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Anti-windup Design for a Class of Multivariable Nonlinear Control Systems: an LMI-based Approach

M. Z. Oliveira, J. M. Gomes da Silva Jr, D. Coutinho, S. Tarbouriech

Abstract— **This paper focuses on the problem of static anti-windup design for a class of multivariable nonlinear systems subject to actuator saturation. More precisely, a convex approach is proposed to compute a static anti-windup gain which ensures regional stability for the closed-loop system assuming that a nonlinear dynamic output feedback controller is previously designed to stabilize the nonlinear system. The results are based on the differential-algebraic representation of rational systems and a modified sector bound condition is applied to model the saturation effects. From these elements, LMI based conditions are devised to compute an anti-windup gain for enlarging the closed-loop region of attraction. A numerical example is given to illustrate the proposed method.**

I. Introduction

The general principle of the anti-windup technique is the introduction of an extra feedback loop in a pre-designed control system to mitigate the effects caused by saturation. In spite of existing many different techniques and approaches, the majority of the results regards linear models (see [1]–[3] and references therein) and focuses on the performance improvement. Moreover, it is shown that through the design of anti-windup compensators, we can also enlarge the region of attraction of the closed-loop system, as illustrated in [4], [5] for linear systems. However, if the anti-windup mechanism is designed based on the linear approximation of the nonlinear dynamics, it can lead to a poor behavior when implemented on the original nonlinear control system. Further, the computed region of attraction of the closed-loop system considering the linear approximation may be highly modified by the nonlinear dynamics. In general, one cannot ensure that a region of stability computed considering the linear approximation will be valid for the actual nonlinear system.

In addition, only few works have addressed the anti-windup synthesis problem for nonlinear systems subject to saturating actuators. We can cite, for instance, the references [6], [7] which consider anti-windup synthesis for linear-parameter varying systems, [8] which proposes anti-windup methods for Euler-Lagrange systems, and [9] which considers an adaptive control design. We can also cite some related works dealing with an anti-windup architecture for systems with nonlinear dynamic inversion (NDI) such as the references [10]–[14].

On the other hand, a key problem to characterize the stability of nonlinear systems is to determine a non conservative estimate of the system region of attraction. In general, the estimates are obtained from Lyapunov domains (see, for instance, the references [15]–[19]. In this context, in [20] a dynamic anti-windup compensator is proposed for the class of quadratic systems aiming at enlarge the estimate of the region of attraction. More recently, an approach to compute an anti-windup gain has been proposed in [21] for the class of rational nonlinear systems subject to actuator saturation. It turns out that in [21], the method is based on a non-convex condition, although the problem solution is obtained from LMI relaxations, and the multivariable case is not addressed.

In light of the above scenario, this paper aims at devising a numerical and tractable technique to design static anti-windup compensators for a class of nonlinear systems subject to actuator saturation. The class of systems considered in this paper covers all systems modeled by rational differential equations. We emphasize that a large class of systems can be embedded in this setup such as quadratic systems, polynomial systems and rational systems. Further, the proposed technique can deal with more complex nonlinearities by means of additional algebraic constraints and/or change of variables (see, e.g., [22], [23]). In particular, the method to be presented in the sequel applies a differential algebraic representation (DAR) of nonlinear systems letting to cast Lyapunov based stability conditions in terms of a finite set of state-dependent linear matrix inequalities which can be numerically solved at the vertices of a given polytope of admissible states. To deal with the saturation nonlinearity, a modified version of the generalized sector bound condition proposed in [5] is also considered. From these elements, we derive regional stabilizing conditions directly in LMI form.In addition, an LMI based optimization problem is devised to compute an anti-windup gain in order to obtain a maximized region of asymptotic stability, which implicitly leads to the maximization of the basin of attraction of the closed-loop system. This work can be seen as a further development of our previous result proposed in [21], where the main differences and advantages are: (a) the conditions are directly cast in terms of LMIs avoiding iterative relaxation schemes, and (b) it allows to consider multivariable nonlinear control systems in a straightforward way.

