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Interpolation in Output-Feedback Tube-Based Robust MPC

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Abstract— **The theoretical framework of tube-based robust model predictive control (MPC) for linear systems subject to bounded, additive disturbances has recently drawn attention. This paper considers an extension of this framework, specifically the use of interpolation methods for the terminal controller, which can increase the overall controller's region of attraction for a modest increase in complexity. Standard interpolation-based robust MPC guarantees robust asymptotic convergence of the closed-loop system to a robust invariant set. This paper shows that, by choosing a modified cost function and control law, robust exponential convergence can also be guaranteed.**

I. Introduction

Tube-based robust model predictive control (TBRMPC) [1], [2] is a framework for robust model predictive control (MPC) of linear systems that are subject to bounded, additive disturbances. It builds upon the theory of set invariance [3], [4]. The approach was later also extended to the output-feedback problem [5].

One of the main advantages of TBRMPC is its comparatively low on-line complexity, at the expense of being more conservative than exact min-max MPC formulations [6], [7]. A drawback that may arise with TBRMPC is that, when the terminal controller is optimized for local performance, the region of attraction of the resulting overall controller may be quite small. Although one can increase the region of attraction by increasing the prediction horizon, this comes at a potentially high computational cost.

In [8], [9], the authors show that using a model predictive controller for tracking enlarges the region of attraction. Another way of achieving this while retaining favorable computational properties is to interpolate between a number of pre-computed linear terminal controllers [10], [11], [12], [13]. This can essentially be seen as "on-line tuning" a time-varying terminal controller, allowing for both a large region of attraction and good local performance. Motivated by these ideas, interpolation-based robust MPC (IMPC) has been proposed in [14], [15], [16].

This papers combines the main ideas from IMPC and TBRMPC. In particular, a new controller is proposed that features an enlarged region of attraction while being computationally tractable. While sharing many similarities, the newly presented controller differs from IMPC in terms of cost function and terminal constraint: IMPC uses the set-based robust MPC approach from [17], whereas the controller presented here is a variant of to the one in [5]. As a result, it is possible to prove robust exponential stability of a robust positively invariant set for the closed-loop system, while IMPC only guarantees robust asymptotic stability of such a set. In addition, the chosen formulation simplifies controller synthesis in that it involves only the computation of a positively invariant set rather than a robust positively invariant set in a higher dimensional state space. Finally, it is shown that the controller recovers optimality (i.e. the performance of standard TRMPC) for a known subset of its overall region of attraction.

Outline: Section II provides a brief review of output-feedback TBRMPC and IMPC. Section III introduces the controller and presents the main theoretical results. A case study provided in section IV illustrates important properties of the controller. Section V concludes this paper.

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II. Background

A. Notation and basic definitions

In this paper, $S_1 \oplus S_2 = \{s_1 + s_2 \mid s_1 \in S_1, s_2 \in S_2\}$ and $S_1 \ominus S_2 = \{x \in \mathbb{R}^n \mid x + s_2 \in S_1, \forall\, s_2 \in S_2\}$ denote Minkowski set addition and Pontryagin difference [18], respectively. A set S is positively invariant (PI) for the system $x(k+1) = f(x(k))$ if for all $x(0) \in S$ the solution $x(k) \in S$ for all $k \geq 0$; S is robust positively invariant (RPI) for the system $x(k+1) = f(x(k), w(k))$ if for all $x(0) \in S$ the solution $x(k) \in S$ for all $k \geq 0$ and all $w(k) \in W$ [19]. A polyhedron is the intersection of a finite number of open and/or closed half-spaces and a polytope is a closed and bounded polyhedron. $\mathrm{Proj}_x(S) := \{x \mid \exists\, y \text{ s.t. } [x\ y]^T \in S\}$ denotes the projection of a set S on the x-space, Convh denotes the convex hull, $\succ$ denotes positive definiteness and $\mathbf{0}$ ($\mathbf{I}$) is the zero (identity) matrix of appropriate dimension. Furthermore, $||x||_Q := \sqrt{x^T Q x}$.

B. Output-Feedback Tube-Based Robust MPC

This section provides a brief review of the output-feedback tube-based robust MPC framework [5]. Consider the discrete-time linear time-invariant system

$$\begin{aligned} x^+ &= Ax + Bu + w \\ y &= Cx + v, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the current state and control input, respectively. The successor state x^+ and the current measured output $y \in \mathbb{R}^p$ are affected by unknown but bounded additive disturbances $w \in \mathcal{W} \subset \mathbb{R}^n$ and $v \in \mathcal{V} \subset \mathbb{R}^p$. T simplify computation both $\mathcal{W}$ and $\mathcal{V}$ are assumed to be polytopic and to contain the origin in their respective interior. A, B and C are system matrices of appropriate dimension. State and control input of the system are subject to the constraints

$$x \in \mathbb{X}, \quad u \in \mathbb{U}, \tag{2}$$

where $\mathbb{X}\subseteq\mathbb{R}^n$ is polyhedral, $\mathbb{U}\subseteq\mathbb{R}^m$ is polytopic, and both $\mathbb{X}$ and $\mathbb{U}$ contain the origin in their respective interior. It is assumed in the following that (A, B) is controllable and that (A, C) is observable.

