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On the passivity approach to quantized coordination problems

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***Abstract*— We investigate a passivity approach to collective coordination problems in the presence of quantized measurements and show that coordination tasks can be achieved in a practical sense for a large class of passive systems. Both static and time-varying graphs are considered. The results are then specialized to some particular coordination problems and compared with existing results.**

I. INTRODUCTION

In the very active area of consensus, synchronization and coordinated control there has been an increasing interest in the use of quantized measurements and control ([16], [21], [13], [4], [17], [5] and references therein). As a matter of fact, since these problems investigate systems or agents which are distributed over a network, it is very likely that the agents must exchange information over a digital communication channel and quantization is one of the basic limitations induced by finite bandwidth channels. To cope with this limitation, measurements are processed by quantizers, i.e. discontinuous maps taking values in a discrete or finite set. Another reason to consider quantized measurements descends from the use of coarse sensors.

The use of quantized measurements induces a partition of the space of measurements: whenever the measurement function crosses the boundary between two adjacent sets of the partition, a new value is broadcast through the channel. As a consequence, when the networked system under consideration evolves in continuous time, as it is often the case with e.g. problems of coordinated motion, the use of quantized measurements results in a completely asynchronous exchange of information among the agents of the network. Despite the asynchronous information exchange and the use of a discrete set of information values, meaningful examples of synchronization or coordination can be obtained ([12], [7], [10]).

In view of the several contributions to quantized coordination problems available for discrete-time systems ([16], [21], [13], [4], [17], [5]), one may wonder whether it would be more convenient simply to derive the sampled-data model of the system and then apply the discrete-time results. Due to the distributed nature of the system, a sampled-data approach to the design of coordinated motion algorithms presents a few drawbacks: it might require synchronous sampling at all the nodes of the network and consequent accurate synchronization of all the node clocks; it might also require fast sampling rates, which may not be feasible in a networked system with a large number of nodes and connections. Finally, the sampled-data model may not fully preserve some of the features of the original model. For these reasons, we focus here on continuous-time coordination problems under quantized measurements.

Despite the unquestionable interest of the results in papers such as ([12], [7], [10]), they present an important limitation: they focus on agents with simple dynamics and on consensus problems. The goal of this paper is to investigate the potentials of an approach to coordinated motion which might take into account simultaneously complex dynamics for the agents of the network, advanced cooperative tasks and quantized measurements. This motivates us to focus on the passivity approach to coordinated motion problems proposed in [1]. In that paper, the author has shown how a number of coordination tasks could be achieved for a class of passive nonlinear systems and has been using this approach for related problems in subsequent work ([2], [3]). Others have been exploiting passivity in synchronization and coordination tasks ([23], [25], [8], [26] to name a few). On the other hand, the passivity approach naturally lends itself to deal with the presence of quantized measurements. As a matter of fact, the presence of quantized measurements can be taken into account by introducing in the feedback law static discontinuous maps (the previously recalled quantizers). In the approach of [1], these maps play the role of multivariable nonlinearities which are designed to achieve the desired coordination task under appropriate conditions. Although in the case of quantized measurements these conditions are not fulfilled due to the discontinuous nature of the quantizers, one can argue that an approximate or "practical" ([7]) coordination task is achievable under suitably modified conditions. This is the idea which is pursued in this paper. In the case of a control system with a single communication channel this was studied in [6].

The main contribution of the paper is to provide a passivity approach to coordinated control problems in the presence of quantization: some of the results of [1] are extended to deal with the case of quantized measurements. Because the latter introduce discontinuities in the system, a rigorous analysis is carried out relying on notions and tools from nonsmooth control theory and differential inclusions. Both static and time-varying graphs are considered. The results are then specialized to some particular coordination problems and compared with existing results. Although the passivity

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approach allows to consider a large variety of coordination control problems, in this paper for the sake of simplicity we mainly focus on agreement problems.

The passivity approach to coordination problems is recalled in Section II. In Section III the coordination control problem in the presence of uniform quantizers is formulated and the main results in the case of static graphs are presented along with some examples. The case of coordination with logarithmically quantized measurements and time-varying graphs is studied in Section IV. In Section V conclusions are drawn. In the Appendix some technical tools are reviewed.

II. THE PASSIVITY APPROACH TO THE COORDINATION PROBLEM

We recall in this section the passivity approach to coordination problems of [1], to which we refer the reader for more details.

Consider N agents connected over an undirected graph $G = (V, E)$, where V is a set of N nodes and $E \subseteq V \times V$ is a set of M edges connecting the nodes. Each agent i, with $i = 1, 2, \dots, N$, is associated to the node i of the graph and the edges which connects that node to other nodes of the graph describe which agents are communicating. We assume there are N variables $x_i \in \mathbb{R}^p$ which must be coordinated. Possibly after a preliminary feedback which uses information available locally, the agent is assumed to be strictly passive from the control input u_i to the velocity error $y_i = \dot{x}_i - v$, where $v(t)$ is the prescribed reference velocity of the formation, i.e. the velocity to which all the agent should converge asymptotically. More precisely, it is assumed that each agent's dynamics can be described as:

$$\begin{array}{rcll} \dot{x}_i & = & y_i + v(t) & \\ \dot{\xi}_i & = & f_i(\xi_i) + g_i(\xi_i)u_i & \\ y_i & = & h_i(\xi_i), & i = 1, 2, \dots, N, \end{array} \tag{1}$$

with $\xi_i \in \mathbb{R}^{n_i}$ a state variable, $u_i \in \mathbb{R}^p$, $f_i(\mathbf{0}) = \mathbf{0}$, $g_i(\mathbf{0}) \neq 0$, $h_i(\mathbf{0}) = \mathbf{0}$, and f_i, g_i, h_i locally Lipschitz. Moreover, by strict passivity we intend that there exists a continuously differentiable storage function $S_i : \mathbb{R}^{n_i} \to \mathbb{R}$ which is positive definite and radially unbounded and for which

$$\frac{\partial S_i}{\partial \xi_i}(f_i(\xi_i) + g_i(\xi_i)u_i) \leq -W_i(\xi_i) + y_i^T u_i \tag{2}$$

for all $\xi_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^p$, and where $W_i(\xi_i)$ is a positive definite function. Concisely, the systems (1) are written as

$$\begin{array}{l} \dot{\xi} = \underbrace{\begin{pmatrix} f_1(\xi_1) \\ \vdots \\ f_N(\xi_N) \end{pmatrix}}_{f(\xi)} + \underbrace{\begin{pmatrix} g_1(\xi_1) & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & g_N(\xi_N) \end{pmatrix}}_{g(\xi)} u \\ \dot{x} = \underbrace{\begin{pmatrix} h_1(\xi_1) \\ \vdots \\ h_N(\xi_N) \end{pmatrix}}_{h(\xi)} + \underbrace{\begin{pmatrix} v \\ \vdots \\ v \end{pmatrix}}_{\mathbf{1}_N \otimes v} \end{array} \tag{3}$$

where $\xi = (\xi_1^T \dots \xi_N^T)^T$, $\mathbf{1}_N$ is the N-dimensional vector whose entries are all equal to 1 and the symbol $\otimes$ denotes the Kronecker product of matrices (see Appendix A for a definition).

Label one end of each edge in E by a positive sign and the other one by a negative sign. Now, consider the kth edge in E, with $k \in \{1, 2, \dots, M\}$, and let i, j be the two nodes connected by the edge. Then, the variable z_k which describes the difference between the variables x_i, x_j which must be coordinated can be defined as follows:

$$z_k = \begin{cases} x_i - x_j & \text{if } i \text{ is the positive end of the edge } k \\ x_j - x_i & \text{if } i \text{ is the negative end of the edge } k\ . \end{cases}$$

Recall also that the incidence matrix D associated with the graph G is the $N \times M$ matrix such that

$$d_{ik} = \begin{cases} +1 & \text{if node } i \text{ is the positive end of edge } k \\ -1 & \text{if node } i \text{ is the negative end of edge } k \\ \ \ 0 & \text{otherwise} \end{cases}$$

By the definition of D,

$$z = [z_1^T \dots z_M^T]^T = (D^T \otimes I_p)x\ . \tag{4}$$

The formation control problem consists of designing each control law u_i, with $i = 1, 2, \dots, N$, in such a way that it uses only the information available to the agent i and guarantees the following two specifications:
(i) $\lim_{t\to\infty} |\dot{x}_i(t) - v(t)| = 0$ for each $i = 1, 2, \dots, N$, with $v(t)$ a bounded and piece-wise continuous reference velocity for the formation;
(ii) $z_k(t) \to \mathcal{A}_k$ as $t \to \infty$ for each $k = 1, 2, \dots, M$, where $\mathcal{A}_k \subset \mathbb{R}^p$ are the prescribed sets of convergence[1].
The feedback laws proposed in [1] to solve the problem formulated above is:

$$u_i = -\sum_{k=1}^{M} d_{ik}\psi_k(z_k)\ ,\ i = 1, 2, \dots, N \tag{5}$$

where the maps $\psi_k : \mathbb{R}^p \to \mathbb{R}^p$ are to be designed. Observe that, as required, each control law u_i uses only information which is available to the agent i: indeed, $d_{ik} \neq 0$ if and only if the edge k connects i to one of its neighbors. In compact form, (5) can be rewritten as

