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Constrained Actuator Coordination by Virtual State Governing

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Abstract— **The coordination of multiple constrained actuators is relevant to several practical systems, including those in automotive and aerospace applications. Often, the usage of a specific group of actuators is to be minimized either because of its operating cost or because of undesired side-effects. In some cases, controllers for each actuator are already available and rather than redesigning the whole control strategy, a coordination scheme can be introduced to regulate the interaction between the different actuator controllers and to enforce system-wide constraints. In this paper we propose a design for such a coordination strategy in the case where two sets of actuators are available, each with a pre-designed and non-modifiable state-feedback controller. The obtained control strategy is shown to recursively enforce constraints on the actuators and on the system state, to be asymptotically stable, and to use the set of expensive actuators only for finite time. An example of satellite attitude control is shown.**

I. Introduction

Redundant actuation is typical of many applications. Some examples in automotive domain include engine idle speed control [1], [2], where spark timing and throttle are used to regulate engine speed; hybrid electric vehicles, where total wheel power is delivered by the engine/fuel cell and a battery [3]; and vehicle cornering stabilization achieved by active steering and differential braking [4]. In aerospace applications, related examples include aircraft control [5], where multiple aerodynamic surfaces may be coordinated to result in desired attitude moments, and spacecraft attitude control by combining reaction wheels and thrusters [6].

As a consequence, a major challenge for control becomes the coordination of several constrained actuators to achieve a common control objective, while possibly minimizing the use of an “expensive” set of actuators, due to high operating cost, efficiency loss, or undesired side effects. For instance, in vehicle cornering active steering may be preferred to braking, to reduce intrusiveness. In spacecraft attitude control, the use of momentum exchange devices is preferred to the use of thrusters which consume fuel.

Several control techniques have been proposed for dealing with control of systems with *redundant* actuators, including loop shaping H_∞ control [7], and model predictive control [8]. These techniques take a holistic approach, where a single controller is designed to command the actuators, while accounting for the plant dynamics and all the actuators.

A different approach, that separates the control design in two stages, is control allocation [9]. A higher level controller generates virtual commands, which are then allocated to the available constrained actuators by another control algorithm, often based on the solution a constrained least-squares problem [10].

A third approach, which is the one pursued in this paper, starts from already available controllers for each actuator, and implements a coordination scheme. This is of particular interest in applications where the control strategies and software for the individual actuators are already available, and need to be combined without redesigning and re-validating the entire control system. For instance, in vehicle cornering stabilization by active front steering and differential braking, rather than completely redesigning the vehicle stability control architecture [4], the legacy active steering and braking control algorithms can be retained, while only the coordination scheme needs to be introduced. In another example, the International Space Station (ISS) the actuator configuration may be changing, since ISS can use the attitude control means of the currently docked spacecrafts (e.g., Progress or ESA ATV). Finally, the coordination strategy may also be used to enforce pointwise-in-time state and control constraints that are ignored in the single actuator control design. In this way, if the constraints change either during the life cycle of the system or due to particular external conditions only the coordination scheme needs to be re-designed.

In this paper, in Section II we consider the case when two actuators (or two groups of actuators) are available, one of which is to be used only when strictly necessary, possibly due to higher operating cost, efficiency loss, or undesired side-effects. We assume that two single actuator state-feedback controllers were designed without specifically taking into account either control or state constraints. We then develop a coordination strategy that decomposes the system state into “virtual” states that are responded to by the two individual controllers. The decomposition into virtual states, based on constraint-admissible invariant sets [11], ensures that the constraints are satisfied and the use of the expensive actuator is reduced. The obtained control strategy is a form of discrete-time controller state resetting [12], [13], and it is related to the reference governor approach of [14], [15], which however generates a virtual reference -rather than a virtual state- and does not address actuator coordination

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issues.

In Section III we show that the proposed coordination strategy based on virtual state decomposition provides several desirable properties, including closed-loop asymptotic stability, recursive feasibility, and finite-time usage of the expensive actuator. This last property is a major differences of the proposed strategy with respect to other constrained controllers. An application example in aerospace control is presented in Section IV and the future extensions are briefly discussed in Section V.

Notation and Basic Definitions: In what follows $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_{0+}$, and $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{0+}$ denote the sets of integers, positive and non-negative integers, and the sets of reals, positive and non-negative reals, respectively. $\|\cdot\|$ denotes the vector norm. Relational operator between vectors are applied componentwise, while for matrices they indicate (semi)definiteness. The notation $[A]_k$ where A is a vector, indicates the k^{th} component of vector A. I_n is the identity matrix in $\mathbb{R}^{n\times n}$ and 0 indicate a matrix of appropriate dimensions entirely composed of zeros. Given sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}\oplus\mathcal{B}$ is the Minkowski sum, while $\mathrm{int}(\mathcal{A})$ indicates the interior of $\mathcal{A}$.

Definition 1: A function $\alpha:\mathbb{R}_+\to\mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0)=0$. It *belongs to class* $\mathcal{K}_\infty$ [16] if $\alpha\in\mathcal{K}$ and $\alpha(s)\to\infty$ when $s\to\infty$.