A standard Luenberger observer of the form

$$\begin{aligned}\hat{x}^+ &= A\hat{x} + Bu + L(y-\hat{y})\\ \hat{y} &= C\hat{x},\end{aligned} \tag{3}$$

is used to estimate x, where $\hat{x}\in\mathbb{R}^n$ is the current observer state, $\hat{x}^+$ the successor state estimate, $\hat{y}\in\mathbb{R}^p$ the current output estimate and $L\in\mathbb{R}^{n\times p}$ the observer feedback gain.

The underlying strategy of TBRMPC is to control an artificial nominal system

$$\bar{x}^+ = A\bar{x} + B\bar{u} \tag{4}$$

in such a way that the actual system (1) is guaranteed to satisfy its constraints (2) for all possible disturbance sequences $\mathbf{w}:=\{w(0), w(1),\dots\}$ and $\mathbf{v}:=\{v(0), v(1),\dots\}$.

The control law

$$u = \bar{u} + K(\hat{x}-\bar{x}) \tag{5}$$

consists of a predicted nominal control action $\bar{u}$ and a feedback component $K(\hat{x}-\bar{x})$, where the disturbance rejection controller K is chosen such that $A_K = A+BK$ is Hurwitz.

Defining the state estimation error $e_e := x-\hat{x}$ and the error $e_c := \hat{x}-\bar{x}$ between observer state and state of the nominal system, the actual system state can be expressed as $x = \hat{x} + e_e = \bar{x} + e_c + e_e = \bar{x} + e$.

Suppose now that the errors e_c and e_e can be bounded by RPI sets $\mathcal{E}_c$ and $\mathcal{E}_e$, respectively ($\mathcal{E}_c$ and $\mathcal{E}_e$ can be determined as in [5], [20]), and let $\mathcal{E} := \mathcal{E}_c \oplus \mathcal{E}_e$. It is shown in [5] that if the control problem for the nominal system (4) is solved for the tightened constraints

$$\bar{\mathbb{X}} := \mathbb{X}\ominus\mathcal{E}, \qquad \bar{\mathbb{U}} := \mathbb{U}\ominus K\mathcal{E}_c, \tag{6}$$

the use of the feedback policy (5) will ensure persistent constraint satisfaction for the controlled uncertain system (1).

For the problem to be well-posed, the following must hold:

Assumption 1: There exist $K\in\mathbb{R}^{m\times n}$ and $L\in\mathbb{R}^{n\times p}$ such that $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ exist and contain the origin.

C. Interpolation-Based MPC

Let $K_0,\dots,K_{\nu-1}$, be ν stabilizing linear controllers designed for the nominal system (4) satisfying the following:

Assumption 2: K_0 is the infinite horizon LQR controller for system (4) for cost matrices $Q\succ\mathbf{0}$ and $R\succ\mathbf{0}$. Furthermore, $K_1,\dots,K_{\nu-1}$ are stabilizing linear state-feedback controllers for system (4), i.e. $\rho(A+BK_p)<1$.

Note that Assumption 2 implies that for each K_p there exists $P_p\succ\mathbf{0}$ such that $A_{K_p}^T P_p A_{K_p} - P_p = -Q-K_p^T R K_p$. With such P_p, the infinite horizon cost of the trajectory of the unconstrained closed-loop system $\bar{x}^+=(A+BK_p)\bar{x}$ starting from an initial state $\bar{x}_0$ is given by $V_\infty(\bar{x}_0) = \bar{x}_0^T P_p \bar{x}_0$.

Following [21], an interpolated control law of the form

$$\bar{u} = \kappa^{ip}(\bar{x}) = \sum_{p=0}^{\nu-1} K_p\tilde{x}^p \tag{7}$$

is used as the terminal control law, where the nominal state

$$\bar{x} = \sum_{p=0}^{\nu-1}\tilde{x}^p \tag{8}$$

is decomposed into ν slack state variables $\tilde{x}^p$. The closed-loop nominal system under the control law (7) can be written as $\bar{x}^+=A_{K_0}\bar{x}+\sum_{p=1}^{\nu-1}(A_{K_p}-A_{K_0})\tilde{x}^p$, where $A_{K_p}:=A+BK_p$. Introducing auxiliary systems $(\tilde{x}^p)^+:=A_{K_p}\tilde{x}^p$, one can form an augmented closed-loop system [22] as

$$\begin{bmatrix}\bar{x}^+\\ (\tilde{x}^1)^+\\ \vdots\\ (\tilde{x}^{\nu-1})^+\end{bmatrix} = \begin{bmatrix}A_{K_0} & A_{K_1}-A_{K_0} & \dots & A_{K_{\nu-1}}-A_{K_0}\\ \mathbf{0} & A_{K_1} & \dots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \dots & A_{K_{\nu-1}}\end{bmatrix}\cdot\begin{bmatrix}\bar{x}\\ \tilde{x}^1\\ \vdots\\ \tilde{x}^{\nu-1}\end{bmatrix} \tag{9}$$

which is subject to the constraints

$$\bar{x}\in\bar{\mathbb{X}}, \qquad \bar{u} = K_0\bar{x} + \sum_{p=1}^{\nu-1}(K_p-K_0)\,\tilde{x}^p \in\bar{\mathbb{U}}. \tag{10}$$

