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Finite Frequency Approaches to H_∞ Filtering for Continuous-Time State-Delayed Systems

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Abstract— **This paper is concerned with the problem of H_∞ filter design for linear continuous-time state-delayed systems with finite frequency specifications. The developed approaches in the paper are to design a filter guaranteeing an H_∞ performance bound in a finite frequency range for delayed systems. To reduce conservatism, delay-partitioning idea is exploited to derive a new finite frequency bounded real lemma (BRL). By utilizing the generalized Kalman-Yakubovich-Popov lemma and projection lemma, the conditions on the existence of H_∞ filters for different finite frequency ranges are unified in terms of solving a set of linear matrix inequalities.**

I. INTRODUCTION

State estimation has significant meanings in theory and practical engineering in control and signal processing field. If the external disturbance satisfies the Gaussian white noise assumption, and $a\ priori$ information of the statistic about the external disturbance is available, the well-known Kalman filtering is the optimal estimation scheme [12]. But in practice, this rigid condition is not always satisfied [1]. In such cases, many other estimation strategies have been developed, such as H_∞ filtering [3], $\mathcal{L}_2$-$\mathcal{L}_\infty$ filtering [13], and $\mathcal{L}_1$ filtering [16]. During the past two decades, many efforts have been made to H_∞ filter design for various systems [19], [20], [22]. Moreover, it is well-known that actuators, sensors and transmission lines in feedback loops especially in networked controlled systems always introduce after-affect more or less, and delays often deteriorate the system performance and even could be the source of instability. Therefore, more and more attention has been focused on the problem of H_∞ filtering for various time-delay systems [8], [14], [15], [17], [21], [23].

One the other hand, it should be pointed out that the objective of all the H_∞ filtering results mentioned previously is to minimize the H_∞ norm of the filtering error system in the entire frequency (EF) range. If the frequency range of external noise is known but finite, the "global" optimization in the EF range will bring about overdesign, which causes the above H_∞ filtering methods to be much conservative. A milestone in looking for techniques to overcome this obstacle is the generalized Kalman-Yakubovich-Popov (GKYP) lemma [10], which has been shown to be as an effective fundamental tool in various engineering design problems with finite frequency specifications, such as feedback control synthesis [11], structure design integration [9]. For discrete-time delayed systems, [24] investigated the finite frequency $\mathcal{H}_\infty$ filtering by the GKYP lemma and proposed a design method in terms of linear matrix inequalities (LMIs). However, to the authors' knowledge, *finite frequency H_∞ filtering for continuous-time delayed systems* hasn't been addressed in the literature and it is not an easy work to extend the results in [24] to the discrete-time setting. Moreover, the results there still leave much room for further improvement. These aspects motivate our research.

In the paper, we will study the problem of H_∞ filtering for continuous-time state-delayed systems with finite frequency specifications. The disturbance is assumed to reside in a known low/middle/high frequency (LF/MF/HF) range. By virtue of the generalized KYP lemma [10] and projection lemma [5], firstly a finite frequency bounded real lemma (BRL) is proposed for analyzing finite frequency H_∞ performance of continuous-time systems with state delay and then filter design methods are obtained in terms of solving a set of LMIs.

Notation: $\mathcal{N}_X$ is arbitrary matrices whose columns form a basis of the nullspace of X. $\mathbf{I}$ denotes an identity matrix with appropriate dimension, and $\mathbf{H}_n$ stands for the set of $n \times n$ Hermitian matrices. For matrices Φ and P, $\Phi \otimes P$ means the Kronecker product. In block matrices, we use "$*$" to denote the terms that can be induced by conjugate symmetry. In addition, $\mathrm{He}\{A\}$ indicates $A^* + A$, and $\sigma_{\max}(\bullet)$ denotes maximum singular value of transfer function. For $G \in \mathbb{C}^{n\times m}$ and $\Pi \in \mathbf{H}_{n+m}$, a function $\upsilon : \mathbb{C}^{m\times n} \times \mathbf{H}_{n+m} \to \mathbf{H}_m$ is defined by

$$\upsilon(G,\Pi) \triangleq \begin{bmatrix} G \\ \mathbf{I}_m \end{bmatrix}^* \Pi \begin{bmatrix} G \\ \mathbf{I}_m \end{bmatrix}. \qquad (1)$$

II. PROBLEM FORMULATION AND PRELIMINARIES

A. Problem Statement

Consider the following linear time-invariant continuous-time system with a state delay:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-d) + Bw(t), \\ y(t) &= Cx(t) + C_d x(t-d) + Dw(t), \\ z(t) &= Hx(t) + H_d x(t-d) + Lw(t), \\ x(t) &= \phi(t),\ \forall t = [-d, 0], \end{aligned} \qquad (2)$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state vector; $y(t) \in \mathbb{R}^{n_y}$ is the measured output; $z(t) \in \mathbb{R}^{n_z}$ is the signal to be estimated;