$$u = [u_1^T \dots u_N^T]^T = -(D \otimes I_p)\psi(z)\ , \tag{6}$$

having set $\psi(z) = [\psi_1(z)^T \dots \psi_M(z)^T]^T$ and where z is as in (4). Before ending the section, we recall that the system below with input $\dot{x}$ and output $-u$, namely (see Figure 2 in [1] for a pictorial representation of the system)

$$\begin{array}{rcl} \dot{z} & = & (D^T \otimes I_p)\dot{x} \\ -u & = & (D \otimes I_p)\psi(z) \end{array} \tag{7}$$

is passive with storage function $\sum_{k=1}^{M} P_k(z_k)$, where P_k is required to be a $\mathcal{C}^2$ nonnegative function such that

$$\nabla P_k(z_k) = \psi_k(z_k)\ . \tag{8}$$

[1]We refer the interested reader to [1] for examples of sets $\mathcal{A}_k$ related to some coordination problems. The sets $\mathcal{A}_k$ which are of interest in this paper will be introduced below.

The requirement on P_k to be $\mathcal{C}^2$ is removed in the next section. The function $P_k(z_k)$ is chosen in such a way that the region where the variable z_k must converge for the system to achieve the prescribed coordination task coincides with the set of the global minima of $P_k(z_k)$. Hence, the coordination task guides the design of $P_k(z_k)$ which in turn allows to determine the control functions (5) via (8). The functions $P_k(z_k)$ in the case of agreement problems via quantized control laws will be designed below.

III. Quantized coordination control

A. Quantized measurements

In this paper we are interested in control laws which use quantized measurements. For each $k = 1, 2, \ldots, M$, instead of z_k, the measurements $\mathrm{q}_{\mathbf{\Delta}}(z_k)$ are available, where $\mathrm{q}_{\mathbf{\Delta}}$ is the quantizer map. For the sake of simplicity, in this section we focus on uniform quantizers and static graphs. We refer the reader to Section IV for the case of logarithmic quantizers.

Given a positive real number Δ we let $\mathrm{q}_\Delta : \mathbb{R} \to \mathbb{Z}\Delta$ be the function

$$\mathrm{q}_\Delta(\zeta) = \Delta \left\lfloor \frac{\zeta}{\Delta} + \frac{1}{2} \right\rfloor \tag{9}$$

with $\lfloor \cdot \rfloor$ the floor function and $\frac{1}{\Delta}$ the precision of the quantizer. As $\Delta \to 0$, $\mathrm{q}_\Delta(\zeta) \to \zeta$. The vector of quantized measurements corresponding to z_k is $\mathrm{q}_{\mathbf{\Delta}_k}(z_k) = (\mathrm{q}_{\Delta_{k1}}(z_{k1}) \ldots \mathrm{q}_{\Delta_{kp}}(z_{kp}))^T$, with $\mathbf{\Delta}_k = (\Delta_{k1} \ldots \Delta_{kp})^T$ a vector of positive real numbers. Hence, each entry of z_k is quantized independently of the others and the quantized information is then used in the control law.

B. A practical agreement problem

Despite the generality allowed by that the passivity approach, for the sake of simplicity we focus here on an agreement problem. By an agreement problem it is meant a special case of coordination in which all the variables x_i connected by a path converge to each other. In the problem formulation in Section II, this amounts to have $\mathcal{A}_k = \{\mathbf{0}\}$ for all $k = 1, 2, \ldots, M$. When using quantized measurements, however, it is a well established fact ([16], [12], [7]) that a coordination algorithm may only lead to a practical agreement result, meaning that each variable z_k might converge to a compact set of the origin, rather than to the origin itself. Motivated by this observation, we set in this paper a weaker convergence goal, namely for each $k = 1, 2, \ldots, M$ we ask the target set $\mathcal{A}_k$ to be of the form:

$$\mathcal{A}_k = \bigtimes_{j=1}^{p} [-a_{kj}, a_{kj}] \tag{10}$$

where $\mathbf{a}_k = (a_{k1} \ldots a_{kp})^T$ is a vector of positive constants and the symbol $\times$ denotes the Cartesian product. Then the design procedure of Section II prescribes to choose a potential function $P_k(z_k)$ which is radially unbounded on its domain of definition and such that

$$P_k(z_k) = 0 \text{ and } \nabla P_k(z_k) = \mathbf{0} \quad \text{if and only if} \quad z_k \in \mathcal{A}_k. \tag{11}$$