Defining the augmented state

$$\bar{x}^E := \begin{bmatrix}\bar{x}^T & (\tilde{x}^1)^T & \dots & (\tilde{x}^{\nu-1})^T\end{bmatrix}^T, \tag{11}$$

the augmented system (9) can be written as $(\bar{x}^E)^+:=A^E\bar{x}^E$. From its block diagonal structure and Assumption 2 it is easy to verify that A^E is Hurwitz. Hence, provided that Assumption 1 holds, the maximal positively invariant (MPI) set Ω_∞^E for the augmented system (9) exists and contains a nonempty region around the origin [23], [24]. Furthermore, since $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ are polytopic and (9) is linear, Ω_∞^E is also polytopic. Denote by $\bar{\mathbb{X}}_f$ the projection of Ω_∞^E onto the $\bar{x}$-space, i.e. $\bar{\mathbb{X}}_f := \mathrm{Proj}_{\bar{x}}(\Omega_\infty^E)$.

Proposition 1: Let $K_0,\dots,K_{\nu-1}$ satisfy Assumption 2, and let Ω_∞^E be the MPI set for system (9) subject to (10). Then, $\bar{\mathbb{X}}_f$ is a constraint admissible PI set for the system $\bar{x}^+=A\bar{x}+B\kappa^{ip}(\bar{x})$ subject to constraints $\bar{x}\in\bar{\mathbb{X}}$ and $\bar{u}\in\bar{\mathbb{U}}$.

Proof: Proposition 1 is a special case of Proposition 1 in [22]. ■

Proposition 2: Let $\Omega_\infty^0,\dots,\Omega_\infty^{\nu-1}$ be the ν MPI sets corresponding to the closed-loop systems $\bar{x}^+=A_{K_p}\bar{x}$ subject to $\bar{x}\in\bar{\mathbb{X}}$ and $\bar{u}\in\bar{\mathbb{U}}$. Then, $\bar{\mathbb{X}}_f \supseteq \mathrm{Convh}(\Omega_\infty^0,\dots,\Omega_\infty^{\nu-1})$.

Proof: Consider the case when only the p^*th slack state variable $\tilde{x}^{p^*}$ is non-zero. Then (9) can be reduced to

$$\begin{bmatrix}\bar{x}^+\\ \bar{x}^+\end{bmatrix} = \begin{bmatrix}A_{K_0} & A_{K_{p^*}}-A_{K_0}\\ \mathbf{0} & A_{K_{p^*}}\end{bmatrix}\cdot\begin{bmatrix}\bar{x}\\ \bar{x}\end{bmatrix}, \tag{12}$$

since $\tilde{x}^p = \mathbf{0}$ for all $p\neq p^*$. It is easy to see that (12) is equivalent to $\bar{x}^+=A_{K_{p^*}}\bar{x}$, for which the associated MPI set is $\Omega_\infty^{p^*}$. Thus, $\mathrm{Proj}_{\bar{x}}(\Omega_\infty^E)\supseteq\Omega_\infty^{p^*}$. Since (9) is linear and the constraints $\bar{x}\in\bar{\mathbb{X}}$ and $\bar{u}\in\bar{\mathbb{U}}$ are polytopic, it follows that $\bar{\mathbb{X}}_f\supseteq\mathrm{Convh}(\Omega_\infty^0,\dots,\Omega_\infty^{\nu-1})$. ■

Proposition 2 states that $\bar{\mathbb{X}}_f$ contains the convex hull of all ν MPI sets Ω_∞^p for the respective closed-loop systems under the feedback controllers K_p. Hence, using an interpolated terminal controller of the form (7) can significantly enlarge $\bar{\mathbb{X}}_f$ and consequently the overall region of attraction. Note that in contrast to [22], the set $\bar{\mathbb{X}}_f$ is obtained not from a RPI set but rather from a (nominal) PI set. In many cases, this can greatly simplify the computation of $\bar{\mathbb{X}}_f$.

III. The Controller

In this section, the ideas of IMPC are combined with those of TBRMPC in order to synthesize a refined tube-based robust model predictive controller. The benefit of using interpolation is a potentially larger region of attraction than that of TBRMPC for a comparable on-line computational complexity. The general idea for the controller presented in the following is drawn from the one described in [14], [15], [16], which uses a similar interpolation method and guarantees robust asymptotic stability of a RPI set. By using a different cost function and control law, the controller proposed here can be shown to also guarantee robust exponential stability of a RPI set.

This paper considers the general output-feedback case. A state-feedback controller is easily obtained by setting $C = \mathbf{I}$ and $v = \mathbf{0}$ in (1), yielding $e_e = \mathbf{0}$ or, equivalently, $x = \hat{x}$.

A. Cost Function and Control Law

Consider the cost function

$$V_N(\hat{x};\bar{x}_0,\bar{\mathbf{u}},\tilde{\mathbf{x}}) := \sum_{i=0}^{N-1} ||\bar{x}_i||_Q^2 + ||\bar{u}_i||_R^2 + \sum_{p=0}^{\nu} ||\tilde{x}_N^p||_{P_p}^2, \tag{13}$$

where $\tilde{\mathbf{x}} := \{\tilde{x}_N^0,\dots,\tilde{x}_N^{\nu-1}\}$ denotes the set of slack state variables into which the terminal state $\bar{x}_N$ is decomposed, $\bar{\mathbf{u}} := \{\bar{u}_0,\bar{u}_1,\dots\}$ denotes the sequence of predicted optimal nominal control actions, and the weighting matrices Q, R and $P_0,\dots,P_{\nu-1}$ satisfy Assumption 2.

Let $\bar{\Phi}(k;\bar{x},\bar{\mathbf{u}})$ denote the solution of (4) at time k controlled by the sequence $\bar{\mathbf{u}}$ when the initial state at time 0 is $\bar{x}$. The set of admissible nominal control sequences for a given nominal initial state $\bar{x}_0$ is given by

$$\begin{aligned}\mathcal{U}_N(\bar{x}_0) = \big\{\bar{\mathbf{u}} \mid \bar{u}_i \in \bar{\mathbb{U}},\ \bar{\Phi}(i;\bar{x}_0,\bar{\mathbf{u}}) \in \bar{\mathbb{X}} \text{ for } i \in \mathbb{I}_N, \\ \bar{\Phi}(N;\bar{x}_0,\bar{\mathbf{u}}) \in \bar{\mathbb{X}}_f\big\},\end{aligned} \tag{14}$$

where $\mathbb{I}_N := \{0,\dots,N-1\}$ and $\bar{\mathbb{X}}_f = \mathrm{Proj}_{\bar{x}}(\Omega_\infty^E)$. At each time step, given an estimate $\hat{x}$ of the current system state x, the following optimization problem is solved on-line:

$$\begin{aligned}V_N^*(\hat{x}) = \min_{\bar{x}_0,\bar{\mathbf{u}},\tilde{\mathbf{x}}_N} \Big\{V_N(\hat{x};\bar{x}_0,\bar{\mathbf{u}},\tilde{\mathbf{x}}) \bigm| \bar{\mathbf{u}} \in \mathcal{U}_N(\bar{x}_0), \\ \bar{x}_0 \in \{\hat{x}\}\oplus(-\mathcal{E}_c),\ \bar{x}_N = \sum_{p=0}^{\nu-1}\tilde{x}_N^p,\ \tilde{x}_N^E \in \Omega_\infty^E\Big\},\end{aligned} \tag{15}$$

with $\tilde{x}_N^E$ as defined in (11). It is easy to see that (15) is a Quadratic Program (QP).

Let $\bar{x}_0^*(\hat{x})$, $\bar{\mathbf{u}}^*(\hat{x})$ and $\tilde{\mathbf{x}}^*(\hat{x})$ denote the minimizers of (15). The domain of the value function $V_N(\cdot)$ is

$$\hat{\mathcal{X}}_N = \{\hat{x} \mid \exists\, \bar{x}_0 \ \text{ s.t. } \ \bar{x}_0 \in \{\hat{x}\}\oplus(-\mathcal{E}_c),\ \mathcal{U}_N(\bar{x}_0) \neq \emptyset\}. \tag{16}$$

Define $\bar{\mathcal{X}}_N := \{\bar{x} \mid \mathcal{U}_N(\bar{x}) \neq \emptyset\}$ as the set of admissible nominal initial states. It is easy to verify that $\hat{\mathcal{X}}_N = \bar{\mathcal{X}}_N \oplus \mathcal{E}_c$.

Applying only the first element of the optimizer $\bar{\mathbf{u}}^*(\hat{x})$ obtained from (15) at each time step in a Receding Horizon fashion yields the Model Predictive Control law

$$\kappa_N^{ip}(\hat{x}) := \bar{u}_0^*(\hat{x}) + K(\hat{x} - \bar{x}_0^*(\hat{x})). \tag{17}$$

Remark 1: Note that for $\nu > 1$, the on-line computation still involves solving a QP even for the trivial prediction horizon $N = 0$. This is because the decomposition (8) of $\bar{x}_0$ is performed on-line. In this case the controller essentially degenerates to the one proposed in [11].