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$w(t) \in \mathbb{R}^{n_w}$ is the noise input belonging to $\mathcal{L}_2[0,\infty)$ and the frequency of $w(t)$ resides in a known but finite range $\mathbf{W}$; d is the time-invariant and known state delay; $\phi(t)$ is the initial state vector function over $[-d,0]$; A, A_d, B, C, C_d, D, H, H_d and L are known real system matrices and time-constant with appropriate dimension. The finite frequency range $\mathbf{W}$ is assumed to be the general LF/MF/HF range, that is,

$$\mathbf{W} \triangleq \begin{cases} \{\omega \in \mathbb{R} |\, |\omega| \leq \omega_l, \omega_l \geq 0\}, & \text{(LF)} \\ \{\omega \in \mathbb{R} |\, \omega_1 \leq \omega \leq \omega_2, 0 \leq \omega_1 \leq \omega_2\}, & \text{(MF)} \\ \{\omega \in \mathbb{R} |\, |\omega| \geq \omega_h, \omega_h \geq 0\}, & \text{(HF)} \end{cases} \tag{3}$$

We desire to design a full-order filter with its output $z_F(t)$ being the estimation of the signal $z(t)$. Consider the following state-space realization of a desired filter:

$$\begin{aligned} \dot{x}_F(t) &= A_F x_F(t) + B_F y(t),\ x_F(0) = \mathbf{0}, \\ z_F(t) &= C_F x_F(t) + D_F y(t), \end{aligned} \tag{4}$$

where $x_F(t) \in \mathbb{R}^n$ is the filter state vector; $y(t)$ is the measured output in original system in (2) as the input of filter; A_F, B_F, C_F and D_F with appropriate dimensions are real filter matrices to be determined. Augmenting the system in (2) to include the filter states in (4), we have the following filtering error system:

$$\begin{aligned} \dot{\xi}(t) &= \bar{A}\xi(t) + \bar{A}_d K \xi(t-d) + \bar{B} w(t), \\ e(t) &= \bar{C}\xi(t) + \bar{C}_d K \xi(t-d) + \bar{D} w(t), \\ \xi(t) &= [\phi^T(t), 0]^T,\ \forall t = [-d, 0], \end{aligned} \tag{5}$$

where $e(t) \triangleq z(t) - z_F(t)$, $\bar{x}(k) = \begin{bmatrix} x^T(k) & x_F^T(k) \end{bmatrix}^T$ and

$$\bar{A} = \begin{bmatrix} A & \mathbf{0} \\ B_F C & A_F \end{bmatrix}, \bar{A}_d = \begin{bmatrix} A_d \\ B_F C_d \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ B_F D \end{bmatrix},$$
$$\bar{C} = \begin{bmatrix} H - D_F C & -C_F \end{bmatrix},\ \bar{C}_d = H_d - D_F C_d,$$
$$\bar{D} = L - D_F D,\ K = \begin{bmatrix} \mathbf{I}_{n_x} & \mathbf{0}_{n_x \times n_x} \end{bmatrix}.$$

Accordingly, the transfer function of the filtering error system in (5) is given by

$$G_s \triangleq (\bar{C} + \mathrm{e}^{-ds}\bar{C}_d K)(s\mathbf{I} - \bar{A} - \mathrm{e}^{-ds}\bar{A}_d K)^{-1}\bar{B} + \bar{D} \tag{6}$$

with $s = j\omega$ being the Laplace operator. We can always find a set $\mathbf{S}$ of complex number such that $s \in \mathbf{S}$ is the equivalent characterization of $\omega \in \mathbf{W}$ in (3) with $s = j\omega$. For brevity and convenience, both $s \in \mathbf{S}$ and $\omega \in \mathbf{W}$ denote the finite frequency specifications in the context.

The finite frequency H_∞ filtering problem for continuous-time delayed systems is formulated as: Find a filter of the state-space realization form in (4) for the state-delayed system in (2) such that

(i) the filtering error system in (5) is asymptotically stable;

(ii) under zero-initial condition, the following finite frequency index holds:

$$\sup \sigma_{\max}(G_s) < \gamma \quad \forall s \in \mathbf{S} \tag{7}$$

for a given proper positive scalar γ, or equivalently, the following index holds:

$$\|e\|_2 < \gamma \|w\|_2 \tag{8}$$

for all nonzero $w(t) \in \mathcal{L}_2[0,\infty)$ with the frequency of $w(t)$ satisfying $\omega \in \mathbf{W}$ and a given proper positive scalar γ.

