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The role of nonminimum phase zeros in the transient response of multivariable systems

Robert Schmid and Amit Pandey

Abstract— **Nonminimum phase zeros are known to cause fundamental performance limitations in the system transient response, particularly undershoot and overshoot. Much of the available literature deals with scalar-input scalar-output systems. In this paper we consider the role of nonminimum phase zeros on the transient response of multivariable systems. We explore via examples the extent to which such zeros may imply undershoot or overshoot in the transient response.**

I. Introduction

The role of system zeros on the control systems performance of linear time-invariant (LTI) systems has been studied for many decades. Numerous studies have reported fundamental performance limitations arising from nonminimum phase (NMP) zeros. A recent comprehensive survey of the impact of system zeros on control system performance was given in [2].

Much of the existing literature on overshoot and undershoot is concerned with single-input single-output (SISO) systems. [8] showed that an LTI SISO continuous-time system has an undershooting step response if it contains at least one real nonminimum phase zero. A lower bound for the size of the undershoot was also given, and this result was extended in [4] where SISO systems with two real nonminimum phase zeros are considered and a lower bound for the minimum undershoot is given.

Papers offering analytic results on the system overshoot include [6], which considered third-order continuous-time SISO systems, and gave necessary and sufficient conditions in terms of the closed-loop poles for which the step response would be nonovershooting. In [14] it was shown that for a continuous-time SISO system with two nonminimum phase real zeros (right-hand complex plane), the step response must overshoot if the settling time is sufficiently small.

In contrast with the extensive literature for SISO systems, to date there have been few papers offering analysis or design methods for undershoot or overshoot in the step response of multi-input multi-output (MIMO) systems. The paper [3] considered MIMO systems subject to dynamic output feedback, and gave a lower bound on the system undershoot and interaction for systems with at least one real NMP zero. A recent contribution offering design methods for MIMO systems is [10], which gave a state feedback controller to yield a nonovershooting step response for LTI MIMO systems; the design method was applicable to some nonminimum phase systems, and could be applied to both continuous-time and discrete-time systems. Very recently it was shown in [11], [12] how the method could be adapted to obtain a nonundershooting step response. The design method given in both these papers makes use of the classic eigenstructure assignment algorithm of [9].

In this paper we review a number of classic results indicating the effect of nonminimum phase zeros on the system transient response for SISO systems, and consider the extent to which these effects must also be observed for MIMO systems with nonminimum phase zeros. The methods of [10]-[12] will be used to obtain the examples.

II. Problem Formulation

We will consider LTI MIMO systems with state space representation

$$\Sigma : \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t), & x(0) = x_0 \in \mathbb{R}^n, \\ y(t) = C\,x(t) + D\,u(t), \end{cases} \tag{1}$$

where, for all $t \geq 0$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the output, and A, B, C and D are appropriate dimensional constant matrices. We assume that B has full column rank and C has full row rank. System Σ is also assumed to be right invertible, stabilizable and has no invariant zeros at the origin.

The following method for designing a tracking controller for a step reference signal $r \in \mathbf{R}^p$ is standard: choose a feedback gain matrix F such that $A + B\,F$ is stable. Two vectors $x_{\rm ss} \in \mathbb{R}^n$ and $u_{\rm ss} \in \mathbb{R}^m$ exist that satisfy

$$0 = A\,x_{\rm ss} + B\,u_{\rm ss} \tag{2}$$

$$r = C\,x_{\rm ss} + D\,u_{\rm ss} \tag{3}$$

for any $r \in \mathbb{R}^p$. Applying the state feedback control law

$$u(t) = F\left(x(t) - x_{\rm ss}\right) + u_{\rm ss} \tag{4}$$

to (1) yields x converging to x_{ss} and y converging to r as t goes to infinity. We say that the transient response overshoots (respectively, undershoots) if it exhibits overshoot (or undershoot) in any one of the output components.