B. Controller Properties

Assumption 3: Suppose $\mathcal{E}_e$ and $\mathcal{E}_c$ are RPI sets as defined in section II-B such that Assumption 1 holds. Let $\{K_0,\dots,K_{\nu-1}\}$ be a set of linear state-feedback controllers for the nominal system (4) satisfying Assumption 2 for the same weighting matrices Q and R as in (13). Furthermore, let Ω_∞^0 be the MPI set for the system $\bar{x}^+ = (A + BK_0)\bar{x}$ subject to the constraints $x \in \bar{\mathbb{X}}$ and $u \in \bar{\mathbb{U}}$, and let Ω_∞^E be the MPI set for system (9) subject to the constraints (10).

Lemma 1 (Persistent feasibility): Suppose that Assumption 3 holds. Then, for any initial system and observer states $x(0)$ and $\hat{x}(0) \in \hat{\mathcal{X}}_N$ that satisfy $x(0) - \hat{x}(0) \in \mathcal{E}_e$, the resulting state trajectory $\mathbf{x} := \{x(0), x(1),\dots\}$ and sequence of control inputs $\mathbf{u} := \{u(0), u(1),\dots\}$ of the perturbed closed-loop system $x^+ = Ax + B\kappa_N^{ip}(\hat{x}) + w$ are persistently feasible, i.e. it holds that $x(t) \in \mathbb{X}$ and $u(t) \in \mathbb{U}$ for all $t \geq 0$, provided that $w(t) \in \mathcal{W}$ and $v(t) \in \mathcal{V}$.

Proof: Since $\hat{x} - \bar{x}_0^* \in \mathcal{E}_c$ is an explicit constraint in (15), the result follows directly from Theorem 1 in [5]. ■

Theorem 1 (Robust exponential stability): Suppose that Assumption 3 holds. Then, if initial system and observer states satisfy $x(0) - \hat{x}(0) \in \mathcal{E}_e$, the set $\mathcal{E} = \mathcal{E}_e \oplus \mathcal{E}_c$ is robustly exponentially stable for the perturbed closed-loop system $x^+ = Ax + B\kappa_N^{ip}(\hat{x}) + w$ with a region of attraction $\mathcal{X}_N := \hat{\mathcal{X}}_N \ominus \mathcal{E}_e$.

Proof: The assumption $x(0) - \hat{x}(0) \in \mathcal{E}_e$ guarantees that $\hat{x}(0) \in \hat{\mathcal{X}}_N$ for all $x(0) \in \mathcal{X}_N$. Denote by $V_N^*(\hat{x})$ the cost obtained from solving (15) for the current state estimate $\hat{x} \in \hat{\mathcal{X}}_N$. At the next time step, the cost $V_N(\hat{x}^+)$ for the feasible control sequence $\bar{\mathbf{u}}^+ = \{\bar{u}_1^*,\dots,\bar{u}_{N-1}^*,\kappa^{ip}(\bar{x}_N^*)\}$ and the feasible initial state $\bar{x}_0^+ = \bar{x}_1^*$ is

$$\begin{aligned}
V_N(\hat{x}^+) &= \sum_{i=1}^{N} ||\bar{x}_i^*||_Q^2 + ||\bar{u}_i^*||_R^2 + \sum_{p=0}^{\nu-1} ||\, A_{K_p}\tilde{x}_N^{p,*}||_{P_p}^2 \\
&\leq V_N^*(\hat{x}) + \sum_{p=0}^{\nu-1} ||\tilde{x}_N^{p,*}||_Q^2 + \sum_{p=0}^{\nu-1} ||K_p\tilde{x}_N^{p,*}||_R^2 - ||\bar{x}_0^*||_Q^2 \\
&\qquad - ||\bar{u}_0^*||_R^2 + \sum_{p=0}^{\nu-1} ||\, A_{K_p}\tilde{x}_N^{p,*}||_{P_p}^2 - ||\,\tilde{x}_N^{p,*}||_{P_p}^2 \\
&\ \ \vdots \\
&= \sum_{p=0}^{\nu-1} (\tilde{x}_N^{p,*})^T (A_{K_p}^T P_p A_{K_p} - P_p + Q + K_P^T R K_p)(\tilde{x}_N^{p,*}) \\
&\qquad + V_N^*(\hat{x}) - ||\bar{x}_0^*||_Q^2 - ||\bar{u}_0^*||_R^2 \\
&= V_N^*(\hat{x}) - ||\bar{x}_0^*||_Q^2 - ||\bar{u}_0^*||_R^2
\end{aligned}$$

Since the control sequence $\bar{\mathbf{u}}^{+}$ and the initial state $\bar{x}_0^{+}$ are feasible but not necessarily optimal, it holds that

$$V_N^*(\hat{x}^+) - V_N^*(\hat{x}) \leq -||\bar{x}_0^*||_Q^2 - ||\bar{u}_0^*||_R^2. \tag{18}$$