TABLE I

THE VALUES OF Ψ FOR LF/MF/HF

$\mathbf{W}$	LF ($\lvert\omega\rvert \leq \omega_l$)	MF ($\omega_1 \leq \omega \leq \omega_2$)	HF ($\lvert\omega\rvert \geq \omega_h$)
Ψ	$\begin{bmatrix} -1 & 0 \\ 0 & \omega_l^2 \end{bmatrix}$	$\begin{bmatrix} -1 & j\omega_c \\ -j\omega_c & -\omega_1\omega_2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -\omega_h^2 \end{bmatrix}$

B. Preliminaries

Define the finite frequency set $\mathbf{S}$ as

$$\mathbf{S}(\Phi, \Psi) \triangleq \{s \in \mathbb{C} | \upsilon(s, \Phi) = 0,\ \upsilon(s, \Psi) \geq 0\}. \tag{9}$$

According to [10], let $\Phi \triangleq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then for all $s \in \mathbf{S}(\Phi, 0)$, $s = j\omega$ with $\omega \in \mathbb{R}$, and $\mathbf{W}$ in (3) can be characterized by Ψ, which is shown in Table I, where $\omega_c = (\omega_1 + \omega_2)/2$. For technical reasons, we assume $\infty \in \mathbf{S}$ if $\mathbf{W}$ is unbounded.

To conclude this section, we introduce the following two essential lemmas.

Lemma 1 (GKYP Lemma [10]): Let $\Theta \in \mathbf{H}_{n_1+n_2}$, $F \in \mathbb{C}^{2n_1 \times (n_1+n_2)}$ and Φ, $\Psi \in \mathbf{H}_2$ be given such that $\mathbf{S}$ in (9) represents curves on the complex plane. Define

$$\Gamma_s \triangleq \begin{cases} \begin{bmatrix} \mathbf{I}_{n_1} & -s\mathbf{I}_{n_1} \end{bmatrix}, & (s \in \mathbb{C}), \\ \begin{bmatrix} 0 & -\mathbf{I}_{n_1} \end{bmatrix}, & (s = \infty). \end{cases} \tag{10}$$

Then

$$\mathcal{N}_{\Gamma_s F}^* \Theta \mathcal{N}_{\Gamma_s F} < 0 \ \forall s \in \mathbf{S}(\Phi, \Psi)$$

is equivalent to

$$F^*(\Phi \otimes P + \Psi \otimes Q)F + \Theta < 0 \ \exists P, Q \in \mathbf{H}_{n_1}, Q > 0.$$

Lemma 2 (Projection Lemma [5]): Let X, Z, Σ be given. There exists a matrix Y satisfying

$$\mathrm{He}(X^* Y Z) + \Sigma < 0$$

if and only if the following projection inequalities hold:

$$\mathcal{N}_X^* \Sigma \mathcal{N}_X < 0,\ \mathcal{N}_Z^* \Sigma \mathcal{N}_Z < 0.$$

III. MAIN RESULTS

A. Delay-dependent finite frequency BRL

First we present the following finite frequency BRL ensuring the specifications in (i) and (ii).

Theorem 1: Consider the system in (2) and suppose that a scalar $\gamma > 0$, and an integer $m > 0$ and a scalar $\tau > 0$ satisfying $d = m\tau$ are given. For a specified filter, the filtering error system in (5) is asymptotically stable with an H_∞ performance bound γ in the frequency range $\mathbf{S}$, if there exist matrices $P_1 \in \mathbf{H}_{2n_x}$, $0 < P_2 \in \mathbf{H}_{2n_x}$, $0 < Q \in \mathbf{H}_{2n_x}$, $R_1 = \begin{bmatrix} R_{1,1} & R_{1,2} \\ R_{1,2}^* & R_{1,3} \end{bmatrix} \in \mathbf{H}_{mn_x}$, $0 < R_2 = \begin{bmatrix} R_{2,1} & R_{2,2} \\ R_{2,2}^* & R_{2,3} \end{bmatrix} \in \mathbf{H}_{mn_x}$, $0 < S_j \in \mathbf{H}_{n_x}$ and $Y_j \in \mathbb{C}^{4n_x \times 2n_x}$, $j = 1, 2$, such that

$$\begin{bmatrix} -\mathbf{I}_{n_z} & M \\ M^* & \Sigma + \mathrm{He}(X_1^* Y_1 Z_1) \end{bmatrix} < 0, \tag{11}$$