Definition 2: A function $\mathcal{V}:\mathbb{R}^n\to\mathbb{R}_{0+}$ is a Lyapunov function [16] for system $x(k+1)=f(x(k))$ in the positive invariant set $\mathcal{X}\subseteq\mathbb{R}^n$ if there exist functions $\overline{\alpha},\underline{\alpha},\alpha_\Delta\in\mathcal{K}_\infty$ such that for all $x\in\mathcal{X}$,

$$\underline{\alpha}(\|x\|)\le\mathcal{V}(x)\le\overline{\alpha}(\|x\|),$$
$$\Delta\mathcal{V}(x)=\mathcal{V}(f(x))-\mathcal{V}(x)\le-\alpha_\Delta(\|x\|).$$

Result 1 ([16]): Let $\mathcal{V}:\mathbb{R}^n\to\mathbb{R}_{0+}$ be a Lyapunov function for system $x(k+1)=f(x(k))$ in the positive invariant set $\mathcal{X}\subseteq\mathbb{R}^n$, then $x(k+1)=f(x(k))$ is asymptotically stable in $\mathcal{X}$.

Definition 3: Given system $x(k+1)=f(x(k))$, $y(k)=h(x(k))$, and the output set $y\in\mathcal{Y}$, the maximum constraint-admissible set $\mathcal{O}_\infty$ [11] is the largest set of states such that if $x(k)\in\mathcal{O}_\infty$, then $y(t)\in\mathcal{Y}$ for all $t\ge k$.

Result 2 ([11]): Given an asymptotically stable discrete-time system $x(k+1)=Ax(k)$, $y(k)=Cx(k)$, where (A,C) is observable, and constraints $y\in\mathcal{Y}$, where $\mathcal{Y}$ is a compact polyhedron and $0\in\mathrm{int}(\mathcal{Y})$, the $\mathcal{O}_\infty$ set is finitely determined, i.e., it consists of a finite number of inequalities, and $0\in\mathrm{int}(\mathcal{O}_\infty)$.

II. Actuator Management by Virtual State Decomposition

We consider a linear system with two vector inputs

$$\begin{aligned} x(k+1) &= Ax(k)+B_1u_1(k)+B_2u_2(k)\ , &\text{(1a)}\\ y(k) &= Cx(k)\ , &\text{(1b)}\end{aligned}$$

where $x\in\mathbb{R}^n$ is the state vector, $u_1\in\mathbb{R}^{m_1}$, $u_2\in\mathbb{R}^{m_2}$ are input vectors, $y\in\mathbb{R}^p$ is the output vector. Being vectors, each of u_1,u_2 may represent several inputs, hence even though we will often refer to u_i as i^{th} actuator input, it is understood that it may refer to a *group* of actuators. Linear inequality constraints on u_1, u_2 and (possibly) on y are also given, i.e., we want to ensure that $u_i\in\mathcal{C}_{u,i}$, $i=\{1,2\}$, and (possibly) $y\in\mathcal{C}_y$, where $\mathcal{C}_y,\mathcal{C}_{u,i}$, $i=\{1,2\}$ are given polyhedra. Two non-modifiable controllers are given by

$$u_1=K_1x,\quad u_2=K_2x \tag{2}$$

where K_i, $i\in\{1,2\}$ are designed separately, so that each enforces asymptotic stability of $x(k+1)=(A+B_iK_i)x(k)$, yet no stability guarantee is given for $x(k+1)=Ax(k)+B_1K_1x(k)+B_2K_2x(k)$.

We aim at designing a control law $g:\mathbb{R}^n\to\mathbb{R}^{2n}$, that solves the following control problem

Problem 1: Given system (1) and pre-assigned controllers (2), design a control law $g:\mathbb{R}^n\to\mathbb{R}^{2n}$,

$$g(x)=\begin{bmatrix} g_1(x)\\ g_2(x)\end{bmatrix}=\begin{bmatrix} x_1\\ x_2\end{bmatrix} \tag{3}$$

that provides "virtual states" to controllers (2),

$$u_1=K_1x_1,\quad u_2=K_2x_2, \tag{4}$$

such that the closed-loop system

$$x(k+1)=Ax(k)+B_1K_1x_1(k)+B_2K_2x(k) \tag{5}$$

(i) enforces the input and (possibly) output constraints, (ii) is Asymptotically Stable (AS), (iii) $u_2\neq 0$ only when "strictly necessary", and only for a finite time interval. □

The closed-loop plant evolution is as follows: At time $k\in\mathbb{Z}_{0+}$, from $x=x(k)$, $x_i(k)=x_i$, $i\in\{1,2\}$ are generated by (3), then $u_i(k)=u_i$, $i=\{1,2\}$ are computed by (4), so that the closed-loop system evolves as

$$x(k+1)=Ax(k)+B_1K_1g_1(x(k))+B_2K_2g_2(x(k)). \tag{6}$$

Then the process is repeated starting from $x(k+1)$ at time $(k+1)\in\mathbb{Z}_+$. Note that control architecture (3), (4) is a (nonlinear) static state-feedback.