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M. Z. Oliveira is with the Department of Electrical Engineering, UFRGS, Av. Osvaldo Aranha 103, 90035-190 Porto Alegre-RS, Brazil, and with University of Caxias do Sul - UCS, Rua Francisco Getulio Vargas 1131, 95050-560, Caxias do Sul-RS, Brazil. zardo@ece.ufrgs.br

J. M. Gomes da Silva Jr is with the Department of Electrical Engineering, UFRGS, Av. Osvaldo Aranha 103, 90035-190 Porto Alegre-RS, Brazil. jmgomes@ece.ufrgs.br

D. Coutinho is with the Department of Automation and Systems, UFSC, PO Box 476, 88040-900, Florianópolis, Brazil, daniel.coutinho@ufsc.br

S. Tarbouriech is with the CNRS; LAAS; 7 Avenue Colonel Roche, F-31077 Toulouse, France. Université de Toulouse; UPS; INSA; INP; ISAE; LAAS; F-31077 Toulouse, France, tarbour@laas.fr

The paper is organized as follows. Section II introduces the problem to be addressed in the paper. Section III provides preliminary results concerning the system representation, the Lyapunov theory, and the modified sector bound condition. The main result is presented in Section IV, where the computation of the anti-windup gain is obtained by means of an optimization problem. An illustrative example is given in Section V demonstrating the potentialities of the proposed approach. Section VI ends the paper with some concluding remarks.

Notation: I_n is the $n \times n$ identity matrix and 0 may either denote the scalar zero or a matrix of zeros with appropriate dimensions. For a real matrix H, H' denotes its transpose and $H > 0$ means that H is symmetric and positive definite. For a block matrix, the symbol $\star$ represents symmetric blocks outside the main diagonal block. For a given polytope Φ, $\mathcal{V}(\Phi)$ is the set of vertices of Φ. Matrix and vector dimensions are omitted whenever they can be inferred from the context.

II. Problem Statement

Consider the following class of nonlinear control systems:

$$\begin{array}{rcl} \dot{x}(t) & = & f_x(x(t)) + g(x(t))sat(v_c(t)) \\ y(t) & = & H_{y_x}x(t) \end{array} \tag{1}$$

where $x \in \mathcal{B}_x \subset \mathbb{R}^n$ denotes the state vector; $y \in \mathbb{R}^{n_y}$ is the measured output; $v_c \in \mathbb{R}^{n_v}$ is the control input; $sat(\cdot)$ is the classical unit saturation function, i.e., $sat(v_c(t)) := sign(v_c(t))\min\{|v_c(t)|, 1\}$; and $H_{y_x} \in \mathbb{R}^{n_y \times n}$ is a constant matrix. It is assumed that $f_x, g \; : \; \mathbb{R}^n \mapsto \mathbb{R}^n$ are rational functions of x satisfying the conditions for the existence and uniqueness of solution for all $x \in \mathcal{B}_x$.

In addition, we assume that a dynamic output stabilizing compensator:

$$\begin{array}{rcl} \dot{\eta}(t) & = & f_\eta(\eta(t), y(t)) \\ v_c(t) & = & H_{v_\eta}\eta(t) + H_{v_y}y(t) \end{array} \tag{2}$$

is designed to guarantee some performance requirements and the stability of the closed-loop system (1)-(2) in the absence of control saturation, where $\eta \in \mathcal{B}_\eta \subset \mathbb{R}^{n_c}$ denotes the controller state; $y(t)$ is the controller input; $v_c(t)$ is the controller output; $f_\eta \; : \; \mathbb{R}^{n_c} \times \mathbb{R}^{n_y} \mapsto \mathbb{R}^{n_c}$ is a rational function of η and y satisfying the conditions for existence and solutions for all $\eta \in \mathcal{B}_\eta$; and $H_{v_\eta} \in \mathbb{R}^{n_v \times n_c}$, $H_{v_y} \in \mathbb{R}^{n_v \times n_y}$ are constant matrices.

In view of the undesirable effects of windup caused by input saturation, an anti-windup gain is added to the controller. Thus, considering the dynamic controller and the anti-windup strategy, the closed-loop system reads:

$$\begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))sat(v_c(t)) \\ y(t) & = & H_{y_x}x(t) \\ \dot{\eta}(t) & = & f_\eta(\eta(t), y(t)) + E_c(sat(v_c(t)) - v_c(t)) \\ v_c(t) & = & H_{v_\eta}\eta(t) + H_{v_y}y(t) \end{array} \tag{3}$$

where $E_c \in \mathbb{R}^{n_c \times n_v}$ is a constant matrix representing the anti-windup gain to be determined.

Considering the above setup, we aim at determining the anti-windup gain E_c such that the region of asymptotic stability of the closed-loop system is enlarged.