$$\Xi_2 + \mathrm{He}(X_2^* Y_2 Z_2) < 0, \quad (12)$$

where

$$M \triangleq [\ \mathbf{0}_{n_z\times 2n_x} \quad M_1\ ],\ \Sigma \triangleq \mathrm{diag}\{\Xi_1+\Xi_Q, -\gamma^2\mathbf{I}_{n_w}\},$$
$$X_1 \triangleq [\ \mathbf{I}_{4n_x} \quad \mathbf{0}_{4n_x\times(mn_x+n_w)}\ ],\ Z_1 \triangleq [\ -\mathbf{I}_{2n_x} \quad M_2\ ],$$
$$M_1 \triangleq [\ \bar{C} \quad \bar{C}_d E_d \quad \bar{D}\ ],\ M_2 \triangleq [\ \bar{A} \quad \bar{A}_d E_d \quad \bar{B}\ ],$$
$$X_2 \triangleq [\ \mathbf{I}_{4n_x} \quad \mathbf{0}_{4n_x\times mn_x}\ ],\ Z_2 \triangleq [\ -\mathbf{I}_{2n_x} \quad N\ ],$$
$$\Xi_Q \triangleq \begin{bmatrix} \Psi\otimes Q & \mathbf{0}_{4n_x\times mn_x} \\ \mathbf{0}_{mn_x\times 4n_x} & \mathbf{0}_{mn_x\times mn_x} \end{bmatrix},\ N \triangleq [\ \bar{A} \quad \bar{A}_d E_d\ ],$$
$$\Xi_j \triangleq \begin{bmatrix} \tau^2 K^T S_j K & P_j & \mathbf{0}_{2n_x\times mn_x} \\ P_j & \Xi_{j,1} & \Xi_{j,2} \\ \mathbf{0}_{mn_x\times 2n_x} & \Xi_{j,2}^* & \Xi_{j,3} \end{bmatrix},$$
$$\Xi_{j,1} \triangleq K^T[-S_j + R_{j,1}]K,\ \Xi_{j,2} \triangleq K^T[R_{j,2}E_R + S_j E_S],$$
$$\Xi_{j,3} \triangleq E_R^T R_{j,3} E_R - R_j - E_S^T S_j E_S,\ j=1,2,$$
$$E_R \triangleq [\ \mathbf{I}_{(m-1)n_x} \quad \mathbf{0}_{(m-1)n_x\times n_x}\ ],$$
$$E_S \triangleq [\ \mathbf{I}_{n_x} \quad \mathbf{0}_{n_x\times(m-1)n_x}\ ].$$

Proof: For introducing the delay partitioning idea in [7], we fraction d into m subintervals, i.e., $d = m\tau$ with m being a positive integer, and define:

$$T_s \triangleq [\ T_{1s}^* \quad T_{2s}^* \quad \mathbf{I}_{n_w}\ ]^*,$$
$$T_{1s} \triangleq (s\mathbf{I} - \bar{A} - \mathrm{e}^{-\tau s}\bar{A}_d K)^{-1}\bar{B},$$
$$T_{2s} \triangleq [\ (\mathrm{e}^{-\tau s}KT_{1s})^* \quad \cdots \quad (\mathrm{e}^{-m\tau s}KT_{1s})^*\ ]^*. \quad (13)$$

It follows that

$$G_s = \underbrace{[\ \bar{C} \quad \bar{C}_d E_d \quad \bar{D}\ ]}_{M_1} T_s,\ sT_{1s} = \underbrace{[\ \bar{A} \quad \bar{A}_d E_d \quad \bar{B}\ ]}_{M_2} T_s, \quad (14)$$

where $E_d \triangleq [\ \mathbf{0}_{n_x\times(m-1)n_x} \quad \mathbf{I}_{n_x}\ ]$. By defining

$$\Pi \triangleq \begin{bmatrix} \mathbf{I}_{n_z} & \mathbf{0} \\ \mathbf{0} & -\gamma^2\mathbf{I}_{n_w} \end{bmatrix},\ E_1 \triangleq [\ \mathbf{0}_{n_w\times(m+2)n_x} \quad \mathbf{I}_{n_w}\ ],$$

the finite frequency H_∞ specification in (7) can be rewritten as the following FDI:

$$\upsilon(G,\Pi) < 0\ \forall s\in\mathbf{S}.$$

From (14), this can be further written as

$$T_s^*\left(M_1^*M_1 - \gamma^2 E_1^* E_1\right)T_s < 0\quad \forall s\in\mathbf{S}. \quad (15)$$

Now we show that (15) follows from (11). Due to $s = j\omega$, for any $P_1\in\mathbf{H}_{2n_x}$, $R_1\in\mathbf{H}_{mn_x}$, and $0 < S_1\in\mathbf{H}_{n_x}$, we have the following relations:

$$\Delta_1 \triangleq (s^*+s)T_{1s}^* P_1 T_{1s} \equiv 0, \quad (16)$$
$$\Delta_2 \triangleq [(\mathrm{e}^{\tau s})^*\mathrm{e}^{\tau s} - 1]\,T_{2s}^* R_1 T_{2s} \equiv 0, \quad (17)$$
$$\begin{aligned}\Delta_3 &\triangleq [\tau^2 s^* s - 1 + \left(\mathrm{e}^{-\tau s}\right)^* + \mathrm{e}^{-\tau s} - \left(\mathrm{e}^{-\tau s}\right)^*\mathrm{e}^{j\tau s}]T_{1s}^* \\ &\quad\times K^T S_1 K T_{1s} \\ &= \left[\tau^2\omega^2 - 2 + 2\cos(\tau\omega)\right] T_{1s}^* K^T S_1 K T_{1s} \\ &= 4\left[\left(\frac{\tau\omega}{2}\right)^2 - \sin^2\left(\frac{\tau\omega}{2}\right)\right] T_{1s}^* K^T S_1 K T_{1s} \geq 0. \end{aligned} \quad (18)$$