The rationale for $g(\cdot)$ is motivated by considering a system with two groups of redundant actuators where each actuator group is capable of stabilizing the system on its own, yet without accounting for constraints. In addition, the actuators in one group, for instance the second, are "expensive" to use. The control function $g(\cdot)$ generates a virtual state that is used to modulate the control action of each actuator in order to enforce the constraints, to guarantee asymptotic stability, and to minimize the use of the "expensive" actuators, in a sense that is clarified in Section II-B.

A. Virtual State Computation

For the design of (3), the following assumptions are made.

Assumption 1: The pairs (A,B_i), for $i=1,2$, are controllable, and (A,C) is observable. □

Assumption 2: K_1, K_2 are designed such that systems $x_i(k+1)=(A+B_iK_i)x_i(k)$, $i=1,2$, are asymptotically stable. □

Assumption 3: Sets $\mathcal{C}_{u,i}$, $i=1,2$ are finitely determined, compact and $0\in\mathrm{int}(\mathcal{C}_{u,i})$, $i=1,2$. If $\mathcal{C}_y\subset\mathbb{R}^p$, $\mathcal{C}_y$ is finitely determined, compact and $0\in\mathrm{int}(\mathcal{C}_y)$. □

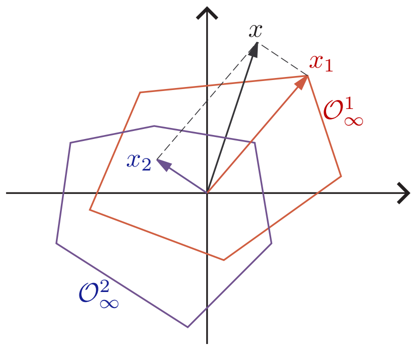

Fig. 1. Graphical representation of the decomposition operated by the virtual state controller.

Consider a virtual state controller $g(\cdot)$ that at time k generates $x_1(k), x_2(k)$, such that $x(k) = x_1(k) + x_2(k)$. Then

$$x(k+1) = (A + B_1K_1)x_1(k) + (A + B_2K_2)x_2(k), \quad (7)$$

and (1) appears to be decomposed into two subsystems

$$\Sigma_1 \;:\; x_1(k+1) = Ax_1(k) + B_1K_1x_1(k), \quad (8a)$$
$$\Sigma_2 \;:\; x_2(k+1) = Ax_2(k) + B_2K_2x_2(k). \quad (8b)$$

Consider first the case where only input constraints are present, i.e., $\mathcal{C}_y \equiv \mathbb{R}^p$ can be ignored, and let $\mathcal{O}^i_\infty$ be the maximum constraint-admissible set for the system Σ_i, subject to $u_i \in \mathcal{C}_{u,i}$, $i = 1, 2$.

Given $x(k)$, consider the decomposition problem

$$\min_{x_1,x_2} \quad J(x_1, x_2) \quad (9a)$$
$$\text{s.t.} \quad x_1 + x_2 = x(k) \quad (9b)$$
$$x_i(k) \in \mathcal{O}^i_\infty,\; i = 1, 2 \quad (9c)$$

where $J(\cdot)$ is a cost function. Constraint (9a) decomposes the state in two vectors, each to be provided to one of the predefined controllers, i.e., each used for feedback by one of the available actuators. Cost function (9a) minimizes the use of the expensive actuator, while (9c) ensures that the decomposition (9b) satisfies the constraints at every future time instant. The effect of (9) is depicted in Figure II-A. The state vector $x(k) = x$ is decomposed into $x_1 \in \mathcal{O}^1_\infty$, $x_2 \in \mathcal{O}^2_\infty$ such that $x_1 + x_2 = x$, and (9a) is minimized.

Let $\left[x_1^*(x(k))' \; x_2^*(x(k))' \right]'$ be the optimizer of (9), and define

$$g(x(k)) = \begin{bmatrix} g_1(x(k)) \\ g_2(x(k)) \end{bmatrix} = \begin{bmatrix} x_1^*(x(k)) \\ x_2^*(x(k)) \end{bmatrix}. \quad (10)$$

For simplicity, in what follows we call $x_i(k) = x_i^*(x(k)) = g_i(x(k))$. Let $\mathcal{X}_\mathrm{f}$ be the set of states such that (9) is feasible, i.e.,

$$\mathcal{X}_\mathrm{f} \triangleq \{x \in \mathbb{R}^n : \text{ (9) is feasible for } x(k) = x\}.$$

Proposition 1: Optimization problem (9) admits a feasible solution for all $x(k) \in (\mathcal{O}^1_\infty \oplus \mathcal{O}^2_\infty)$. For system (6), if (9) admits a feasible solution at time $k \in \mathbb{Z}_{0+}$, then it admits a feasible solution at all instants $t \geq k$, $t \in \mathbb{Z}_{0+}$, i.e., $\mathcal{X}_\mathrm{f}$ is positively invariant for (6) with $g(\cdot)$ defined by (10). □

Proposition 1 can be proved using the feasible set properties due to (9b), and the fact that recursive feasibility is guaranteed by the $\mathcal{O}_\infty$-set properties.