Note that $V_N^*(\hat{x}) = 0$ for all $\hat{x} \in \mathcal{E}_c$, as the trivial choice $\bar{\mathbf{u}}^* = \{0,\ldots,0\}$, $\bar{\mathbf{x}}^* = \{0,\ldots,0\}$, $\tilde{\mathbf{x}}_N^* = \{0,\ldots,0\}$ is feasible for all $\hat{x} \in \mathcal{E}_c$. On the other hand, the constraint $\hat{x} - \bar{x}_0^* \in \mathcal{E}_c$ implies that $\bar{x}_0^* \neq 0, \forall \hat{x} \notin \mathcal{E}_c$. Together with (18) this yields robust exponential stability of $\mathcal{E}_c$ for the state estimate $\hat{x}$. Since $x(0) - \hat{x}(0) \in \mathcal{E}_e$, it follows from Proposition 1 in [5] that $x \in \hat{x} \oplus \mathcal{E}_e \subseteq \mathcal{E}_c \oplus \mathcal{E}_e = \mathcal{E}$, which proves robust exponential stability of $\mathcal{E}$ for the actual system state x. ■

Lemma 2 (Increased attractivity): Suppose Assumption 3 holds. Then, $\hat{\mathcal{X}}_N \supseteq \mathrm{Convh}(\hat{\mathcal{X}}_N^0, \ldots, \hat{\mathcal{X}}_N^{\nu-1})$, where $\hat{\mathcal{X}}_N^p$ denotes the region of attraction of state estimates of a TBRMPC controller that employs K_p as its terminal controller.

Proof: Define the one-step controllable set $\mathrm{Ctrl}(\Omega) := \{\bar{x} \in \bar{\mathbb{X}} \mid \exists\, \bar{u} \in \bar{\mathbb{U}} \text{ s.t. } A\bar{x} + B\bar{u} \in \Omega\}$ of a target set Ω for the nominal system (4). Clearly, the region of attraction of nominal states of the controller is $\bar{\mathcal{X}}_N = \mathrm{Ctrl}^N(\bar{\mathbb{X}}_f)$, where $\mathrm{Ctrl}^N(\cdot)$ indicates N recursive operations of $\mathrm{Ctrl}(\cdot)$. Together with Proposition 2 it follows from convexity of the sets $\bar{\mathbb{X}}$, $\bar{\mathbb{U}}$ and Ω_∞^p, $p = 1, \ldots, \nu-1$, and linearity of the system (4) that $\bar{\mathcal{X}}_N(\bar{\mathbb{X}}_f) \supseteq \mathrm{Convh}(\bar{\mathcal{X}}_N^0(\bar{\mathbb{X}}_f), \ldots, \bar{\mathcal{X}}_N^{\nu-1}(\bar{\mathbb{X}}_f))$. Noting that $\hat{\mathcal{X}}_N = \bar{\mathcal{X}}_N \oplus \mathcal{E}_c$ completes the argument. ■

Theorem 2 (Local optimality): Suppose Assumption 3 holds. Then, for all $\hat{x} \in \hat{\mathcal{X}}_N^0$, the control law $\kappa_N^{ip}(\hat{x})$ yields the same closed-loop performance as a standard TBRMPC controller $\kappa_N(\hat{x})$ that employs the unconstrained infinite horizon optimal controller K_0 as its terminal controller.

Proof: Consider $\sum_{p=0}^{\nu-1} ||\tilde{x}_N^p||_{P_p}^2$, the terminal cost component in (13). Since K_0 is the infinite horizon unconstrained optimal controller it holds that $P_p \succ P_0$ for all $p \neq 0$. Hence,

$$\begin{aligned} V_N(\hat{x}) &= \sum_{i=1}^{N} ||\bar{x}_i^*||_Q^2 + ||\bar{u}_i^*||_R^2 + \sum_{p=0}^{\nu-1} ||\tilde{x}_N^p||_{P_p}^2 \\ &\geq \sum_{i=1}^{N} ||\bar{x}_i^*||_Q^2 + ||\bar{u}_i^*||_R^2 + \sum_{p=0}^{\nu-1} ||\tilde{x}_N^p||_{P_0}^2 \\ &\ \vdots \\ &= \sum_{i=1}^{N} ||\bar{x}_i^*||_Q^2 + ||\bar{u}_i^*||_R^2 + ||\bar{x}_N||_{P_0}^2 \end{aligned} \tag{19}$$

Therefore, if feasible, the optimal values of the slack state variables are $\tilde{x}_N^{0,*} = \bar{x}_N^*$ and $\tilde{x}_N^{p,*} = 0$ for all $p \neq 0$. This combination is feasible for all $\bar{x}_N^* \in \Omega_\infty^0$ and hence for all $\bar{x}_0^* \in \bar{\mathcal{X}}_N^0$. In this case the cost function (13) is reduced to that of standard TBRMPC [2]. Consequently, $\kappa_N(\hat{x}) = \kappa_N^{ip}(\hat{x})$ for all $\hat{x} \in \hat{\mathcal{X}}_N^0$, i.e. TBRMPC is recovered within $\hat{\mathcal{X}}_N^0$. ■