Defining

$$E_2 \triangleq [\ \mathbf{I}_{2n_x} \quad \mathbf{0}_{2n_x\times(mn_x+n_w)}\ ],\ F_1 \triangleq [\ M_2^* \quad E_2^*\ ]^*,$$
$$E_3 \triangleq [\ \mathbf{0}_{mn_x\times 2n_x} \quad \mathbf{I}_{mn_x} \quad \mathbf{0}_{mn_x\times n_w}\ ],\ F_2 \triangleq [\ F_1^* \quad E_3^*\ ]^*$$

and combining (16)-(18), we have

$$F_2^*\Xi_1 F_2 = \Delta_1 + \Delta_2 + \Delta_3 \geq 0. \quad (19)$$

Thus the FDI in (15) is ensured by

$$T_s^*(M_1^*M_1 - \gamma^2 E_1^*E_1 + F_2^*\Xi_1 F_2)T_s < 0\ \forall s\in\mathbf{S}. \quad (20)$$

Further, since $\Gamma_s F T_s = \mathbf{0}$ with Γ_s defined in (10) and $n_1 = 2n_x$ can be derived from its definition, one knows that the FDI in (20) holds if the following one holds:

$$\mathcal{N}_{\Gamma_s F}^*\left(M_1^*M_1 - \gamma^2 E_1^*E_1 + F_2^*\Xi_1F_2\right)\mathcal{N}_{\Gamma_s F} < 0\ \forall s\in\mathbf{S}. \quad (21)$$

By Lemma 1 with $\Theta = M_1^*M_1 - \gamma^2E_1^*E_1 + F_2^*\Xi_1F_2$, (21) is equivalent to that the LMI

$$\Theta + F_1^*\left(\Phi\otimes P + \Psi\otimes Q\right)F_1 < 0 \quad (22)$$

holds for some $P, Q\in\mathbf{H}_{2n_x}$ and $Q > 0$. Redefining P_1 by $P_1 + P$ and performing some routine matrix manipulations to the LMI in (22), we have (22) can be exactly rewritten as

$$\begin{bmatrix} M_2 \\ \mathbf{I}_{(2+m)n_x+n_w} \end{bmatrix}^*(\Sigma + \Sigma_{M_1})\begin{bmatrix} M_2 \\ \mathbf{I}_{(2+m)n_x+n_w} \end{bmatrix} < 0, \quad (23)$$

where $\Sigma_{M_1} \triangleq \begin{bmatrix} \mathbf{0}_{2n_x\times 2n_x} & \mathbf{0}_{2n_x\times[(2+m)n_x+n_w]} \\ \mathbf{0}_{[(2+m)n_x+n_w]\times 2n_x} & M_1^*M_1 \end{bmatrix}$.

Note that one can set

$$\mathcal{N}_{X_1} = \begin{bmatrix} \mathbf{0}_{4n_x\times(mn_x+n_w)} \\ \mathbf{I}_{mn_x+n_w} \end{bmatrix},\ \mathcal{N}_{Z_1} = \begin{bmatrix} M_2 \\ \mathbf{I}_{(2+m)n_x+n_w} \end{bmatrix}.$$

By this setting and Lemma 2, we know that (23) and

$$\mathcal{N}_{X_1}^*\left(\Sigma + \Sigma_{M_1}\right)\mathcal{N}_{X_1} < 0 \quad (24)$$

hold if and only if there exists $Y_1\in\mathbb{C}^{4n_x\times 2n_x}$ such that

$$\Sigma + \Sigma_{M_1} + \mathrm{He}(X_1^*Y_1Z_1) < 0 \quad (25)$$

holds. Further by carefully calculating (24), we have that

$$\mathcal{N}_{X_1}^*\left(\Sigma + \Sigma_{M_1}\right)\mathcal{N}_{X_1} = \begin{bmatrix} \Xi_{j,3} + \left(\bar{C}_dE_d\right)^*\bar{C}_dE_d & \left(\bar{C}_dE_d\right)^*\bar{D} \\ \bar{D}^*\bar{C}_dE_d & \bar{D}^*\bar{D} - \gamma^2\mathbf{I}_{n_w} \end{bmatrix} < 0 \quad (26)$$

naturally holds from (23), which means that (23) is equivalent to (25). Then by the Schur complement [2], it follows that (25) is exactly (11). According to the above derivation, (11) is a sufficient condition ensuring (15), which completes the first part of the proof.