III. Preliminaries

This section presents some basic results needed to derive an LMI-based method to address the anti-windup computation as stated in Section II. In this sense, we present in the following the Differential Algebraic Representation (DAR) of nonlinear systems and some results regarding the inclusion of ellipsoids in polytopic domains. Then, we recall a generalized version of the modified sector bound condition, proposed in [5], which will be useful for dealing with the saturation nonlinearity.

A. Differential Algebraic Representation – DAR

Firstly, define the deadzone nonlinearity as follows

$$\psi(v_c(t)) \stackrel{\triangle}{=} v_c(t) - sat(v_c(t)), \tag{4}$$

and rewrite system (3) as:

$$\begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))v_c(t) - g(x(t))\psi(v_c(t)) \\ y(t) & = & H_{y_x}x(t) \\ \dot{\eta}(t) & = & f_\eta(\eta(t), y(t)) - E_c\psi(v_c(t)) \\ v_c(t) & = & H_{v_\eta}\eta(t) + H_{v_y}y(t). \end{array} \tag{5}$$

We consider the following Differential Algebraic Representation (DAR) for the system defined in (5):

$$\begin{array}{rcl} \dot{x}(t) & = & A_1x(t) + A_2\eta(t) + A_3z(t) + A_4\psi(v_c(t)) \\ \dot{\eta}(t) & = & C_1x(t) + C_2\eta(t) + C_3z(t) - E_c\psi(v_c(t)) \\ 0 & = & \Omega_1x(t) + \Omega_2\eta(t) + \Omega_bz(t) + \Omega_c\psi(v_c(t)). \end{array} \tag{6}$$

where $z \in \mathbb{R}^{n_z}$ is an auxiliary nonlinear vector function of (x, η, ψ) containing rational and polynomial terms (having terms of order equal or larger than two) of $f_x(x) + g(x)v_c - g(x)\psi(v_c)$ and of $f_\eta(x)$; and $A_1 \in \mathbb{R}^{n \times n}$, $A_2 \in \mathbb{R}^{n \times n_c}$, $A_3 \in \mathbb{R}^{n \times n_z}$, $A_4 \in \mathbb{R}^{n \times n_v}$, $C_1 \in \mathbb{R}^{n_c \times n}$, $C_2 \in \mathbb{R}^{n_c \times n_c}$, $C_3 \in \mathbb{R}^{n_c \times n_z}$, $E_c \in \mathbb{R}^{n_c \times n_v}$, $\Omega_1 \in \mathbb{R}^{n_z \times n}$, $\Omega_2 \in \mathbb{R}^{n_z \times n_c}$, $\Omega_b \in \mathbb{R}^{n_z \times n_z}$, and $\Omega_c \in \mathbb{R}^{n_z \times n_v}$ are affine matrix functions of (x, η).

Considering $\xi(t) = [x(t)' \; \eta(t)']' \in \mathcal{B}_\xi \subset \mathbb{R}^{n_\xi}$, $\mathcal{B}_\xi = \{\xi \in \mathbb{R}^{n_\xi} \; ; \; x \in \mathcal{B}_x \text{ and } \eta \in \mathcal{B}_\eta\}$, with $n_\xi = n + n_c$, we can rewrite (6) as follows:

$$\begin{array}{rcl} \dot{\xi}(t) & = & \mathcal{A}_a\xi(t) + \mathcal{A}_bz(t) + (\mathcal{A}_c - \mathcal{W}E_c)\psi(v_c(t)) \\ 0 & = & \Omega_a\xi(t) + \Omega_bz(t) + \Omega_c\psi(v_c(t)) \end{array} \tag{7}$$

with

$$\mathcal{A}_a = \begin{bmatrix} A_1 & A_2 \\ C_1 & C_2 \end{bmatrix}, \; \mathcal{A}_b = \begin{bmatrix} A_3 \\ C_3 \end{bmatrix}, \; \mathcal{A}_c = \begin{bmatrix} A_4 \\ 0 \end{bmatrix},$$

$$\mathcal{W} = \begin{bmatrix} 0_{n \times n_c} \\ I_{n_c} \end{bmatrix}, \; \Omega_a = \begin{bmatrix} \Omega_1 & \Omega_2 \end{bmatrix}.$$

In this case, it should be stressed that $\mathcal{A}_a$, $\mathcal{A}_b$, $\mathcal{A}_c$, Ω_a, Ω_b are affine matrices on ξ.