C. The Role of the Terminal Controller Gains K_p

By choosing the design parameters ν and $K_1, \ldots, K_{\nu-1}$, one can trade off complexity for optimality. A simple way to determine the K_p is to use unconstrained LQR controllers designed for different weighting matrices Q and R [25]. A more systematic way is to use LMI optimization techniques. In this context, [16] propose solving an LMI optimization problem that maximizes the volume of the projection of a positively invariant ellipsoid in the $\bar{x}^E$-space onto the $\bar{x}$-space, where the gains $K_1, \ldots, K_{\nu-1}$ are regarded as optimization variables. Unfortunately, the resulting optimization problem is, in general, subject to non-convex Bilinear Matrix Inequality (BMI) constraints. The question of how to choose the different K_p has been further investigated in [14], [16].

It should be clear from the previous section that interpolation will yield strictly "better" results compared to using a single terminal controller. Specifically, for any given single terminal controller, adding additional terminal controller gains to interpolate between will only lead to an increase in overall controller performance or size of the region of attraction (or both), but never to a decrease in either.

IV. Case Study

Consider the output-feedback double integrator example from [5] with system dynamics

$$\begin{aligned} x^+ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + w \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x + v, \end{aligned} \tag{20}$$

with state constraints $\mathbb{X} = \{x \mid -50 \leq x_i \leq 3,\ i = 1, 2\}$ and control constraints $\mathbb{U} = \{u \mid |u| \leq 3\}$. The state and output disturbances w and v are assumed to be bounded by $\mathcal{W} = \{w \mid ||w||_\infty \leq 0.1\}$ and $\mathcal{V} = \{v \mid |v| \leq 0.05\}$, respectively. The weighting matrices in the cost function (13) are given by $Q = \mathbf{I}_2$ and $R = 0.01$, and the disturbance rejection controller and observer gains are chosen as $K = [-0.7 \ \ -1.0]$ and $L = [1.00 \ \ 0.96]^T$.

For comparison with standard output-feedback TBRMPC (controller A) with prediction horizon $N_A = 13$, two instances (B and C) of the controller from section III were designed for the system (20). Controller B uses $\nu_B = 2$ terminal controllers and a prediction horizon of $N_B = 6$, while controller C uses $\nu_C = 3$ terminal controllers and a prediction horizon of $N_C = 4$. K_0 is the infinite horizon LQR controller for the specified weighting matrices $Q = \mathbf{I}_2$ and $R = 0.01$. The terminal controllers of controller B and C were computed as the infinite horizon LQR controllers for the modified input weights $R_{B,1} = 10$, $R_{C,1} = 1$ and $R_{C,2} = 100$, respectively.

A. Regions of Attraction

The terminal constraint sets $\bar{\mathbb{X}}_f^A$, $\bar{\mathbb{X}}_f^B$ and $\bar{\mathbb{X}}_f^C$ and the corresponding regions of attraction $\hat{\mathcal{X}}_N^A$, $\hat{\mathcal{X}}_N^B$ and $\hat{\mathcal{X}}_N^C$ for state estimates of the three controllers are depicted in Figure 1. Figure 1 also shows the regions of optimality $\hat{\mathcal{X}}_{opt}^B$ and $\hat{\mathcal{X}}_{opt}^C$ for controllers B and C (by definition, the region of optimality of controller A is $\hat{\mathcal{X}}_N^A$). Note that, even for the significantly reduced prediction horizons $N_B = 6$ and $N_C = 4$, the regions of attraction $\hat{\mathcal{X}}_N^B$ and $\hat{\mathcal{X}}_N^C$ of controller B and C are comparable to that of controller A with $N_A = 13$.