For the case of constraints on system's output, let $\mathcal{C}_y \triangleq \{x \in \mathbb{R}^n : \; HCx \leq h\}$, where $H \in \mathbb{R}^{\ell\times p}$, $h \in \mathbb{R}^\ell$. Compute the maximum constraint admissible set, $\bar{\mathcal{O}}^i_\infty$, for each of the augmented systems

$$x_i(k+1) = (A + B_iK_i)x_i(k) \quad (11a)$$
$$\varepsilon_i(k+1) = \varepsilon_i(k) \quad (11b)$$
$$HCx_i(k) \leq \phi + \varepsilon_i(k) \quad (11c)$$
$$0 \leq \varepsilon_i(k) \leq h - \phi \quad (11d)$$

where $i \in \{1, 2\}$, and $\phi \in [0, h]$ is a fixed constant vector, finite yet possibly arbitrarily small, such that $0 \in \text{int}(\mathcal{H})$, $\mathcal{H} = \{x \in \mathbb{R}^n : HCx \leq \phi\}$. Note that $\bar{\mathcal{O}}^i_\infty$ is defined for $(x_i, \varepsilon_i) \in \mathbb{R}^{n+\ell}$. Given ε_i, we call $\mathcal{O}^i_\infty(\varepsilon_i) \triangleq \{x_i \in \mathbb{R}^n : (x_i, \varepsilon_i) \in \mathcal{O}^i_\infty\}$, i.e., the cross-section of $\bar{\mathcal{O}}^i_\infty$ for the given value of ε_i. Note that the standard theory for the $\mathcal{O}_\infty$ set (Result 2) does not apply in this case, since (11a), (11b) is not asymptotically stable. However, we can prove the following general result that ensures that under appropriate assumptions, sets $\bar{\mathcal{O}}^i_\infty$, $i \in \{1, 2\}$ are compact and finitely determined.

Theorem 1: Let A be strictly Schur, (A, C) be observable, and consider the constraints $Hy \leq \phi$, where $H \in \mathbb{R}^{\ell\times p}$, $\phi \in \mathbb{R}^\ell$. Let $\mathcal{Y} = \{y \in \mathbb{R}^\ell : Hy \leq \phi\}$ be a compact set with $0 \in \text{int}(\mathcal{Y})$. Then the set $\mathcal{O}_\infty = \{(x, \varepsilon) \in \mathbb{R}^{n+\ell} : HCA^kx \leq \phi + \varepsilon,\; 0 \leq \varepsilon \leq \varepsilon_\mathrm{max},\; k \in \mathbb{Z}_{0+}\}$ is compact and finitely determined.

The proof of this technical theorem is omitted here. The result can be proved by showing that the feasible set is compact for every $\varepsilon \geq 0$, and then using observability of (A, C) and asymptotic stability of A to prove finite determination and compactness of $\mathcal{O}_\infty$.

As a direct consequence of (11) and of Theorem 1, for any $\varepsilon_i \in [0, h - \phi]$, $0 \in \text{int}(\mathcal{O}^i_\infty(\varepsilon_i))$, $i = 1, 2$, Given $x(k)$, consider the following optimization problem

$$\min_{x_1,x_2,\varepsilon_1,\varepsilon_2} \quad J(x_1, x_2, \varepsilon_1, \varepsilon_2) \quad (12a)$$
$$\text{s.t.} \quad x_1 + x_2 = x(k) \quad (12b)$$
$$x_i \in \mathcal{O}^i_\infty(\varepsilon_i),\; i = 1, 2 \quad (12c)$$
$$\varepsilon_1 + \varepsilon_2 \leq h - 2\phi \quad (12d)$$

let $\left[x_1^*(x(k))' \; x_2^*(x(k))' \; \varepsilon_1^*(x(k))' \; \varepsilon_2^*(x(k))' \right]'$ be the optimizer, and define $g(\cdot)$ again by (10), where ε_1^*, ε_2^* do not (explicitly) appear.

With (12), the controller decomposes not only the state between the different subsystems, but also the constraint bounds. Constraint (12c) ensures $Hx_i(k) \leq \varepsilon_i(k)$, $i \in \{1, 2\}$, and, by constraints (12b),(12d), $Hx(k) = H(x_1(k)+ x_2(k)) \leq \varepsilon_1(k) + \varepsilon_2(k) + 2\phi \leq h$. If (12) is used in place of (9), the set of feasible states is $\mathcal{X}_\mathrm{f} \triangleq \{x \in \mathbb{R}^n : \text{(12) is feasible for } x(k) = x\}$.

Proposition 2: The set $\mathcal{X}_\mathrm{f} = \{x(k) \in \mathbb{R}^n :$ (12) is feasible$\}$ is convex and, if $\bar{\mathcal{O}}^i_\infty$, $i \in \{1,2\}$ are compact and finitely determined, $\mathcal{X}_\mathrm{f}$ is compact and finitely determined. For system (6), if (12) is feasible at time $k \in \mathbb{Z}_{0+}$, then the constraints are satisfied at any time instant $t \geq k$, $t \in \mathbb{Z}_{0+}$, i.e., $\mathcal{X}_\mathrm{f}$ is positively invariant for (6) where $g(\cdot)$ is implemented by (9). □

The proof of Proposition 2 is analogous to the one of Proposition 1.