If this is the case then the control law is chosen via (8). To take into account the presence of quantized measurements, the nonlinearities (8) should take the form

$$\nabla P_k(z_k) = \tilde{\psi}_k(\mathrm{q}_{\mathbf{\Delta}_k}(z_k)) \, . \tag{12}$$

Then a possible function $P_k(z_k)$ with the properties (11) and such that a function $\tilde{\psi}_k$ exists for which (12) holds, is $P_k(z_k) = \sum_{j=1}^{p} \int_0^{z_{kj}} \mathrm{q}_{\Delta_{kj}}(s)ds$. Such a function is defined on all $\mathbb{R}^p$, is radially unbounded and locally Lipschitz. By Rademacher's theorem it is differentiable almost everywhere. In all the points of $\mathbb{R}^p$ where it is differentiable $\nabla P_k(z_k) = \mathrm{q}_{\mathbf{\Delta}_k}(z_k)$ i.e. (12) holds with $\tilde{\psi}_k = \mathrm{Id}$ and $\mathrm{Id} : \mathbb{R}^p \to \mathbb{R}^p$ the identity function. Bearing in mind the definitions (9) and (10), to satisfy the second equality in (11) on all the points of $\mathcal{A}_k$ where $P_k(z_k)$ is differentiable it is necessary and sufficient to set $a_{kj} = \frac{\Delta_{kj}}{2}$, for all $j \in 1, 2, \ldots, p$. With such a choice, the first equality of (11) is also satisfied.

In what follows we examine the evolution of the system (3) under the control law:

$$u_i = -\sum_{k=1}^{M} d_{ik} \mathrm{q}_{\mathbf{\Delta}_k}(z_k) \, , \; i = 1, 2, \ldots, N \tag{13}$$

where $\mathbf{\Delta}_k = \mathbf{a}_k$ for all $k = 1, 2, \ldots, N$ and the vectors $\mathbf{a}_k$ define the target sets $\mathcal{A}_k$.

C. Closed-loop system

Similarly to (6), we write the quantized control law as:

$$u = -(D \otimes I_p)\mathrm{q}(z) \, , \tag{14}$$

where $\mathrm{q}(z) = (\mathrm{q}_{\mathbf{\Delta}_1}(z_1)^T \ldots \mathrm{q}_{\mathbf{\Delta}_p}(z_p)^T)^T$. The closed-loop system then takes the following expression:

$$\begin{array}{rcl} \dot{\xi} & = & f(\xi) + g(\xi)(-(D \otimes I_p)\mathrm{q}(z)) \\ z & = & (D^T \otimes I_p)x \\ \dot{x} & = & h(\xi) + \mathbf{1}_N \otimes v. \end{array} \tag{15}$$

The system above has a discontinuous right-hand side due to the presence of the quantization functions and its analysis requires to introduce a suitable notion of solution. In this paper we adopt Krasowskii solutions. In fact, it was shown in [7] that Carathéodory solutions may not exist for agreement problems. Moreover, Krasowskii solutions include Carathéodory solutions and the results we derive for the former also holds for the latter in case they exist.

Denoted by $\dot{X}(t) = F(t, X)$ the system (15), a function $X(\cdot)$ defined on an interval $I \subset \mathbb{R}$ is a Krasowskii solution to the system on I if it is absolutely continuous and satisfies the differential inclusion ([14])

$$\dot{X}(t) \in \mathcal{K}(F(t, X)) := \bigcap_{\delta > 0} \overline{\mathrm{co}}\,(F(t, B(X, \delta)))$$

for almost every (a.e.) $t \in I$. Since the right-hand side of (15) is locally bounded, local existence of Krasowskii solutions is guaranteed ([14]).

Recalling that $(D^T \otimes I_p)(\mathbf{1}_N \otimes v) = \mathbf{0}$, the system (15) in the coordinates (ξ, z) writes as

$$\begin{array}{rcl} \dot{\xi} & = & f(\xi) + g(\xi)(-(D \otimes I_p)\mathrm{q}(z)) \\ \dot{z} & = & (D^T \otimes I_p)h(\xi). \end{array} \tag{16}$$

Even the system above is discontinuous and again its solutions must be intended in the Krasowskii sense. It is straightforward to verify that, given any Krasowskii solution (x, ξ) to (15), the function $(z, \xi) = ((D^T \otimes I_p)x, \xi)$ is a Krasowskii solution to (16). In what follows we investigate the asymptotic properties of the Krasowskii solutions to (16) and infer stability properties of (15). Below, we state two results whose proofs are omitted for lack of space. The interested reader is referred to [11].
The first statement concerns system (16).

Lemma 1 *Assume that the graph G is connected and that (2) holds. Then, any Krasowskii solution to (16) converges to the set of Krasowskii equilibria*[2]

$$\{(\xi, z) \,:\, \xi = \mathbf{0},\ \mathbf{0} \in (D \otimes I_p)\mathcal{K}(\mathrm{q}(z))\}\ . \qquad (17)$$

Based on the previous lemma, one can show the following:

Theorem 1 *Assume that the graph G is connected and that (2) holds. Let $v : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ be a bounded and piecewise continuous function and $\mathbf{\Delta}_k \in \mathbb{R}^p$, $k = 1, 2, \ldots, M$, be vectors of positive numbers. Then any Krasowskii solution to (15) converges to the set*

$$\{(x, \xi) : \xi = \mathbf{0}\ ,\ z \in (\mathcal{A}_1 \times \ldots \times \mathcal{A}_M), z = (D^T \otimes I_p)x\}, \qquad (18)$$

where the sets $\mathcal{A}_k$'s are as in (10), with $\mathbf{a}_k = \mathbf{\Delta}_k/2$ for all $k = 1, \ldots, M$. Moreover, $\lim_{t\to+\infty}[\dot{x}(t) - \mathbf{1}_N \otimes v(t)] = \mathbf{0}$.

D. Examples

We provide two examples of application of the quantized agreement result described above.

Agreement of single integrators by quantized measurements We specialize Lemma 1 and Theorem 1 to the agreement problem for single integrators. The closed-loop system (15) reduces to

$$\begin{array}{rcl} \dot{x} & = & -(D \otimes I_p)\mathrm{q}(z) \\ z & = & (D^T \otimes I_p)x \end{array} \qquad (19)$$

which using the variables z becomes

$$\dot{z} = -(D^T \otimes I_p)(D \otimes I_p)\mathrm{q}(z) = -(D^T D \otimes I_p)\mathrm{q}(z) \quad (20)$$

In this case, Lemma 1 gives that all the Krasowskii solutions to (20) converge to the set of points $\{z \,:\, \mathbf{0} \in (D \otimes I_p)\mathcal{K}(\mathrm{q}(z))\}$. On the other hand, by Theorem 1, any Krasowskii solution $x(t)$ to (19) is such that $z(t) = (D^T \otimes I_p)x(t)$ converges to $\{x \,:\, \mathbf{0} \in (D \otimes I_p)\mathcal{K}(\mathrm{q}(z)),\ z = (D^T \otimes I_p)x\}$ which is included in the set $\{z \,:\, z \in \mathcal{A}_1 \times \ldots \times \mathcal{A}_p,\ z = (D^T \otimes I_p)x\}$. Let x be any Krasowskii solution to (19) with $z = (D^T \otimes I_p)x$. Take any two variables x_i, x_j whose agents are connected by the edge k. Consider for the sake of simplicity that each quantizer has the same parameter Δ. Then $z_k = x_i - x_j$ converges asymptotically to a square of the origin whose edge is not longer than Δ. If the agents are not connected by an edge but by a path, then each entry of $x_i - x_j$ is in magnitude bounded by $\Delta \cdot d$, with d the diameter of the graph. The result can be compared with Theorem 4 in [12]. One difference is that, while trees are considered in [12], connected graphs are considered here. Moreover, in [12] the scalar states are guaranteed to converge to a ball of radius $\frac{||D^T D|| \sqrt{M}}{\lambda_{min}(D^T D)}\Delta$. Hence, denoted by ρ the ratio $\frac{||D^T D||}{\lambda_{min}(D^T D)}$ and considered the bound $M \leq N - 1$, any two states x_i, x_j may differ for $2\sqrt{N-1}\rho\Delta$. The passivity approach considered here yields that they differ for not more than $d \cdot \Delta$, where d grows as $O(\rho \log(N))$ ([9]) for not complete and regular graphs (graphs with all the nodes having the same degree).

[2] (ξ_0, z_0) is a Krasowskii equilibrium for (16) if the function $(\xi(t), z(t)) = (\xi_0, z_0)$ is a Krasowskii solution to (16) which originates from the initial condition (ξ_0, z_0).

Agreement of double integrators by quantized measurements Consider the case of N agents modeled as

$$\ddot{x}_i = f_i\ , \quad i = 1, 2, \ldots, N, \qquad (21)$$

with $x_i, f_i \in \mathbb{R}^2$, for which we want to solve the agreement problem with quantized measurements. The preliminary feedback ([1])

$$f_i = -K_i(\dot{x}_i - v) + \dot{v} + u_i\ , \quad K_i = K_i^T\ , \qquad (22)$$

with u_i to design, and the change of variables $\xi_i = \dot{x}_i - v$, makes the closed-loop system

$$\begin{array}{rcl} \dot{x}_i & = & \xi_i + v \\ \dot{\xi}_i & = & -K_i\xi_i + u_i \\ y_i & = & \xi_i \end{array}$$

passive with storage function $S_i(\xi_i) = \frac{1}{2}\xi_i^T\xi_i$ and $W_i(\xi_i) = -K_i\xi_i^T\xi_i$. The system above is in the form (1). Theorem 1 guarantees that the Krasowskii solutions of (21), (22), (14) converges asymptotically to the set (18) and that all the agents' velocities converge to v. In other words, the formation achieves practical position agreement and convergence to the prescribed velocity. A related quantized coordination problem for double integrators can be found in [18].

IV. Logarithmic quantizers and agreement control under time-varying graph topology

In this section we consider the quantized agreement control problem (19) in the case of logarithmic quantizers. Moreover, we focus on graphs with a time-varying topology: the results for static graphs are an immediate consequence and are omitted. Logarithmic quantizers are continuous at the origin and this allows us to obtain asymptotic agreement results rather than practical results as in the previous section. Moreover, the construction of the functions involved in the Matrosov Theorem for differential inclusions which is at the basis of the result is very natural when using logarithmic quantizers.
We briefly recall the definition of logarithmic quantizers. Let q_0 be a positive real number, $\delta \in (0, 1)$, and $q_i = \rho^i q_0$ for $i \in \mathbb{Z}$ with $\rho = \frac{1-\delta}{1+\delta}$. Then the map $\mathrm{q}_\ell : \mathbb{R} \to \{\pm q_i \,:\, i \in \mathbb{Z}\} \cup \{0\}$ defined as

$$\mathrm{q}_\ell(y) = \left\{ \begin{array}{lll} q_i & \dfrac{q_i}{1+\delta} < y \leq \dfrac{q_i}{1-\delta}\ , & i \in \mathbb{Z} \\ 0 & y = 0 & \\ -\mathrm{q}_\ell(-y) & y < 0\ , & \end{array} \right. \qquad (23)$$

is the logarithmic quantizer.
The system (19) with a time-varying incidence matrix and logarithmic quantizers becomes

$$\begin{array}{rcl} \dot{x} & = & -(D(t)\otimes I_p)\mathrm{q}_\ell(z) \\ z & = & (D(t)^T\otimes I_p)x \end{array} \tag{24}$$

with $\mathrm{q}_\ell(z) = (\mathrm{q}_\ell(z_{11})\ldots\mathrm{q}_\ell(z_{Mp}))^T$. Let Q be the $(N-1)\times N$ matrix with orthonormal rows orthogonal to the span of $\mathbf{1}_N$, i.e. $Q\mathbf{1}_N = \mathbf{0}_{N-1}$ and $QQ^T = I_{N-1}$. The incidence matrix $D(t)$ is assumed to satisfy the following:

Assumption 1 *$D(t)$ is piece-wise continuous and bounded. Moreover, there exist $\delta, \alpha > 0$ such that for all $t_0 \geq 0$,*

$$\int_{t_0}^{t_0+\delta} QD(t)D(t)^TQ^T dt \geq \alpha I\ . \tag{25}$$

The condition (25), introduced in [1], amounts to a graph uniform connectivity assumption which reminds analogous conditions in [15], [20].
Define the new variable $\zeta = (Q\otimes I_p)x$. By definition of Q, $\zeta = \mathbf{0}$ if and only x belongs to the span of $\mathbf{1}_N\otimes I_p$, i.e. if and only if x lies in the agreement set $x_1 = x_2 = \ldots = x_N$. Using the variable ζ, the system (24) can be rewritten as

$$\dot{\zeta} = -F(t)\mathrm{q}_\ell(F^T(t)\zeta)\ , \tag{26}$$

where $F(t) = QD(t)\otimes I_p$. As a consequence of (25), there exist $\delta, \alpha > 0$ such that for all $t_0 \geq 0$,

$$\int_{t_0}^{t_0+\delta} F(t)F(t)^T dt \geq \alpha I\ . \tag{27}$$

Inspired by [1], we analyze the system (26) above using a Matrosov theorem. Because of the discontinuities induced by the quantizers, we resort to a Matrosov theorem for differential inclusions which is obtained from [24] specializing Theorem 4.1 for hybrid systems therein to differential inclusions. The result is recalled in Appendix B.
Here we prove the following lemma:

Lemma 2 *Let $F(t)$ be a piece-wise continuous and bounded matrix which satisfies (27). Then, the set $\mathcal{A} = \{\mathbf{0}\}$ is uniformly globally asymptotically stable (UGAS) for (26).*