Then, we demonstrate that (12) is an asymptotic stability condition for system (5). Define vectors:

$$g(t) \triangleq \begin{bmatrix} \xi(t) \\ g_1(t) \\ w(t) \end{bmatrix},\ g_1(k) \triangleq \begin{bmatrix} x(t-\tau) \\ \vdots \\ x(t-m\tau) \end{bmatrix}. \quad (27)$$

where m has the same meaning as in the first part of the proof. Construct a Lyapunov-Krasovskii functional (LKF) candidate as

$$V_2(t) \triangleq V_{2,1}(t) + V_{2,2}(t) + V_{2,3}(t), \quad (28)$$

where

$$V_{2,1}(t) \triangleq \xi^*(t)P_2\xi(t),$$
$$V_{2,2}(t) \triangleq \int_{t-\tau}^{t} g_1^*(\eta+\tau)R_2 g_1(\eta+\tau)d\eta,$$
$$V_{2,3}(t) \triangleq \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^*(\eta)S_2\dot{x}(\eta)d\eta d\theta.$$

This part can be completed by following the derivation of Proposition 1 in [7] and further applying Projection lemma, and is omitted for brevity. ■

B. Finite Frequency H_∞ Filter Design

Based on Theorem 1, we have the following sufficient condition on the existence of finite frequency H_∞ filters for the delayed system in (2).

Theorem 2: Consider the system in (2) and suppose that a scalar $\gamma > 0$, and an integer $m > 0$ and a scalar $\tau > 0$ satisfying $d = m\tau$ are given. A filter of form (4) exists such that the filtering error system in (5) is asymptotically stable and of an H_∞ performance bound γ in the frequency range $\mathbf{S}$, if there exist routine matrices $\mathcal{Y}_{j,1} \in \mathbb{C}^{n_x\times n_x}$, $\mathcal{Y}_{j,2} \in \mathbb{C}^{n_x\times n_x}$, $\mathcal{Y}_{j,3} \in \mathbb{C}^{n_x\times n_x}$, $\mathcal{Y}_{j,4} \in \mathbb{C}^{n_x\times n_x}$, $\mathcal{Y}_5 \in \mathbb{C}^{n_x\times n_x}$, $\mathcal{A}_\mathcal{F} \in \mathbb{C}^{n_x\times n_x}$, $\mathcal{B}_\mathcal{F} \in \mathbb{C}^{n_x\times n_y}$, $\mathcal{C}_\mathcal{F} \in \mathbb{C}^{n_z\times n_x}$, $\mathcal{D}_\mathcal{F} \in \mathbb{C}^{n_z\times n_y}$, and matrices $P_1 = \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{1,2}^* & P_{1,3}\end{bmatrix} \in \mathbf{H}_{2n_x}$, $0 < P_2 = \begin{bmatrix} P_{2,1} & P_{2,2} \\ P_{2,2}^* & P_{2,3}\end{bmatrix} \in \mathbf{H}_{2n_x}$, $0 < Q = \begin{bmatrix} Q_{1,1} & Q_{1,2} \\ Q_{1,2}^* & Q_{1,3}\end{bmatrix} \in \mathbf{H}_{2n_x}$, $R_1 = \begin{bmatrix} R_{1,1} & R_{1,2} \\ R_{1,2}^* & R_{1,3}\end{bmatrix} \in \mathbf{H}_{mn_x}$, $0 < R_2 = \begin{bmatrix} R_{2,1} & R_{2,2} \\ R_{2,2}^* & R_{2,3}\end{bmatrix} \in \mathbf{H}_{mn_x}$, $0 < S_j \in \mathbf{H}_{n_x}$, $j = 1, 2$, such that the following LMIs hold for scalars ε_j, $j = 1, \cdots, 4$:

$$\begin{bmatrix} -\mathbf{I}_{n_z} & \mathcal{M} \\ \mathcal{M}^* & \Sigma + \text{He}(\Upsilon_1)\end{bmatrix} < 0, \tag{29}$$
$$\Xi_2 + \text{He}(\Upsilon_2) < 0, \tag{30}$$

where

$$\mathcal{M} \triangleq [\ \mathbf{0}\quad \mathbf{0}\quad H - \mathcal{D}_\mathcal{F}C\quad -\mathcal{C}_\mathcal{F}\quad (H_d - \mathcal{D}_\mathcal{F}C_d)\,E_d\quad L - \mathcal{D}_\mathcal{F}D\ ],$$

$$\Upsilon_1 \triangleq \begin{bmatrix} -\mathcal{Y}_{1,1} & -\mathcal{Y}_5 & \mathcal{Y}_{1,1}A + \mathcal{B}_\mathcal{F}C & \mathcal{A}_\mathcal{F} & (\mathcal{Y}_{1,1}A_d + \mathcal{B}_\mathcal{F}C_d)E_d & \mathcal{Y}_{1,1}B + \mathcal{B}_\mathcal{F}D \\ -\mathcal{Y}_{1,2} & -\mathcal{Y}_5 & \mathcal{Y}_{1,2}A + \mathcal{B}_\mathcal{F}C & \mathcal{A}_\mathcal{F} & (\mathcal{Y}_{1,2}A_d + \mathcal{B}_\mathcal{F}C_d)E_d & \mathcal{Y}_{1,2}B + \mathcal{B}_\mathcal{F}D \\ -\mathcal{Y}_{1,3} & -\varepsilon_1\mathcal{Y}_5 & \mathcal{Y}_{1,3}A + \varepsilon_1\mathcal{B}_\mathcal{F}C & \varepsilon_1\mathcal{A}_\mathcal{F} & (\mathcal{Y}_{1,3}A_d + \varepsilon_1\mathcal{B}_\mathcal{F}C_d)E_d & \mathcal{Y}_{1,3}B + \varepsilon_1\mathcal{B}_\mathcal{F}D \\ -\mathcal{Y}_{1,4} & -\varepsilon_2\mathcal{Y}_5 & \mathcal{Y}_{1,4}A + \varepsilon_2\mathcal{B}_\mathcal{F}C & \varepsilon_2\mathcal{A}_\mathcal{F} & (\mathcal{Y}_{1,4}A_d + \varepsilon_2\mathcal{B}_\mathcal{F}C_d)E_d & \mathcal{Y}_{1,4}B + \varepsilon_2\mathcal{B}_\mathcal{F}D \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},$$