B. Selection of the Cost Function

The choice of cost function (9) (or (12)) is important for achieving objectives (ii) and (iii) in Problem 1.

The purpose of the control strategy (10) is to minimize the use of the "expensive" actuators, let us say u_2 for (1). In (9), (12), a possible choice is to minimize $\|K_2 x_2\|^2$, or in other words

$$J(x_1, x_2) = J(x_2) = x_2' K_2' K_2 x_2. \tag{13}$$

While (13) has a clear physical meaning in terms of pointwise minimization of actuator energy, in the general case it lacks desirable properties, and it may not lead to the minimization of the overall use of u_2 over time. In particular, (13) may not be strictly convex with respect to x_2, e.g., when $m_2 < n$. Strict convexity with respect to x_2 is important to achieve stability, as it will be clear later. In what follows we will show that by choosing a weighted version of this cost,

$$J(\cdot) = J(x_2) = \|P_2^{1/2} x_2\|^2 = x_2' P_2 x_2, \tag{14}$$

the desired closed-loop system properties can be obtained. The weight P_2 in (14) is chosen so that $x_2' P_2 x_2$ is a Lyapunov function for system $x_2(k+1) = (A+B_2K_2)x_2(k)$. With cost (14), (10) selects x_2 that belongs to the minimum achievable level set of the Lyapunov function of Σ_2. Thus, (10) minimizes the "energy" in the closed-loop system $x_2(k+1) = (A + B_2K_2)x_2(k)$, which corresponds to the closed-loop dynamics generated by the expensive actuator.

Since for linear systems subject to linear constraints the sets $\mathcal{O}^i_\infty$, $i = 1, 2$ are polyhedra, by choosing (14) as cost function, optimization problems (9), (12) are quadratic programs, for which efficient algorithms exist [17]. Basing on multiparametric programming results [18], we also note that function (9) is piecewise affine and can be computed online, thus reducing the on-line implementation to the evaluation of affine expressions and inequalities.

III. Closed-Loop Asymptotic Stability

In this section we analyze the stability of the closed-loop system (6), where $g(\cdot)$ is defined by (10), (12). In what follows we will often refer to (9), while it is understood that (12) is used, when output constraints are present. Due to limited space we cannot report here the complete proofs. We only sketch the proofs of the essential results.

Consider the control strategy described in Section II, let $\mu_i = \|A + B_iK_i\|$ be the induced 2-norm of the closed-loop matrix, and $\mathcal{V}_i(x_i)$ be Lyapunov functions for $x_i(k+1) = (A + B_iK_i)x_i(k)$, for $i = 1, 2$. We want to prove that there exists $\gamma > 0$ such that the function $\mathcal{V}(x) = \mathcal{V}_1(x_1) + \gamma\mathcal{V}(x_2)$ is a Lyapunov function for $x(k+1) = Ax(k) + B_1K_1g_1(x(k)) + B_2K_2g_2(x(k))$, where $g_i(x(k))$, $i = 1, 2$, are defined by (10). Since systems in (8) are linear, we consider Lyapunov functions $\mathcal{V}_i(x_i) = x_i' P_i x_i$, $i = 1, 2$, where $P_i > 0$, such that $(A + B_iK_i)' P_i (A + B_iK_i) - P_i = -Q_i$, for some $Q_i > 0$, $i \in \{1, 2\}$. We make the following assumption.

Assumption 4: Let either at least one of $\mathcal{O}^i_\infty$, $i = 1, 2$ be bounded, or the minimum of (9) be bounded at $k = 0$.

First, we introduce the two following lemmas, whose proofs are technical and not reported here for conciseness.

Lemma 1: Along the closed loop trajectories of (6) with $g(\cdot)$ defined by (10) and cost function (14), there exists $c_i > 0$, $i = 1, 2$, such that for all $k \in \mathbb{Z}_{0+}$, $\|x_i(k)\| \leq c_i\|x(k)\|$, $i = 1, 2$. □

Lemma 2: Let $x_i(k) \in \mathcal{O}^i_\infty$, $i = 1, 2$ be given, let $\tilde{x}_2(k+1) = (A+B_2K_2)x_2(k)$, and solve (2) with cost function (14) to obtain $x_2(k + 1)$. Then there exists $c \in \mathbb{R}_+$ such that for all $k \in \mathbb{Z}_{0+}$, $\|x_2(k + 1) - \tilde{x}_2(k + 1)\| \leq c\|\tilde{x}_2(k + 1)\| \leq c\|x_2(k)\|$. □

Theorem 2: Let $\mathcal{V}_i(x_i) = x_i' P_i x_i$ be a Lyapunov function for $x_i(k+1) = (A+B_iK_i)x_i(k)$. Then, there exists $\gamma \in \mathbb{R}_+$ such that $\mathcal{V}(x) = \mathcal{V}_1(x_1) + \gamma\mathcal{V}_2(x_2)$ is a Lyapunov function within $\mathcal{X}_\mathrm{f}$ for closed-loop system (6), with $g(x(k))$ defined by (10) and cost function (14). □