Moreover, we can rewrite $v_c(t)$ as follows:

$$v_c(t) = \begin{bmatrix} H_{v_y}H_{y_x} & H_{v_\eta} \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} = K\xi(t),$$

where $K \in \mathbb{R}^{n_v \times n_\xi}$ is a constant matrix.

Regarding system (7), we assume that:

(**A1**) the origin ($\xi = 0$) is a (locally) asymptotically stable equilibrium point; and

(**A2**) the domain $\mathcal{B}_\xi$ is a given polytope containing the origin with known vertices.

To guarantee that the DAR in (7) is well posed (i.e., the uniqueness of the solution $\xi(t)$ is ensured), we further consider that:

(**A3**) the matrix function $\Omega_b(\xi)$ has full rank for all $\xi \in \mathcal{B}_\xi$.

Notice from **A3** that the auxiliary vector $z(t)$ can be eliminated from (7) leading to the original system representation in (5) by means of

$$z = -\Omega_b(\xi)^{-1}(\Omega_b \xi(t) + \Omega_c \psi(v_c(t))) . \quad (8)$$

For further details on the above nonlinear decompositions, the reader may refer to [22] and [23].

To assess the local stability of system (7), we consider a quadratic Lyapunov function:

$$V(\xi) = \xi' P \xi \ , \ P = P' > 0 \quad (9)$$

where $P \in \mathbb{R}^{n_\xi \times n_\xi}$, and the following normalized level set

$$\mathcal{R} = \{\xi \in \mathcal{B}_\xi : \xi' P \xi \leq 1\} \ . \quad (10)$$

From the Lyapunov theory, if $V(\xi)$ satisfies the conditions for asymptotic stability for all $x \in \mathcal{B}_\xi$ and $\mathcal{R} \subset \mathcal{B}_\xi$, then $\mathcal{R}$ as above defined is an estimate of the system region of attraction [15].

B. Polytope of Admissible States

We consider in this paper that $\mathcal{B}_\xi$ is a given polytopic region containing the origin in its interior. Hence, $\mathcal{B}_\xi$ can be described by a set of scalar inequalities as follows:

$$\mathcal{B}_\xi = \{\xi \in \mathbb{R}^{n_\xi} \ : \ q_r' \xi \leq 1, \ r = 1, \ldots, n_e\} \ , \quad (11)$$

where $q_r \in \mathbb{R}^{n_\xi}$, $r = 1, \ldots, n_e$, are given vectors defining the n_e faces of $\mathcal{B}_\xi$. For convenience, $\mathcal{B}_\xi$ can be alternatively described by the convex hull of its vertices, where the notation $\mathcal{V}(\mathcal{B}_\xi)$ denotes the set of vertices of $\mathcal{B}_\xi$.

Notice that the set $\mathcal{R}$ is included in the region $\mathcal{B}_\xi$ if the following condition is satisfied [24]:

$$\begin{bmatrix} P & q_r \\ q_r' & 1 \end{bmatrix} \geq 0, \quad (12)$$

for $r = 1, \ldots, n_e$.

C. Generalized Sector Bound Condition

Consider a matrix $G \in \mathbb{R}^{n_v \times n_\xi}$. Define now the following set

$$\mathcal{S} \stackrel{\triangle}{=} \{\xi \in \mathbb{R}^{n_\xi} : |(K_{(i)} - G_{(i)})\xi| \leq 1, \, i = 1, \ldots, n_v\} \, , \quad (13)$$

where $K_{(i)}$ and $G_{(i)}$ stand for the i-th row of K and G, respectively.

From the deadzone nonlinearity $\psi(v_c)$ in (4) and the set $\mathcal{S}$ as above defined, the following Lemma can be stated [5].

Lemma 1: If $\xi \in \mathcal{S}$ then the relation

$$\psi(v_c)' T[\psi(v_c) - G\xi] \leq 0 \quad (14)$$

is verified for any matrix $T \in \mathbb{R}^{n_v \times n_v}$ diagonal and positive definite.

Considering deadzone nonlinearities, the relation (14) can be viewed as a generalized sector condition which encompasses the classical one used, for instance, in [25] and [26]. The generalized sector condition is known to be less conservative than the classical one when assessing the stability of systems subject to actuator saturation [5].