$$\Upsilon_2 \triangleq \begin{bmatrix} -\mathcal{Y}_{2,1} & -\mathcal{Y}_5 & \mathcal{Y}_{2,1}A + \mathcal{B}_\mathcal{F}C & \mathcal{A}_\mathcal{F} & (\mathcal{Y}_{2,1}A_d + \mathcal{B}_\mathcal{F}C_d)E_d \\ -\mathcal{Y}_{2,2} & -\mathcal{Y}_5 & \mathcal{Y}_{2,2}A + \mathcal{B}_\mathcal{F}C & \mathcal{A}_\mathcal{F} & (\mathcal{Y}_{2,2}A_d + \mathcal{B}_\mathcal{F}C_d)E_d \\ -\mathcal{Y}_{2,3} & -\varepsilon_3\mathcal{Y}_5 & \mathcal{Y}_{2,3}A + \varepsilon_3\mathcal{B}_\mathcal{F}C & \varepsilon_3\mathcal{A}_\mathcal{F} & (\mathcal{Y}_{2,3}A_d + \varepsilon_3\mathcal{B}_\mathcal{F}C_d)E_d \\ -\mathcal{Y}_{2,4} & -\varepsilon_4\mathcal{Y}_5 & \mathcal{Y}_{2,4}A + \varepsilon_4\mathcal{B}_\mathcal{F}C & \varepsilon_4\mathcal{A}_\mathcal{F} & (\mathcal{Y}_{2,4}A_d + \varepsilon_4\mathcal{B}_\mathcal{F}C_d)E_d \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},$$

TABLE II
ACHIEVED MINIMUM γ^* FOR DIFFERENT DELAY d AND FINITE FREQUENCY RANGES ($\varepsilon_1, \varepsilon_2 = 5, \varepsilon_3, \varepsilon_4 = 1$ FOR THEOREM 2)

Method	Frequency (rad/s)	d = 0.06	d = 0.6	d = 1.2
[6]	$0 \le \omega < \infty$	0.0760	1.7877	∞
[23]	$0 \le \omega < \infty$	0.0760	1.7876	∞
[18]	$0 \le \omega < \infty$	0.0760	1.7876	∞
Theorem 2 ($Q = 0$)	$0 \le \omega < \infty$	0.0744	1.3240	7.8350
Theorem 2 (LF)	$0 \le \omega \le 1.5$	0.0460	0.4875	2.2511
Theorem 2 (MF)	$1 \le \omega \le 20$	0.0703	1.3184	7.6450
Theorem 2 (HF)	$10 \le \omega < \infty$	0.0387	0.0872	0.0872

and Σ and Ξ_2 are in (11) and (12), respectively. Moreover, if the previous conditions are satisfied, an acceptable state-space realization of H_∞ filter is given by

$$A_F = \mathcal{Y}_5^{-1}\mathcal{A}_\mathcal{F}, B_F = \mathcal{Y}_5^{-1}\mathcal{B}_\mathcal{F}, C_F = \mathcal{C}_\mathcal{F}, D_F = \mathcal{D}_\mathcal{F}. \tag{31}$$

Proof: The proof is omitted for brevity. A similar proof can be found in [18]. ■

Remark 1: Noting that there are several tuning parameters ε_j involved in the inequality conditions in (29), (30), if these tuning parameters are prescribed in advance, the inequality conditions in Theorem 2 are LMIs. For ensuring that the LMIs in Theorem 2 have feasible solutions, it is possible to search the scalar parameter combination by using some numerical optimization search algorithm [4], [18], while this isn't the concern of this paper. In the later section, to design H_∞ filter for the illustrative examples, we just specify $\varepsilon_j, j = 1, \cdots, 4$ that can give feasible solutions.

Remark 2: [7] shows that larger m generates larger allowable maximum time delay d for the stability criterion. However, the decision variables would increase sharply, which means heavier computational burden. A reasonable trade-off of the contradiction between less conservatism and lighter computational burden is to let $m = 2$. As is shown in the example, $m = 2$ suffices to yield results better than that by the existing methods.