Proof sketch: The set of feasible states $\mathcal{X}_\mathrm{f}$ is positively invariant for the closed loop dynamics by Proposition 1 (or Proposition 2). The existence of $\overline{\alpha}, \underline{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}(\|x\|) \leq \mathcal{V}(x) \leq \overline{\alpha}(\|x\|)$ is proved by using Lemma 1 and standard manipulations. The proof that there exists $\alpha_\Delta(\|x\|) \in \mathcal{K}_\infty$ such that $\Delta\mathcal{V}(x) \leq -\alpha_\Delta(\|x\|)$ is demonstrated by analyzing the closed-loop system as if composed of two parts: the continuous evolution of the two subsystems (8) resulting in $\tilde{x}_i = (A + B_iK_i)x_i$, $i = 1, 2$, then the reset of the subsystem states by (10), i.e, $x_i^+ = g_i(\tilde{x}_1, \tilde{x}_2)$, $i = 1, 2$, where $x^+ = x_1^+ + x_2^+$. As a consequence,

$$\begin{aligned}\Delta\mathcal{V}(x) = \mathcal{V}(x^+) - \mathcal{V}(x) = \\ \Delta\tilde{\mathcal{V}}(x_1, x_2, \tilde{x}_1, \tilde{x}_2) + \phi(\tilde{x}_1, \tilde{x}_2, x_1^+, x_2^+)\end{aligned} \tag{15}$$

where $\Delta\tilde{\mathcal{V}}(x_1, x_2, \tilde{x}_1, \tilde{x}_2)$ is the change of the Lyapunov function value during the continuous execution of the subsystems, while $\phi(\tilde{x}_1, \tilde{x}_2, x_1, x_2)$ is the change of $\mathcal{V}(x)$ induced by the reset operated by (10).

Since $\mathcal{V}_i(x)$, $i = 1, 2$ are Lyapunov functions for subsystems in (8), it follows that $\tilde{x}_i' P_i \tilde{x}_i - x_i' P_i x_i = -x_i' Q_i x_i$, $Q_i > 0$, $i = 1, 2$. Thus in (15), for any $\gamma > 0$, whenever $\|x_1\| + \|x_2\| > 0$,

$$\begin{aligned}\Delta\tilde{\mathcal{V}}(x_1, x_2, \tilde{x}_1, \tilde{x}_2) = \\ -x_1' Q_1 x_1 - \gamma x_2' Q_2 x_2 < 0.\end{aligned}$$

In (15), the term related to the reset is

$$\begin{aligned}\phi(\tilde{x}_1, \tilde{x}_2, x_1^+, x_2^+) = (\mathcal{V}_1(x_1^+) - \mathcal{V}_1(\tilde{x}_1)) + \\ \gamma(\mathcal{V}_2(x_2^+) - \mathcal{V}_2(\tilde{x}_2)).\end{aligned}$$

Since $\tilde{x}_i$, $i = 1, 2$ is feasible for problem (9) with cost function (14), while x_i^+, $i = 1, 2$ is the optimizer,

$$\mathcal{V}_2(x_2^+) - \mathcal{V}_2(\tilde{x}_2) \leq 0.$$

Thus, the following bound on (15) holds

$$-\Delta\mathcal{V}(x) \geq \varphi(x_1, x_2, \tilde{x}_1, x_1^+) = \\ x_1' Q_1 x_1 + \gamma x_2' Q_2 x_2 - (\mathcal{V}_1(x_1^+) - \mathcal{V}_1(\tilde{x}_1)).$$

The fact that φ is lower bounded by a class-$\mathcal{K}$ function is demonstrated using Lemma 2 and optimality of x_1^+, x_2^+ to show that there exists γ for which the decrease of $\mathcal{V}_1 + \gamma\mathcal{V}_2$ during the continuous evolution always offsets the increase of $\mathcal{V}_1$ during the reset. This demonstrated by algebraic manipulations and bounding. □

Corollary 1: Consider system (6) where $g(\cdot)$ is defined by (10) and cost function (14) is used. Let $x(0)$ be such that (9) has a feasible solution with finite cost. Then, the closed-loop trajectory is such that there exists a finite index $\bar{k} \in \mathbb{Z}_{0+}$ such that $x_2(k) = 0$, for all $k \geq \bar{k}$. □

The proof of Corollary 1 follows from the existence of $\alpha_\Delta(\|x\|) \in \mathcal{K}_\infty$, such that $\Delta\mathcal{V}(x) \leq -\alpha_\Delta(\|x\|)$ and by the fact that $\mathcal{O}_\infty^1$ is closed, compact, invariant and $0 \in \mathrm{int}(\mathcal{O}_\infty^1)$.