In this paper we are concerned with the system invariant zeros, which we define as follows:

Definition 2.1: [7] Let

$$P_\Sigma(s) := \begin{bmatrix} A - s\,I & B \\ C & D \end{bmatrix} \tag{5}$$

Robert Schmid is with the Department of Electrical and Electronic Engineering, The University of Melbourne, Australia. Amit Pandey is with the Department of Electrical and Electronic Engineering, The University of Auckland, New Zealand. E-mail: rschmid@unimelb.edu.au; amitpandey.ent@gmail.com

denote the system matrix pencil. The $z_0 \in \mathbb{C}$ is an invariant zero of Σ if $\operatorname{rank} P_\Sigma(z_0) < \operatorname{normrank} P_\Sigma(s)$

It is interesting to consider how many zeros Σ has, in general. Since Σ is square, the number of invariant zeros is $n - p + \operatorname{rank} D$. Hence, if D is of full rank, system Σ has n zeros. Alternatively, if Σ is strictly proper ($D = 0$) and CB is nonsingular, then Σ has $n - p$ zeros [7]. We say that Σ is a *minimum phase* system if it has either no zeros, or else any zeros lie within the left-hand complex plane. Zeros lying in the right-hand complex plane are referred to as *non-minimum phase zeros* of Σ. There are numerous classic SISO results in the literature that link the presence of real nonminimum phase zeros with the transient behaviour of the step response. In this paper we investigate the transient behaviour of MIMO systems with real nonminimum phase zeros. The papers [10]-[12] gave methods for designing the gain matrix F to avoid overshoot and undershoot, and in this paper we will use these methods to obtain closed loop systems that exhibit the desired transient response.

III. Nonovershooting state feedback controller design method of Schmid and Ntogramatzidis

We now briefly describe these design methods. Applying the control law (4) to Σ and changing coordinates with $\xi(t) := x(t) - x_{\rm ss}$ yields

$$\Sigma_{hom}: \left\{ \begin{array}{rcl} \dot\xi(t) & = & (A + B\,F)\,\xi(t), \quad \xi(0) = x_0 - x_{\rm ss} \\ y(t) & = & (C + D\,F)\,\xi(t) + r \end{array} \right.$$

In terms of the distinct closed-loop eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $v_1, \ldots, v_n$, the error term $\varepsilon = r - y$ is

$$\varepsilon(t) = \sum_{i=1}^{n} (C + D\,F)\,v_i\,\alpha_i\,e^{\lambda_i\,t}$$

where $[\alpha_1\ \alpha_2\ \ldots\ \alpha_n]^\top = [v_1\ v_2\ \ldots\ v_n]^{-1}(x_0 - x_{ss})$. If we can obtain closed loop eigenvectors v_i satisfying

$$(C + D\,F)\,v_i = \left\{ \begin{array}{ll} e_i & i = 1, \ldots, p \\ 0 & i = p+1, \ldots, n \end{array} \right.$$

where $\{e_1, \ldots, e_p\}$ are the canonical basis vectors for $\mathbb{R}^p$, then

$$\begin{aligned} \varepsilon(t) &= \sum_{i=1}^{n} (C + D\,F)\,v_i\,\alpha_i\,e^{\lambda_i\,t} \\ &= \sum_{i=1}^{p} e_i \alpha_i e^{\lambda_i t} \\ &= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \alpha_1 e^{\lambda_1 t} + \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \alpha_2 e^{\lambda_2 t} + \ldots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \alpha_p e^{\lambda_p t} \end{aligned}$$

If each $\lambda_i \in \mathbb{R}^-$ for $i \in \{1, \ldots, p\}$, then since each component of $\epsilon(t)$ contains only one exponential, $\epsilon(t)$ cannot change sign in any component, and hence y tracks r without overshoot from all x_0.