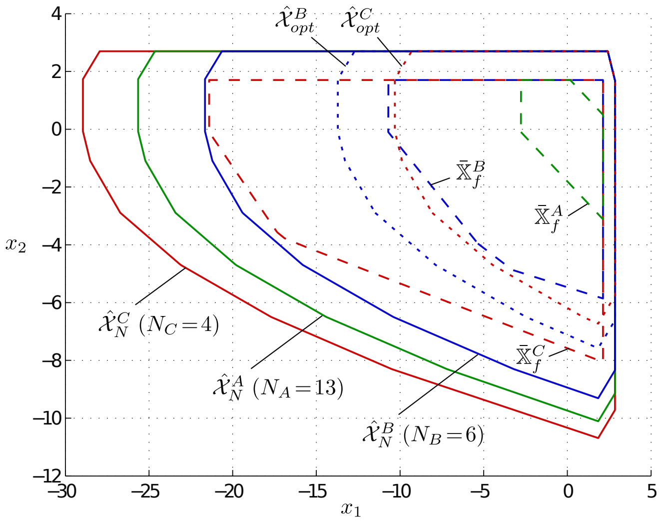

Fig. 1. Terminal sets & regions of attraction for controllers A, B and C

B. Controller Complexity and Performance

Table I shows that the QPs of controller B and C involve a significantly lower number of variables and constraints than that of controller A. However, the reduction in the number of variables N_{var} from B to C is only minor, while the number of constraints N_{con} actually grows. This is because with each additional K_p, the dimension of the augmented system (9) is increased by n, which generally results in a terminal set Ω_∞^E of higher complexity. It is therefore necessary to find a tradeoff between the number ν of terminal controllers and the prediction horizon N.

For a simulation horizon of $N_{sim} = 15$, the closed-loop system for each of the three controllers was simulated for the same 100 randomly generated initial conditions scattered over $\hat{\mathcal{X}}_N^B$. To ensure comparability of performance, the random disturbance sequences $\mathbf{w}$ and $\mathbf{v}$ were the same for all three controllers. For each controller, the on-line computation time for each of the 1500 single solutions of the optimization problem (15) was determined. From this data, the minimal (t_{min}), maximal (t_{max}), and average (t_{avg}) computation time was extracted and reported in Table I. The implementation, which was not specifically optimized for speed (e.g. by warm-starting the solver), is based on the interior-point algorithm of `QPC` [26], running on a on a 2.5 Ghz Intel Core2 Duo CPU. The off-line polytopic computations of this example were performed using the MPT-Toolbox [27].

TABLE I

SIMULATION RESULTS (TIME IN ms)

	N_{var}	N_{con}	t_{min}	t_{avg}	t_{max}	$\max \left\| \frac{J-J_A}{J_A} \right\|$
A	41	117	4.5	5.3	18.7	0
B	22	76	2.3	2.8	12.9	0.0034 %
C	18	79	2.2	2.6	8.5	0.2198 %

Table I reports that the average computation time t_{avg} for controller B (C) is only about 53% (49%) of that for controller A. Maximal and minimal computation times t_{min} and t_{max} for controllers B and C have also been significantly reduced. Although Theorem 2 guarantees local optimality for all three controllers, it is not obvious how the reduced prediction horizons N_A and N_B together with the modified terminal costs affect the control performance during transients. Fortunately, it turns out that, at least for the example in this case study, the effect is very small. Consider to this end the actual cost $J(\mathbf{x},\mathbf{u}) := \sum_{k=0}^{N_{sim}} ||x(k)||_Q^2 + ||u(k)||_R^2$ of a state trajectory $\mathbf{x}$ driven by the control sequence $\mathbf{u}$ with simulation horizon N_{sim}. The last column of Table I contains the maximal relative difference between $J(\mathbf{x},\mathbf{u})$ and the cost $J_A(\mathbf{x}_A,\mathbf{u}_A)$ of the "true optimal trajectory" (the one resulting from controller A) over all 100 initial conditions. A maximal relative difference in the trajectory cost of less than 0.01% for controller B, and of about 0.2% for controller C shows that the performance loss is indeed very small in this case.

V. CONCLUSION

This paper proposes a refined formulation of a tube-based robust model predictive controller that uses interpolation over multiple, precomputed linear terminal controller gains in order to enlarge the closed-loop system's region of attraction. The controller was shown to guarantee persistent feasibility, robust exponential stability of a robust positively invariant set, increased attractivity as well as local optimality. One of its main benefits is that its on-line computation only amounts to solving a Quadratic Program.

Despite many similarities, the controller differs from the one discussed in [14], [15], [16] (in the following: "Sui's Controller") and features some important advantages. The most apparent difference concerns the choice of cost function and control law: Sui's Controller uses the control parametrization from [17] of the form $u_i = Kx + c_i$ for the first $N-1$ time steps. The controller presented in this paper instead follows [5] in using a control law of the form $\kappa_N^{ip}(\hat{x}) := \bar{u}_0^*(\hat{x}) + K(\hat{x} - \bar{x}_0^*(\hat{x}))$. As a result, by separating the evolution of actual system, observer system and virtual nominal system, controller synthesis is simplified. Specifically, the newly proposed controller only involves the computation of a positively invariant set for system (9) subject to appropriately tightened constraints, whereas Sui's Controller requires the computation of a *robust* positively invariant set in the augmented state space, a task which is significantly more involved.

Furthermore, allowing the initial state $\bar{x}_0$ of the nominal system to differ from the system state estimate $\hat{x}$, it is possible to prove robust *exponential* stability rather than robust asymptotic stability of a robust invariant set for the closed-loop system. Finally, the local optimality property is treated rigorously in this paper, yielding the result that optimal TBRMPC performance is recovered for all $\hat{x} \in \hat{\mathcal{X}}_N^0$.

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