Proof: First we prove that $\mathcal{A} = \{\mathbf{0}\}$ is UGS. Take $V_1(\zeta) = \frac{1}{2}\zeta^T\zeta$. Consider the differential inclusion associated with (26), namely

$$\dot{\zeta} \in -\mathcal{K}(F\mathrm{q}_\ell(F^T\zeta)) = -F\mathcal{K}(\mathrm{q}_\ell(\phi)) \tag{28}$$

where for the sake of notational economy we have dropped the dependence of F on t, we have defined $\phi = F^T\zeta$ and where the second equality holds by Theorem 1 in [22]. Let Λ be the following set of diagonal matrices: $\Lambda = \{\mathcal{D} \in \mathbb{R}^\nu\times\mathbb{R}^\nu\ :\ \mathcal{D} = \mathrm{diag}(\lambda_1,\ldots,\lambda_\nu)\ , \lambda_i \in [-1,1]\ ,\ \forall i \in \{1,\ldots,\nu\}\}$, where $\nu = M\,p$. Then, $F\mathcal{K}(\mathrm{q}_\ell(\phi)) \subseteq F\{(I+\mathcal{D}\delta)\phi\ :\ \mathcal{D}\in\Lambda\}$ where we have exploited the property that $\mathcal{K}(\mathrm{q}_\ell(\phi)) \subseteq \times_{i=1}^{\nu}\mathcal{K}(\mathrm{q}_\ell(\phi_i))$ ([22]). Hence, for any $f \in \mathcal{K}(F\mathrm{q}_\ell(\phi))$

$$\langle\nabla V_1(\zeta), f\rangle = -\zeta^TF(I+\mathcal{D}\delta)\phi = -\phi^T(I+\mathcal{D}\delta)\phi$$

for some $\mathcal{D}\in\Lambda$. Since $0<\delta<1$ and $\mathcal{D}$ is a diagonal matrix with diagonal elements in $[-1,1]$, it follows that $\langle\nabla V_1(\zeta), f\rangle = -\sum_i(1+\lambda_i\delta)|\phi_i|^2 \leq -\sum_i(1-\delta)|\phi_i|^2 = -(1-\delta)|\phi|^2 =: Y_1(\phi)$, or equivalently $\sup_{f\in\mathcal{K}(F\mathrm{q}_\ell(\phi))}\langle\nabla V_1(\zeta), f\rangle \leq Y_1(\phi) \leq 0$.
Hence, $\mathcal{A}$ is UGS by Theorem 3 in the Appendix.
Pick now $V_2(t,\zeta) = -\zeta^TS(t)\zeta$, with $S(t) := \int_t^\infty \mathrm{e}^{(t-\tau)}F(\tau)F^T(\tau)d\tau$. It is easily proven ([1]) that $||S(t)|| \leq \mu^2$ and $\nabla_t V_2(t,\zeta) = -\zeta^TS(t)\zeta + \phi^T\phi$.
Moreover, for any $f\in\mathcal{K}(F\mathrm{q}_\ell(\phi))$

$$\langle\nabla_\zeta V_2(t,\zeta), f\rangle = -2\zeta^TS(t)F(I+\mathcal{D}\delta)\phi \leq 2|\zeta|\mu^3(1+\delta)|\phi|$$

where the equality holds for some $\mathcal{D}\in\Lambda$. Hence,

$$\begin{array}{c} \sup_{f\in\mathcal{K}(F\mathrm{q}_\ell(\phi))}\langle\nabla_\zeta V_2(t,\zeta), f\rangle + \nabla_t V_2(t,\zeta) \leq \\ 2|\zeta|\mu^3(1+\delta)|\phi| - \zeta^TS(t)\zeta + \phi^T\phi =: Y_2(\zeta,\phi)\ . \end{array}$$

Recall that $Y_1(\phi)\leq 0$. Moreover, $Y_1(\phi) = 0$ implies $\phi = 0$ which in turn implies $Y_2(\zeta,\phi) = -\zeta^TS(t)\zeta \leq 0$. Hence, all the conditions of Theorem 4 are satisfied, and this implies that $\mathcal{A} = \{\mathbf{0}\}$ is UGAS. □

Let now x be a Krasowskii solution to (24). Then $\zeta = (Q\otimes I_p)x$ is a Krasowskii solution to (28). Since $\mathcal{A} = \{\mathbf{0}\}$ is UGAS for (28), then ζ converges to the origin and by definition x must converge to $\mathrm{span}\{\mathbf{1}_N\otimes I_p\}$. We summarize as follows:

Theorem 2 *Let Assumption 1 hold. For any $\delta\in(0,1)$ for any $q_0 > 0$, all the Krasowskii solutions to (24) converge asymptotically to* $\mathrm{span}\{\mathbf{1}_N\otimes I_p\}$.