Similarly to the result of Section III-B, in view of (13), the constraint $\mathcal{R} \subset \mathcal{S}$ is satisfied if the following holds:

$$\begin{bmatrix} P & K_{(i)}' - G_{(i)}' \\ \star & 1 \end{bmatrix} \geq 0 \, , \ \forall i = 1, \ldots, n_v \, . \quad (15)$$

IV. Main Result

In this section, an LMI framework to address the anti-windup synthesis problem stated in Section II is presented.

In this case, by considering the quadratic Lyapunov function defined in (9), it follows

$$\dot{V}(\xi) = \dot{\xi}' P \xi + \xi' P \dot{\xi} \, . \quad (16)$$

Considering the auxiliary vector

$$\zeta_0 = \ [\dot{\xi}(t)' \quad \xi(t)']' \ , \quad (17)$$

we can rewrite (16) as follows:

$$\dot{V}(\xi) = \zeta_0' \Lambda_1 \zeta_0 \, , \quad (18)$$

with

$$\Lambda_1 = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} .$$

In view of Lemma 1, if $\xi \in \mathcal{S}$, then the relation $\psi(v_c)' T[\psi(v_c) - G\xi] \leq 0$ is verified for any matrix T diagonal and positive definite. Hence, if

$$\zeta_0' \Lambda_1 \zeta_0 - 2\psi(v_c)' T[\psi(v_c) - G\xi] < 0 \quad (19)$$

is verified, then $\dot{V}(\xi) < 0$ for all $\xi \in \mathcal{S} \cap \mathcal{B}_\xi$.

Considering the auxiliary vector

$$\zeta = \ [\dot{\xi}(t)' \quad \xi(t)' \quad z(t)' \quad \psi(v_c(t))']' \quad (20)$$

we can rewrite (19) as

$$\zeta' \Lambda_2 \zeta < 0 \quad (21)$$

with

$$\Lambda_2 = \begin{bmatrix} 0 & P & 0 & 0 \\ P & 0 & 0 & G'T \\ 0 & 0 & 0 & 0 \\ 0 & TG & 0 & -2T \end{bmatrix} .$$

Define now the following scalars:

$$\begin{aligned} \beta_1 \ &= \ \dot{\xi}(t)' M_1[-\dot{\xi}(t) + \mathcal{A}_a \xi(t) + \mathcal{A}_b z(t) \\ &\quad +(\mathcal{A}_c - \mathcal{W} E_c)\psi(v_c(t))] \\ \beta_2 \ &= \ \xi(t)' M_2[-\dot{\xi}(t) + \mathcal{A}_a \xi(t) + \mathcal{A}_b z(t) \\ &\quad +(\mathcal{A}_c - \mathcal{W} E_c)\psi(v_c(t))] \\ \beta_3 \ &= \ z(t)' M_3[\Omega_a \xi(t) + \Omega_b z(t) + \Omega_c(\xi)\psi(v_c(t))] \end{aligned} \quad (22)$$

In view of the representation of the system presented in (7), it follows that the equations

$$0 = \beta_1 + \beta_1' \,,\; 0 = \beta_2 + \beta_2' \,,\; 0 = \beta_3 + \beta_3' \qquad (23)$$

are satisfied, for any matrices $M_1 \in \mathbb{R}^{n_\xi \times n_\xi}$, $M_2 \in \mathbb{R}^{n_\xi \times n_\xi}$ and $M_3 \in \mathbb{R}^{n_z \times n_z}$.

From (23), if

$$\zeta' \Lambda_2 \zeta + \beta_1 + \beta_1' + \beta_2 + \beta_2' + \beta_3 + \beta_3' < 0 \qquad (24)$$

holds, then (21) is satisfied.

Observe that we can rewrite (24) as follows

$$\zeta' \Lambda_3(\xi) \zeta < 0 \qquad (25)$$

where

$$\Lambda_3(\xi) = \begin{bmatrix} -M_1 - M_1' & P - M_2 + M_1\mathcal{A}_a & M_1\mathcal{A}_b & M_1\mathcal{A}_c - M_1\mathcal{W}E_c \\ \star & M_2\mathcal{A}_a + \mathcal{A}_a'M_2' & M_2\mathcal{A}_b + \Omega_a'M_3' & M_2\mathcal{A}_c - M_2\mathcal{W}E_c + G'T \\ \star & \star & M_3\Omega_b + \Omega_b'M_3' & M_3\Omega_c \\ \star & \star & \star & -2T \end{bmatrix}.$$