Remark 3: If $Q = 0$, i.e., $\Sigma_Q = 0$, the proposed results in the paper can be directly utilized for EF H_∞ filtering.

IV. AN ILLUSTRATIVE EXAMPLE

In this section, via an illustrative example, we exhibit the effectiveness and advantage of our proposed method.

Example 1: For the continuous-time state-delayed system model in (2), consider the following nominal matrix parameters borrowed from [18], [23] with delay d assumed to be time-invariant.

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.7\end{bmatrix},\ A_d = \begin{bmatrix} -1 & -1 \\ -1 & -1\end{bmatrix},\ B = \begin{bmatrix} -0.5 \\ 2\end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1\end{bmatrix},\ C_d = \begin{bmatrix} 1 & 2\end{bmatrix},\ D = 1,$$
$$H = \begin{bmatrix} 2 & 1\end{bmatrix},\ H_d = \begin{bmatrix} 0 & 0\end{bmatrix},\ L = 0. \tag{32}$$

For different frequency ranges, we design filter (4) by Theorem 2 in the paper and the results in [6], [23] and [18], and then compare the achieved optimal H_∞ performance γ^*. For brevity, we just take $\varepsilon_1, \varepsilon_2 = 5, \varepsilon_3, \varepsilon_4 = 1$ for the results

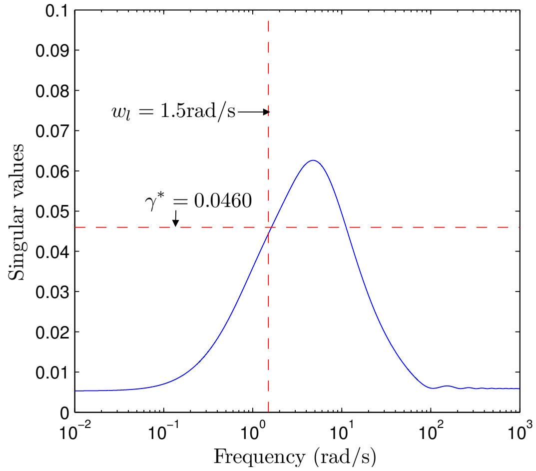

Fig. 1. Singular value curve for $w \leq 1.5$rad/s

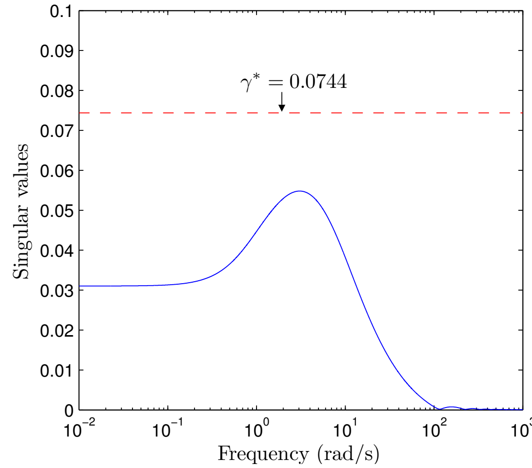

Fig. 2. Singular value curves for the EF range

in this paper and [18]. Table II includes the calculated results, which clearly show that Theorem 2 ($m = 2$), no matter for EF range or FF ranges, yields much better results than that by the existing EF methods in [6], [23] and [18]. Especially when d increases up to 1.2, our method is still able to give a feasible solution while that in [6], [23] and [18] fail.

By Theorem 2 ($m = 2$), the filter state-space realizations $\left[\begin{array}{c|c} A_F & B_F \\ \hline C_F & D_F \end{array}\right]$ for frequency ranges $\omega \leq 1.5$ rad/s and EF range are, respectively,

$$\text{LF}: \left[\begin{array}{cc|c} -3.3045 & -0.0471 & 0.4323 \\ 0.5984 & -5.2810 & -1.7029 \\ \hline -1.9304 & -0.9527 & 0.0059 \end{array}\right], \tag{33}$$

$$\text{EF}: \left[\begin{array}{cc|c} -2.4423 & 0.5103 & 0.5017 \\ -3.9456 & -8.1099 & -2.0596 \\ \hline -2.0000 & -1.0000 & 0 \end{array}\right]. \tag{34}$$

To illustrate to effectiveness of the result in the paper, by connecting above filters to the original system in (32), we depict the singular value curves of the filtering error system transfer functions, respectively, shown in Fig. 1-2. All the singular values in these figures are lower than the achieved H_∞ filtering performance bound γ^* in respectively considered frequency range, which demonstrates the effectiveness of our proposed method.

V. Conclusion

In the paper, we have discussed the problem of H_∞ filtering for continuous-time state-delayed systems with finite frequency specifications. By applying the generalized KYP lemma and projection lemma, LMI-based conditions have been proposed for analyzing and designing finite frequency H_∞ filters for continuous state-delayed systems. Delay-partitioning idea has been introduced to reduce the conservatism caused by delay. The design procedure is cast into solving convex optimization problems, the effectiveness and advantage of which have been shown by an example.

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