Theorem 1 [10]: Assume Σ is square ($m = p$) and has at least $n - p$ distinct invariant zeros in $\mathbb{C}^-$. Let $\{\lambda_1, \ldots, \lambda_n\}$, $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ be such that

1) $$\begin{bmatrix} A - \lambda_i\,I & B \\ C & D \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ e_i \end{bmatrix} \tag{6}$$
where $\lambda_i \in \mathbb{R}^-$ is not a zero of Σ, $\forall\, i \in \{1, \ldots, p\}$ and $\{e_1, \ldots, e_p\}$ are the canonical basis vectors for $\mathbb{R}^p$.
2) $$\begin{bmatrix} A - \lambda_i\,I & B \\ C & D \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$
$\forall\, i \in \{n - p + 1, \ldots, n\}$ since λ_i coincide with zeros of Σ;

Assume $\{v_1, \ldots, v_n\}$ is linearly independent. Let

$$F = [\,w_1\ \ \ldots\ \ w_n\,]\,[\,v_1\ \ \ldots\ \ v_n\,]^{-1}$$

Then, the output $y(t)$ obtained from applying $u(t) = F\,x(t) + (u{\rm ss} - F\,x_{\rm ss})$ tracks r without overshoot from any $x_0 \in \mathbb{R}^n$.

Remark 3.1: As the λ_i for $i \in \{1, \ldots, p\}$ can be chosen to be any distinct stable real numbers (provided they are distinct from the zeros of Σ, and that $\{v_1, \ldots, v_n\}$ is linearly independent), the rate of convergence of y to r can be made to be arbitrarily fast. F is independent of both r and x_0. Hence, the same F can be used to achieve nonovershooting convergence for any r and any x_0. We say that in this case F achieves a *globally nonovershooting* response. The values of r and x_0 enter the control law u only through $x_{\rm ss}$ and $u_{\rm ss}$.

The assumption of $n - p$ invariant zeros in $\mathbb{C}^-$ is quite strong, and [10]-[12] explored several ways to weaken this assumption. For Σ with at least $n - 2p$ distinct finite stable zeros, the above method can be used to constrain the output to have 2 modes per component:

$$\epsilon(t) = \begin{bmatrix} \alpha_{1,1}\,e^{\lambda_{1,1} t} + \alpha_{1,2}\,e^{\lambda_{1,2} t} \\ \alpha_{2,1}\,e^{\lambda_{2,1} t} + \alpha_{2,2}\,e^{\lambda_{2,2} t} \\ \vdots \\ \alpha_{p,1}\,e^{\lambda_{p,1} t} + \alpha_{p,2}\,e^{\lambda_{p,2} t} \end{bmatrix}$$

As the sum of two modes can change sign, it is clear that this time overshoot can be avoided only for some x_0. More generally, for systems with fewer invariant zeros in $\mathbb{C}^-$, the eigenstructure approach could be used to constrain the outputs to be the sum of three or more exponentials. Since overshoot corresponds to the error terms having a real positive root, [10] gave necessary and sufficient conditions under which sum of two or more exponential functions of the form

$$f(t) = \alpha_1 e^{\lambda_1 t} + \cdots + \alpha_n e^{\lambda_n t}$$

would have a root, in terms of the α_i and λ_i. Similarly undershoot corresponds to the error term returning to its initial value at some $t > 0$, i.e. the existence of a $\bar t > 0$ such that $\epsilon(0) = \epsilon(\bar t)$. The papers [11]-[12] gave necessary and sufficient conditions under which such $\bar t$ would exist.

Thus the design method for obtaining the gain matrix F to avoid overshoot and/or undershoot for a given x_0 and r can be summarized as

1) Choose a set of candidate closed loop poles $\mathcal{L}$, consisting of all the available invariant zeros from $\mathbb{C}^-$, and

the remaining modes selected from within a desired interval $[a, b] \subset \mathbb{R}^-$.
2) Solve pencil matrix equations (6)-(7) to obtain eigenvectors $\mathcal{V}$. Check that $\mathcal{V}$ is linearly independent.
3) For given x_0, obtain coordinate vector $\alpha = V^{-1}(x_0 - x_{ss})$ and hence the error term $\epsilon_i(t)$ in each output component as the sum of real exponentials.
4) For each component of the error term, apply the tests to the corresponding α_i and λ_i to see if overshoot or undershoot occur.
5) Keep searching $[a, b]$ until a satisfactory set $\mathcal{L}$ is obtained that avoids overshoot/undershoot in all output components. Then the feedback (4) law using the gain matrix F associated with these $\mathcal{L}$ will yield the desired step response transient behaviour for the given x_0 and r.