We remark that in the theorem above no restriction on the quantization parameter δ is assumed. Moreover, only a uniform connectivity assumption is made on the time-varying graph. These two features mark a major difference with respect to [12], Theorems 5 and 6.

V. Conclusions

The passivity approach to coordinated control problems presents several interesting features such as for instance the possibility to deal with agents which have complex high-dimensional dynamics and with advanced coordination tasks. In this paper we have shown how it also lends itself to take into account the presence of quantized measurements. Using the passivity framework along with appropriate tools from nonsmooth control theory and differential inclusions, we have shown that many of the results of [1] continue to hold in an appropriate sense in the presence of quantized information. We believe that the results presented in the paper are a promising addition to the existing literature on continuous-time consensus and coordinated control under quantization.

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APPENDIX

A. Notation

The Kronecker product of the matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$ is the matrix

$$A \otimes B = \begin{pmatrix} a_{11}\,B & \dots & a_{1n}\,B \\ \vdots & \ddots & \vdots \\ a_{m1}\,B & \dots & a_{mn}\,B \end{pmatrix}.$$

See e.g. [1], [25] for some basic properties.

B. Results for UGS and UGAS of sets in differential inclusions

Consider the differential inclusion:

$$\dot x \in F(t,x)\ , \quad x \in \mathbb{R}^n\ . \tag{29}$$

The following results give sufficient conditions for global uniform stability (UGS) and global uniform asymptotic stability (UGAS) for (29). The interested reader is referred to [19], [24] for a definition of these concepts.

Theorem 3 ([24], Theorem 2.4) *The closed set $\mathcal{A} \subset \mathbb{R}^n$ is UGS for (29) if there exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and class-$\mathcal{K}_\infty$ functions α_1, α_2 such that*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$$

$$\sup_{f\in F(t,x)} \langle \nabla V(x), f\rangle \leq 0\ , \qquad \textit{for all } (t,x) \in \mathbb{R}\times\mathbb{R}^n\ ,$$

where $|x|_{\mathcal{A}}$ denotes the distance of x from $\mathcal{A}$, i.e. $|x|_{\mathcal{A}} = \inf_{\omega\in\mathcal{A}}|x-\omega|$.

In the statement below (Matrosov Theorem) the following notation is in use: $\Omega_{\mathcal{A}}(\delta,\Delta) = \{x \in \mathbb{R}^n : \delta \leq |x|_{\mathcal{A}} \leq \Delta\}$ and $\Upsilon_{\mathcal{A}}(\delta,\Delta) = \mathbb{R}_{\geq 0}\times\Omega_{\mathcal{A}}(\delta,\Delta)$.

Theorem 4 (Matrosov) *Let $\mathcal{A} \subset \mathbb{R}^n$ be a compact set which is UGS for (29). $\mathcal{A}$ is uniformly globally asymptotically stable (UGAS) for (29) if:*
Assumption 1: *There exist $m, s \in \mathbb{Z}_{\geq 1}$ and, for each $0 < \delta < \Delta$,*

- *a number $\mu > 0$*
- *a function $\phi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^s$*
- *continuous functions $Y_i : \Omega_{\mathcal{A}}(\delta,\Delta)\times\mathbb{R}^s \to \mathbb{R}$, $i \in \{1,2,\dots,m\}$*
- *functions $V_i : \mathbb{R}_{\geq 0}\times\mathbb{R}^n \to \mathbb{R}$, $i \in \{1,2,\dots,m\}$, continuously differentiable on an open set of $\Upsilon_{\mathcal{A}}(\delta,\Delta)$*

such that, for each $i \in \{1,2,\dots,m\}$,

$$\max\{|V_i(t,x)|, |\phi(t,x)|\} \leq \mu$$

$$\sup_{f\in F(t,x)} \nabla V_i f(t,x) + \nabla_t V_i(t,x) \leq Y_i(x,\phi(t,x))$$

for all $(t,x) \in \Upsilon_{\mathcal{A}}(\delta,\Delta)$;
Assumption 2: *For each $j \in \{0,1,2,\dots,m\}$*

$$x \in \Omega_{\mathcal{A}}(\delta,\Delta), |\psi| \leq \mu, Y_i(x,\psi) = 0\ \forall i \in \{0,1,2,\dots,j\}$$
$$\Downarrow$$
$$Y_{j+1}(x,\psi) = 0$$

with $Y_0, Y_{m+1} : \mathbb{R}^n\times\mathbb{R}^s \to \{0\}$ identically zero functions.