IV. Application to Spacecraft Attitude Control

We consider the attitude regulation for a satellite where two sets of actuators are available, namely, reaction wheels and thrusters. Reaction wheels, which are powered by solar energy, are relatively inexpensive to operate but have small authority. On the other hand, thrusters have larger authority but they consume fuel, which is available in a limited quantity, and hence their usage shall be minimized. We consider a reference frame aligned with the satellite principal axes and located at the satellite center of mass. We call ϕ, ϑ, ψ the three Euler angles defining the satellite attitude with respect to such frame, and $\dot{\phi}$, $\dot{\vartheta}$, $\dot{\psi}$ are the related angular rates. For small angles, the linearized attitude dynamics are described by [19]

$$\mathbf{J}_{\mathrm{sc}}\dot{\omega}(t) = -\mathbf{J}_{\mathrm{rw}}\alpha(t) + \tau(t) \tag{16}$$

where $\omega = [\, \dot{\phi} \;\dot{\vartheta}\; \dot{\psi} \,]' \in \mathbb{R}^3$ is the vector of angular rates, $\tau = [\, \tau_\phi \; \tau_\vartheta \; \tau_\psi \,] \in \mathbb{R}^3$ is the vector of torques with respect frame axes obtained by the thrusters, and $\alpha = [\, \alpha_\phi \; \alpha_\vartheta \; \alpha_\psi \,]$ is the vector of angular accelerations of the reaction wheels with respect to the frame axes. In (16), $\mathbf{J}_{\mathrm{sc}}, \mathbf{J}_{\mathrm{rw}} \in \mathbb{R}^{3\times 3}$ are the matrices of the moments of inertia of the spacecraft and of the reaction wheels with respect to the frame, respectively. Due to the choice of principal axes as the reference frame, $\mathbf{J}_{\mathrm{sc}}, \mathbf{J}_{\mathrm{rw}}$ are diagonal matrices.

From (16), by taking the angles and the angular rates as states, the vector of thrusters torques and the vector of reaction wheels accelerations as inputs, and by discretizing with sampling period $T_s = 1$s, the system dynamics are

$$x(k+1) = Ax(k) + B_1 u_2(k) + B_2 u_2(k) \tag{17}$$

where $x \in \mathbb{R}^6$ is the state vector, $x = [\, \phi \; \vartheta \; \psi \; \dot{\phi} \; \dot{\vartheta} \; \dot{\psi} \,]'$, and $u_1 = \alpha$, $u_2 = \tau$, $u_1, u_2 \in \mathbb{R}^3$, are the input vectors

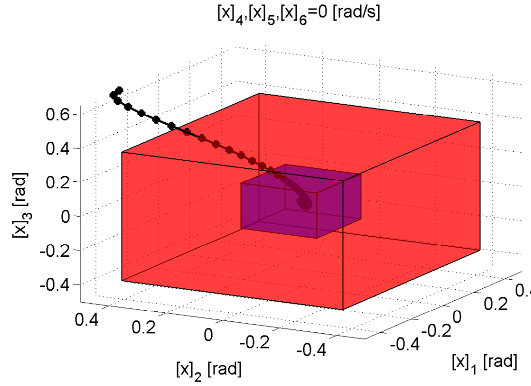

Fig. 2. Trajectory of the satellite attitude angles, and section of $\mathcal{O}_\infty^1, \mathcal{O}_\infty^2$.

corresponding to the two set of actuators, reaction wheels and thrusters, respectively. In (17), $A = \left[\begin{smallmatrix} I_3 & I_3 \\ 0 & I_3 \end{smallmatrix}\right]$, $B_1 = \left[\, 0 \;\; (\mathbf{J}_{\mathrm{sc}}^{-1}\mathbf{J}_{\mathrm{w}})' \,\right]'$, $B_2 = \left[\, 0 \;\; I_3 \,\right]'$. The inputs are subject to saturation, $-0.2 \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right] \leq u_1 \leq 0.2 \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right] \leq u_2 \leq \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]$.

Two controllers are pre-assigned for the actuators, K_i, $i = 1, 2$, that are the solutions of the LQR problems

$$\min \sum_{k=1}^{\infty} x(k)' Q_i x(k) + u(k)' R_i u(k) \tag{18a}$$
$$\text{s.t.} \quad x(k+1) = A_i x(k) + B_i u_i(k), \;\; i = 1, 2 \tag{18b}$$

respectively, where $Q_1 = Q_2 = \left[\begin{smallmatrix} 10\cdot I_3 & 0 \\ 0 & 0.1\cdot I_3 \end{smallmatrix}\right]$, $R_1 = 3.6 \cdot I_3$, $R_2 = I_3$. Note that the pre-assigned controllers do not enforce saturation constraints. The cost function of (9) is implemented by $x_2(k)' P_2 x_2(k)$, where P_2 is the solution of the Riccati equation associated with the LQR problem (18). Sets $\mathcal{O}_\infty^i$, $i = 1, 2$ are computed by using the procedure in [11] and are described by 62 and 34 inequalities, respectively.

We show here the regulation performance starting from initial state $x_0 = [\, -0.2 \;\; 0.5 \;\; 0.64 \;\; -0.075 \;\; 0.0025 \;\; 0.0032 \,]'$. It can be verified that $x_0 \notin \mathcal{O}_\infty^1$, $x_0 \notin \mathcal{O}_\infty^2$, but $x_0 \in (\mathcal{O}_\infty^1 \oplus \mathcal{O}_\infty^2)$. The angles trajectory obtained by applying the virtual state controller from this initial condition is shown in Figure 2, where also the cross-sections of $\mathcal{O}_\infty^1, \mathcal{O}_\infty^2$ for $[x]_i = 0$, $i = 4, 5, 6$ are shown in blue and in red, respectively.

Figure 3 shows the time history of the control inputs, divided in the groups of reaction wheels and thrusters, during a simulation of 60s, where one can see that the actuation constraints are enforced. Since $x_0 \notin \mathcal{O}_\infty^1$, at the beginning both actuators are used, but the thrusters (u_2) are used only for a finite amount of time, and in fact they are shutoff at $t = 19$s. Figure 4 reports the state history and the optimum cost function profile over time. Note that the optimum drops to 0 (the y-axis is in logarithmic scale to better highlight small values) at $t = 19$s, according to Corollary 1. In fact, at $t = 19$s virtual state x_2 is reset to 0, and it is maintained at that value, even though the system state has not yet reached the equilibrium, since from that time on the reaction wheels are sufficient for stabilization. Although not reported here due to space limitations, it can be shown that $\mathcal{V}_1(x_1(k)) = x_1(k)' P x_1(k)$ is not monotonically decreasing

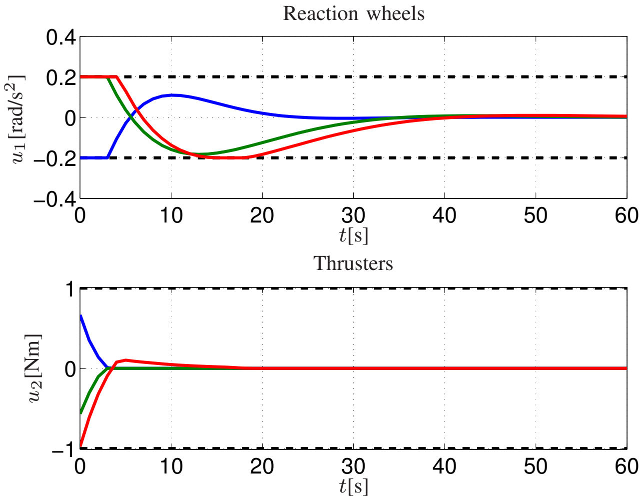

Fig. 3. Time history of the control inputs in the example.

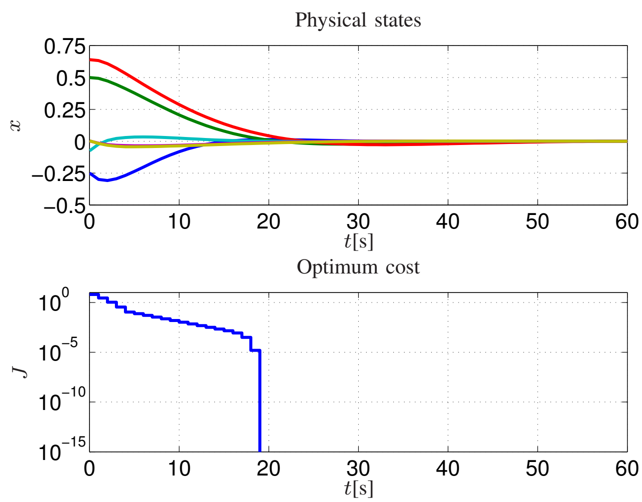

Fig. 4. Time history of the system state and optimum in the example.

in time, due to the effect of (9) on $x_1(k)$, while $\mathcal{V}_2(x_2(k)) = x_2(k)'Px_2(k)$ is monotonically decreasing in time, due to recursive feasibility of (9) and the results of Proposition 1. In addition, it is easy to identify a constant $\gamma > 0$ such that $\mathcal{V}_1(x_1(k))+\gamma\mathcal{V}_2(x_2(k))$ is monotonically decreasing in time, as guaranteed by Theorem 2.

V. CONCLUSION AND FUTURE RESEARCH

We have presented a virtual state governor approach for coordinating two constrained actuators (or actuator groups) in the case when controllers are available for each actuator, individually. Basing on the constraint-admissible invariant set of the plant dynamics in closed-loop with each actuator, and according to a given cost function, the coordination strategy generates virtual states that are fed to the controllers to modulate each actuator's actions. We have indicated a choice of the cost function such that, under reasonable assumptions, the resulting closed-loop response satisfies the constraints, is asymptotically stable, and uses the "expensive" actuators only when needed, and for a finite period of time.

The developments can be generalized to the case of $N \in \mathbb{Z}_+$ actuator groups by using a recursive procedure, for which the stability analysis of Section III can be repeated. However, such analysis becomes more involved in the technicalities, hence an extensive and more precise discussion of the N actuators case will be a subject for a future publication. Furthermore in future research, extensions to other classes of controllers, for instance dynamic feedback using a modeling approach similar to the one used in [20], will be investigated.

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