The conditions on overshoot and undershoot are independent of one another, and thus if only avoiding overshooting is important, we may apply only the tests applicable to overshoot; alternatively we may seek to avoid undershoot only. The design method searches for suitable $\mathcal{L}$ and $\mathcal{V}$, but there is no guarantee that they can always be found, for any given x_0 and r, even if $[a, b] = (-\infty, 0)$ is chosen. Nonetheless in practice the search algorithm provides an effective tool for obtaining a nonundershooting (or nonundershooting) linear controller when they do exist, due to the simplicity of the output function. Also the mathematical tests by which candidate sets of closed loop poles may be tested for suitability are computationally very tractable, allowing for a large number of candidate sets of poles $\mathcal{L}$ to be tested in an efficient manner. Very recently a public domain MATLAB® toolbox, known as **NOUS**, (NonOvershooting and UnderShooting) has been developed to implement the search method of papers [10]-[11]; see [13].

IV. Nonminimum phase zeros and undershoot

In this section we note several results relating real NMP zeros to the transient performance of SISO systems, and employ the above design method to investigate the whether such relationships must also hold for transient behaviour of MIMO systems with real NMP zeros.

Perhaps one of the best known classic results on the relation between nonminimum phase zeros and the transient response is the following:

Theorem 4.1: [8] Let Σ be an LTI stable strictly proper SISO system with at least one real NMP zero. Then the step response must exhibit undershoot.

A natural generalization of this SISO result is the following conjecture for MIMO systems:

Conjecture 4.1: Let Σ be an LTI stable strictly proper square MIMO system with p inputs/outputs, and p real NMP zeros. Then the step response must exhibit undershoot in at least one output component.

Clearly, the case $p = 1$ is given by Theorem 4.1. We show by an example that this conjecture is false for MIMO systems, in general.

Example 4.1: Consider the strictly proper system Σ_1:

```
A = [0 -5 0 6 8 0;0 0 0 -2 0 0;
     6 0 0 0 0 0;0 0 0 0 0 0;
     0 -1 0 -6 3 0;0 0 0 6 0 9]
B = [1 0 0; 7 0 9; 0 4 -5;
     0 0 7; 0 2 10; -2 -1 0]
C = [0 7 0 -2 -3 0;
     0 0 -9 -1 0 0;
     0 0 7  6  5 0]
D = [0 0 0;0 0 0; 0 0 0]
```

System Σ_1 has $n = 6$ states, $p = 3$ inputs and outputs, and $p = 3$ real NMP zeros, at 0.5551, 2.4547 and 9.0000. We assume zero initial conditions and a step reference of $[1,\ 1,\ 1]^T$. Using the **NOUS** toolbox [13] to seek a non-undershooting response, we obtained the gain matrix F as

```
F =
  [24.1   -38.0 -20.9    27.5  2.1 -107.7;
  -143.2 198.5   85.5 -264.3 66.3   681.7;
  -88.0   111.9   45.9 -179.7 51.0   378.1]
```

so that the control law (4), with u_{ss} and x_{ss} obtained from solving (2)-(3), yields transient response curves as shown in Figure 1. The gain matrix F places the closed-loop poles

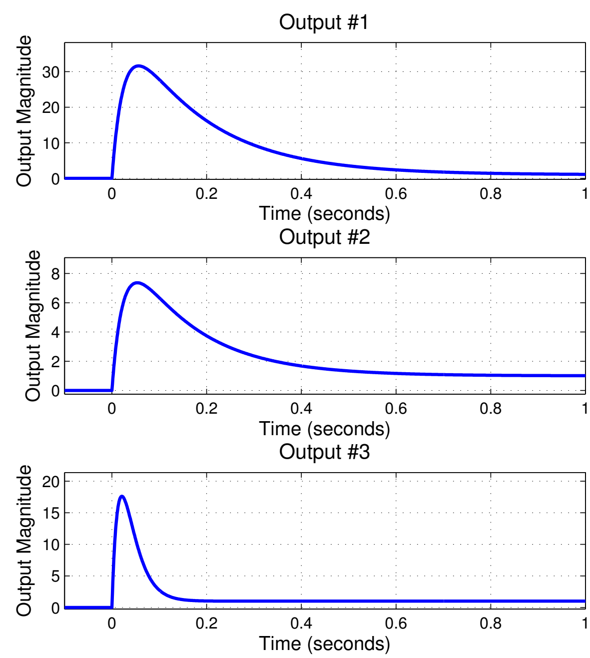

Fig. 1. Outputs from System Σ_1

at $\{-50,\ -47,\ -42,\ -40,\ -7,\ -6\}$, and naturally the closed-loop system has the same zeros as the open loop system, since state feedback has been used. Thus the closed-loop system is stable, strictly proper and square, with $p = 3$ real NMP zeros. We see that with this control law, the step response does not exhibit undershoot in any of its outputs, indicating the conjecture is not valid in general. The authors have also obtained examples to disprove the conjecture for the cases $p = 2$, $p = 4$ and $p = 5$.

V. NONMINIMUM PHASE ZEROS AND OVERSHOOT

Recently [14] established that an LTI stable strictly proper SISO system with at least two real NMP zeros must exhibit overshoot in its step response if the settling time is sufficiently short.

Theorem 5.1: [14] Let Σ be an LTI stable strictly proper SISO system, and assume Σ has two real NMP zeros at $s = z_1$ and $s = z_2$ and any number of additional zeros located anywhere in the complex plane. Let t_s denote the 2% settling time, assume $x_0 = 0$ and $y_\infty > 0$. Then the overshoot y_{os} satisfies the lower bound

$$\frac{y_{os}}{y_\infty} = \left(\frac{0.98}{e^{z_2 t_s} - 1}\right)\left(\frac{1 - z_1 t_s}{z_1 t_s}\right) - 1 \tag{8}$$

For sufficiently small t_s, the bound in (8) is positive, so the step response necessarily exhibits overshoot. As $t_s \to 0$, the bound tends to ∞.

We give an example to show that this does not generalize to MIMO systems.

Example 5.1: Consider Σ_2 with

```
A = [-4,5,0,0;-7,0,-5,0;
     0,-10,4,0;-3,0,0,0]
B = [-2,0;0,-2;-4,-4;-1,0]
C = [0,7,0,5;-2,0,0,0]
D = [0,0;0,0;]
```

System Σ_2 has two real NMP zeros at 1.7857 and 14. We assume zero initial conditions and a step reference of $[1,\ 1]^T$. Using $y_\infty = 1$, we solve (8) for the 2% settling time t_s subject to $y_{os} = 0$. Since we may denote z_1 and z_2 to be either of the two real zeros, we obtain two solutions, $t_s = 0.1131$ and $t_s = 0.0636$. By Theorem 5.1, a SISO system with these two zeros must overshoot if the 2% settling time is less than $t_s = 0.1131$. Using the **NOUS** toolbox to seek a nonovershooting response, we obtained the gain matrix F as

```
 F = [ -1465  -979  492   1376;
        4765   596 -199  -8752]
```

so that the control law (4), with u_{ss} and x_{ss} obtained from solving (2)-(3), yields a transient responses as follows as shown in Figure 2. We see that both outputs are nonovershooting, and achieve a 2% settling time of considerably less than $t_s = 0.1131$. Thus the bound in [14] for SISO systems with two real nonminimum phase zeros does not necessarily apply to MIMO systems.

VI. NONMINIMUM PHASE ZEROS, LOCAL EXTREMA AND ZERO CROSSINGS

In [1], the number of local extrema (turning points) in the step response of a SISO system was linked to the number of system zeros:

Theorem 6.1: [1] For an asymptotically stable, strictly proper SISO system with only real poles and real zeros, the number of extrema in the step response (not including $t = 0$) is greater than or equal to the number of zeros to the right of the right-most pole.

The transient curve is said to have a *zero crossing* if it crosses the time axis at some $t > 0$. In [2] a simple proof was given to show that a SISO system with a NMP real zero must have a zero crossing. More generally, [5] gave

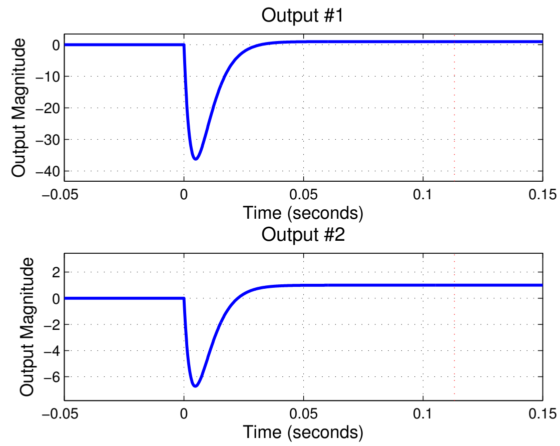

Fig. 2. Outputs from System Σ_2

Theorem 6.2: [5] For an asymptotically stable, strictly proper SISO system with only real poles and real zeros, the number of zero crossings is equal to the number of positive zeros.

We give an example to show that both these results do not generalize to MIMO systems.

Example 6.1: Consider Σ_3 with

```
A = [0 0 -3 0;0 0 0 4;0 6 -10 0;0 -10 0 0]
B = [-5 -5; -5  0;0 -2; 0 1]
C = [-4 0 -5 0;-4 0 -4 0]
D = [0 0;0 0]
```

System Σ_3 has two real NMP zeros at 12.8151 and 2.1849. We assume zero initial conditions and a step reference of $[1,\ 1]^T$. Using the **NOUS** toolbox to seek a monotonic response, we obtained the gain matrix F as

```
F =
 [-6.11   23.14   6.16 -25.37;
   9.24  -15.62  -0.75  18.84]
```

so that the control law (4), with u_{ss} and x_{ss} obtained from solving (2)-(3), yields a transient responses as in Figure 3.

We see that with this control law, the step response is monotonic in both outputs. The gain matrix F places the closed-loop poles at $\{-41, -40, -35, -5\}$ and thus the closed-loop system is stable and strictly proper with only real poles and zeros. This example shows that for MIMO systems, the presence of real NMP system zeros does not imply the outputs must exhibit local extrema or zero crossings.

VII. CONCLUSION

We have surveyed several results from the control systems literature on performance limitations arising from the presence of real minimum phase zeros. Examples have been given to show that these limitations can sometimes be circumvented for MIMO systems when a suitable state feedback controller is used. The examples offer further insight into the role played by real nonminimum phase zeros on the transient performance of MIMO systems. Ongoing work by

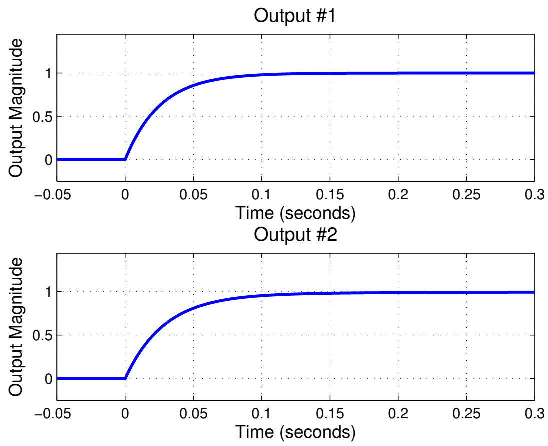

Fig. 3. Outputs from System Σ_3

the authors is aimed at addressing the question of whether a globally monotonic response can be obtained in terms of structural properties of the system matrices (A, B, C, D).

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