Let us assume that M_1 is nonsingular and that $M_2 = M_2' > 0$ and $M_3 = M_3' > 0$. Define now the following matrices $Q_1 = M_1^{-1}$, $Q_2 = M_2^{-1}$, $Q_3 = M_3^{-1}$, $F = T^{-1}$ and

$$\Pi_0 = \begin{bmatrix} Q_1 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 \\ 0 & 0 & Q_3 & 0 \\ 0 & 0 & 0 & F \end{bmatrix}. \qquad (26)$$

Pre- and post-multiplying the condition $\Lambda_3(\xi) < 0$ by Π_0 and Π_0', we have

$$\begin{bmatrix} -Q_1 - Q_1' & Q_1PQ_2 - Q_1 + \mathcal{A}_aQ_2 & \mathcal{A}_bQ_3 & \mathcal{A}_cF - \mathcal{W}E_cF \\ \star & \mathcal{A}_aQ_2 + Q_2\mathcal{A}_a' & \mathcal{A}_bQ_3 + Q_2\Omega_a' & \mathcal{A}_cF - \mathcal{W}E_cF + Q_2G' \\ \star & \star & \Omega_bQ_3 + Q_3\Omega_b' & \Omega_cF \\ \star & \star & \star & -2F \end{bmatrix} < 0.$$

Observe that the above inequality is not an LMI due the term Q_1PQ_2. However, in this case, we can consider $P = M_2$ and it follows that $PQ_2 = I_{n_\xi}$. Besides, for the terms Q_2G' and E_cF, we consider the following change of variables: $G_q' = Q_2G'$ and $E_F = E_cF$. In this case, if

$$\Lambda_4(\xi) < 0\,, \qquad (27)$$

where

$$\Lambda_4(\xi) = \begin{bmatrix} -Q_1 - Q_1' & \mathcal{A}_aQ_2 & \mathcal{A}_bQ_3 & \mathcal{A}_cF - \mathcal{W}E_F \\ \star & \mathcal{A}_aQ_2 + Q_2\mathcal{A}_a' & \mathcal{A}_bQ_3 + Q_2\Omega_a' & \mathcal{A}_cF - \mathcal{W}E_F + G_q' \\ \star & \star & \Omega_bQ_3 + Q_3\Omega_b' & \Omega_cF \\ \star & \star & \star & -2F \end{bmatrix} < 0\,,$$

holds, then (21) is satisfied.

On the other hand, defining $\Pi_1 = \begin{bmatrix} Q_2 & 0 \\ 0 & 1 \end{bmatrix}$, if

$$\Pi_1 \begin{bmatrix} P & q_r \\ q_r' & 1 \end{bmatrix} \Pi_1 \geq 0, \qquad (28)$$

holds, then (12) is satisfied. The above condition can be rewritten as follows:

$$\begin{bmatrix} Q_2 & Q_2q_r \\ q_r'Q_2 & 1 \end{bmatrix} \geq 0, \qquad (29)$$

Similarly, the condition (15) is equivalent to:

$$\begin{bmatrix} Q_2 & Q_2K_{(i)}' - G_{q(i)}' \\ \star & 1 \end{bmatrix} \geq 0\,. \qquad (30)$$

In light of the above, we state the following result.

Theorem 1: Consider system (5) satisfying **A1**-**A2** and its DAR representation (7) satisfying **A3**. If there exist constant matrices $Q_2 = Q_2' > 0$, $Q_3 = Q_3' > 0$, E_F and G_q of appropriate dimensions and a positive diagonal matrix F, satisfying the following matrix inequalities for all $\xi \in \mathcal{V}(\mathcal{B}_\xi)$.

$$\Lambda_4(\xi) < 0\,, \qquad (31)$$

$$\begin{bmatrix} Q_2 & Q_2q_r \\ q_r'Q_2 & 1 \end{bmatrix} \geq 0, \qquad (32)$$

$$\begin{bmatrix} Q_2 & Q_2K_{(i)}' - G_{q(i)}' \\ \star & 1 \end{bmatrix} \geq 0\,. \qquad (33)$$

then the anti-windup gain $E_c = E_F F^{-1}$ is such that for all $\xi(0) \in \mathcal{R}$, with $P = Q_2^{-1}$, the trajectory $\xi(t)$ belongs to $\mathcal{R}$, and approaches the origin as $t \to \infty$, where $\mathcal{R}$ is as given in (10).

Proof. First recall that matrices $\mathcal{A}_a$, $\mathcal{A}_b$, $\mathcal{A}_c$, