$, and $U(\cdot)$ is a map from T to $I\!R^m$.

The main object of consideration in this paper is the Euler-type discretization of (P) with relaxed endpoint constraints:

$$(P_N)\begin{cases} \text{minimize } \varphi_0(x_N(t_1)) \\ \text{subject to} \\ x_N(t+h_N) = x_N(t) + h_N f(t,x_N(t),u_N(t)), \\ x_N(t_0) = x_0 \in I\!R^n, \\ u_N(t) \in U(t), \quad t \in T_N, \\ \varphi_i(x_N(t_1)) \le \alpha_{iN}, \quad i = 1,\dots,r_1, \\ |\varphi_i(x_N(t_1))| \le \delta_{iN}, \quad i = r_1+1,\dots,r_1+r_2, \\ h_N := \dfrac{t_1-t_0}{N}, \quad N \in I\!N := \{1,2,\dots\}, \end{cases}$$

where

$$T_N := \{t_0, t_0+h_N, \dots, t_1-h_N\},$$

$\alpha_{iN} \to 0$ and $\delta_{iN} \downarrow 0$ as $N \to \infty$ for all i. We treat (P_N) as a *sequence* of discrete problems depending on the natural parameter $N = 1,2,\dots$.

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The goal of this paper is to investigate stability of the Pontryagin maximum principle in the process of discrete approximations, which is important for numerical solutions of optimal control problems.

Sequences of discrete-time problems with the length of time discretization tending to zero occupy an intermediate position between continuous-time and discrete-time problems. It is well known that the Pontryagin maximum principle holds for continuous-time problems without any convexity of the admissible velocity sets, while such convexity assumption is essential for the validity of the exact maximum principle for discrete-time systems.

By

$$H(t,x,p,u) := \langle p, f(t,x,u)\rangle, \quad p \in I\!R^n,$$

denote the Hamilton-Pontryagin function for the dynamic systems under consideration.

Let $(\bar{x}_N, \bar{u}_N)$ be an optimal process in (P_N). The following *Approximate Maximum Principle* (AMP) for smooth nonconvex constrained problems (P_N) was established by Mordukhovich ([1], Section 4).

Theorem 1. (Smooth Approximate Maximum Principle) Assume the following:

(a) Function f is continuous with respect to its variables and continuously differentiable with respect to x in a tube containing the optimal trajectories $\bar{x}_N(t)$ for large N;

(b) Each φ_i, $i = 0,\dots,r_1+r_2$ is continuously differentiable around the limiting points of $\{\bar{x}_N(t_1)\}$;

(c) The consistency condition *on the perturbation of the equality constraints:*

$$\lim_{N\to\infty} \frac{h_N}{\delta_{iN}} = 0 \quad \textit{for all} \quad i = r_1+1,\dots,r_1+r_2;$$

(d) The map $U(\cdot)$ is compact-valued and continuous in Hausdorff metric;

(e) The properness *of the sequences of optimal controls $\{\bar{u}_N\}$, which means that for every increasing subsequence $\{N\}$ of natural numbers and every sequence of mesh points $\tau_{\theta(N)} \in T_N$ satisfying $\tau_{\theta(N)} = t_0 + \theta(N)h_N$, $\theta(N) = 0,1,\dots,N-1$, and $\tau_{\theta(N)} \to t \in [t_0,t_1]$ one has*

$$\textit{either } |u_N(\tau_{\theta(N)}) - u_N(\tau_{\theta(N)+q})| \to 0$$
$$\textit{or } |u_N(\tau_{\theta(N)}) - u_N(\tau_{\theta(N)-q})| \to 0$$

as $N \to \infty$ with any natural constant q.

Then for any $\varepsilon > 0$ there exist numbers λ_{iN}, $i = 0,\dots,r_1+r_2$ such that for any sufficiently small h_N we have

$$H(t,\bar{x}_N(t),p_N(t+h_N),\bar{u}_N(t)) \ge \max_{u\in U(t)} H(t,\bar{x}_N(t),p_N(t+h_N),u) - \varepsilon \qquad (1)$$

for all $t \in T_N$,

$$\lambda_{iN}|\varphi_i(\bar{x}_N(t_1)) - \alpha_{iN}| < \varepsilon, \quad i = 1, \ldots, r_1, \tag{2}$$

$$\lambda_{iN} \geq 0, \;\; i = 0, \ldots, r_1, \quad \textit{and} \quad \sum_{i=0}^{r_1+r_2} \lambda_{iN}^2 = 1 \tag{3}$$

for all $N \in I\!N$. *Here* $p_N(t)$, $t \in T_N \cup \{t_1\}$, *is the corresponding trajectory of the adjoint system*

$$p_N(t) = p_N(t+h_N) + h_N \frac{\partial H}{\partial x}(t, \bar{x}_N(t), p_N(t+h_N), \bar{u}_N(t)) \quad t \in T_N, \tag{4}$$

with the endpoint condition

$$p_N(t_1) = -\sum_{i=0}^{r_1+r_2} \lambda_{iN} \nabla \varphi_i(\bar{x}_N(t_1)). \tag{5}$$

Moreover, if endpoint constraints are absent, that is, $\varphi_i = 0$, $i = 1, \ldots, r_1 + r_2$, *the properness condition (e) is not necessary for validity of* (1)-(5).

It could be expected that this result could be generalized to nonsmooth functions φ_i, $i = 0, \ldots, r_1 + r_2$ by replacing condition (5) with

$$-p_N(t_1) \in \sum_{i=0}^{r_1+r_2} \lambda_{iN} \partial \varphi_i(\bar{x}_N(t_1)), \tag{6}$$

similarly to continuous-time problems (e.g., [4], Section 6), where $\partial \varphi$ denotes a relevant generalized derivative such as the subdifferential of convex analysis or the limiting subdifferential ([2], Section 4). However, it is shown in [3] that the above conjecture is not true even in the case of a convex nonsmooth cost function φ_0 and in the absence of constraints. The main objective of this paper is to replace (5) with a similar condition, for which the "nonsmooth" AMP holds true. Such condition is given in the following theorem.

Theorem 2. (Nonsmooth Approximate Maximum Principle) *Assume that the functions defining the cost and the inequality constraints* φ_i, $i = 0, \ldots, r_1$ *are convex, and the functions defining the equality constraints* φ_i, $i = r_1 + 1, \ldots, r_1 + r_2$ *are smooth. Let the sequence* $\bar{x}_N(t_1)$ *converge to some point* $\tilde{x}$ *as* $N \to \infty$ *and let the assumptions (a),(c),(d), and (e) of Theorem 1 hold. Then for any* $\varepsilon > 0$ *there exist numbers* λ_{iN}, $i = 0, \ldots, r_1 + r_2$ *such that for sufficiently small* h_N *conditions* (1)-(4) *hold with the endpoint condition for* p_N *given by*

$$-p_N(t_1) \in \sum_{i=0}^{r_1} \lambda_{iN} \partial \varphi_i(\tilde{x}) + \sum_{i=r_1+1}^{r_1+r_2} \lambda_{iN} \nabla \varphi_i(\tilde{x}) \tag{7}$$

Here and below $\partial \varphi$ denotes the subdifferential of convex analysis. Note that the subdifferentials in (7) are evaluated not at $\bar{x}_N(t_1)$, but at the limiting point $\tilde{x}$. Theorem 2 provides a generalization of a nonsmooth maximum principle ([4], Section 6) from continuous to discrete time.

II. Outline of the Proof of Theorem 2

Throughout the proof we assume that there are no equality constraints, that is, $r_2 = 0$.

Lemma 1. It can be assumed without loss of generality that $\bar{x}_N(t_1)$ is the *unique* point of minimim of φ_0 over the endpoints of admissible trajectories of (P_N).

Proof of Lemma 1. Assume that $\bar{x}_N(t_1)$ is, possibly, not unique point of minimim of φ_0 over the endpoints of admissible trajectories of (P_N). If Theorem 2 is true under the uniqueness assumption, we can consider the problem of minimization of

$$\tilde{\varphi}(x_N(t_1)) := \varphi_0(x_N(t_1)) + |x_N(t_1) - \bar{x}_N(t_1)|^2$$

over admissible trajectories of (P_N). Clearly, $\bar{x}_N(t_1)$ is the unique minimizer of $\tilde{\varphi}(x_N(t_1))$. Applying now Theorem 2 to the optimal process $(\bar{x}_N, \bar{u}_N)$, we deduce, for any $\varepsilon > 0$, the existence of numbers λ_{iN}, $i = 0, \ldots, r_1$ satisfying (2), (3), adjoint trajectory $\tilde{p}_N$ satisfying

$$\tilde{p}_N(t) = \tilde{p}_N(t+h_N) + h_N \frac{\partial H}{\partial x}(t, \bar{x}_N(t), \tilde{p}_N(t+h_N), \bar{u}_N(t)), \quad t \in T_N,$$

with endpoint condition

$$-\tilde{p}_N(t_1) = \sum_{i=0}^{r_1} \lambda_{iN} y_i^* + 2\lambda_{0N}(\tilde{x} - \bar{x}_N(t_1))$$

for some $y_i^* \in \partial \varphi_i(\tilde{x})$, $i = 0, \ldots, r_1$ such that there holds the approximate maximum condition

$$H(t, \bar{x}_N(t), \tilde{p}_N(t+h_N), \bar{u}_N(t)) \geq \max_{u \in U(t)} H(t, \bar{x}_N(t), \tilde{p}_N(t+h_N), u) - \varepsilon, \; t \in T_N$$

Taking into account that $|\tilde{x} - \bar{x}_N(t_1)| < \varepsilon$ for sufficiently large N and defining p_N by (4) with endpoint condition

$$p_N(t_1) := -\sum_{i=0}^{r_1} \lambda_{iN} y_i^* \in -\sum_{i=0}^{r_1} \lambda_{iN} \partial \varphi_i(\tilde{x})$$

we obtain the approximate maximum condition in the form

$$H(t, \bar{x}_N(t), p_N(t+h_N), \bar{u}_N(t)) \geq \max_{u \in U(t)} H(t, \bar{x}_N(t), p_N(t+h_N), u) - M\varepsilon, \; t \in T_N$$

with some constant $M > 0$. Lemma 1 is proved.

Lemma 2. Let Ω be a closed convex set in a finite-dimensional space, sequences $\{z_n\}, \{y_n\}$ are such that $z_n \in \Omega$, $z_n + y_n \notin \Omega$, $y_n \to 0$, and $z_n \to \bar{z}$ as $n \to \infty$. Then there exists a sequence $\{c_n\}$ with $|c_n| = o(|y_n|)$ as $n \to \infty$ such that $\bar{z} + y_n + c_n \notin \operatorname{int} \Omega$ for all sufficiently large n.

Proof of Lemma 2. Let $\alpha_n = \sup\{\alpha \geq 0 | \, z_n + \alpha y_n \in \Omega\}$. The supremum is reached for some $\alpha_n \in [0, 1)$ due to closedness and convexity of Ω. Denote

$$w_n = z_n + \alpha_n y_n$$

By construction w_n belongs to the boundary of Ω and $w_n \to \bar{z}$. It is easy to see that the ray $w_n + \beta y_n = z_n + (\alpha_n + \beta) y_n$,

$\beta > 0$ does not intersect Ω, since otherwise it would contradict the definition of α_n. Let A be the set of the limiting points of $y_n/|y_n|$, that is,

$$A = \{\xi |\, \xi = \lim_{n\to\infty} \frac{y_n}{|y_n|} \text{ along a subsequence}\}$$

and

$$B = \bigcup_{\beta>0} (\bar{z} + \beta A)$$

We claim that

$$B \cap \operatorname{int}\Omega = \emptyset\,. \tag{8}$$

Indeed, otherwise there would exist $\beta^* > 0$ and $\xi \in A$ such that such that

$$\bar{z} + \beta^*\xi \in \operatorname{int}\Omega\,,$$

which would imply that

$$w_n + \beta^* \frac{y_n}{|y_n|} \in \Omega$$

along a subsequence for sufficiently large n, contradicting the assertion that $w_n + \beta y_n$, $\beta > 0$ does not intersect with Ω. We have

$$\bar{z} + y_n = \bar{z} + |y_n| \left(\frac{y_n}{|y_n|}\right) \in \bar{z} + |y_n|A + \tilde{c}_n$$

with $\tilde{c}_n$ of magnitude of order $o(|y_n|)$. This implies that

$$\bar{z} + y_n - \tilde{c}_n \in \bar{z} + |y_n|A \subset B,$$

hence, $\bar{z} + y_n - \tilde{c}_n \notin \operatorname{int}\Omega$ via (8). Setting now $c_n = -\tilde{c}_n$ we complete the proof of Lemma 2.

Proof of Theorem 2.

Denote the constraint vector function

$$\varphi(x) := (\varphi_0(x), \ldots, \varphi_{r_1}(x))$$

Form the set E as

$$E = \{(x, \nu_0, \ldots, \nu_{r_1}) \in I\!R^{n+r_1+1} |\, \varphi_i(x) \le \nu_i,\ i = 0, \ldots, r_1\}$$

E represents a combination of the epigraphs of φ_i; in particular, if the endpoint constraints are not present, E is the epigraph of φ_0. Due to convexity of φ_i, $i = 0, \ldots, r_1$, E is convex.

Take $\tau(N) \in T_N$, $v(N) \in U(\tau(N))$ and consider the needle variation of the optimal control

$$u_N(t) = \begin{cases} v(N), & t = \tau(N) \\ \bar{u}_N(t), & t \in T_N,\, t \ne \tau(N)\,. \end{cases} \tag{9}$$

Denote the corresponding trajectory by $x_N(\cdot)$ and the corresponding increment of trajectory by $\Delta x_N(\cdot) = x_N(\cdot) - \bar{x}_N(\cdot)$. Apply Lemma 2 with $\Omega = E$, $z_N = (\bar{x}_N(t_1), \varphi(\bar{x}_N(t_1)))$ and $y_N = (\Delta x_N(t_1), 0, \ldots, 0)$. By construction

$$z_N \to (\tilde{x}, \varphi(\tilde{x})), \quad y_N \to 0 \text{ as } N \to \infty$$

and

$$z_N + y_N = (\bar{x}_N(t_1) + \Delta x_N(t_1), \varphi(\bar{x}_N(t_1))) \notin E,$$

since, otherwise, we would have $\varphi_i(\bar{x}_N(t_1) + \Delta x_N(t_1)) \le \varphi_i(\bar{x}_N(t_1))$, $i = 0, \ldots, r_1$ contradicting the uniqueness of the minimizer, which can be assumed due to Lemma 1. From Lemma 2 we obtain that

$$(\tilde{x} + \Delta x_N(t_1), \varphi(\tilde{x})) + c_N \notin \operatorname{int} E, \tag{10}$$

for some c_N of order $o(|\Delta x_N(t_1)|) = o(h_N)$.

Denote

$$\begin{aligned}\Omega_N =& \operatorname{co}\{(\tilde{x} + \Delta x_N(t_1), \varphi(\tilde{x}))\} \\ =& (\tilde{x}, \varphi(x)) + \operatorname{co}\{(\Delta x_N(t_1), 0)\},\end{aligned}$$

where convexification is taken with respect to increments $\Delta x_N(t_1)$, corresponding to all admissible needle variations of the optimal control. It is possible to prove the following extension of formula (10): There exists a vector c_N of a magnitude of order $o(h_N)$ such that

$$(\Omega_N + c_N) \cap \operatorname{int} E = \emptyset$$

Via convex separation it can be shown that there exists a nonzero vector $(x_N^*, \lambda_{0N}, \ldots, \lambda_{r_1 N})$ from the normal cone to E at the point $(\tilde{x}, \varphi(\tilde{x})) \in \Omega_N \cap E$ such that

$$\langle x_N^*, \Delta x_N(t_1)\rangle + o(h_N) \le 0\,. \tag{11}$$

Furthermore, due to the structure of the normal cone to E at $(\tilde{x}, \varphi(\tilde{x}))$ we have

$$\lambda_{iN}(\varphi_i(\tilde{x}) - \alpha_{iN}) = 0, \quad i = 1, \ldots, r_1,$$

$$\lambda_{iN} \ge 0,\ \ i = 0, \ldots, r_1, \quad x_N^{*2} + \sum_{i=0}^{r_1} \lambda_{iN}^2 = 1$$

and

$$x_N^* \in -\sum_{i=0}^{r_1} \lambda_{iN} \partial\varphi_i(\tilde{x})$$

(see [1], Theorem 3.3). The last inclusion implies that λ_{iN}, $i = 0, \ldots, r_1$ can be re-normalized to satisfy $\sum_{i=0}^{r_1} \lambda_{iN}^2 = 1$.

Set $p_N(\cdot)$ to be the solution of (4) with endpoint condition $p_N(t_1) = x_N^*$. It can be shown (see, for example, [1], Section 14.2 for details) that (11) implies, for the needle variation (9),

$$\begin{aligned} h_N[&H(\tau, \bar{x}_N(\tau), p_N(\tau + h_N), v) - \\ &H(\tau, \bar{x}_N(\tau), p_N(\tau + h_N), \bar{u}_N(\tau))] + o(h_N) \le 0, \end{aligned}$$

and the approximate maximum condition (1) follows. This completes the proof of Theorem 2.

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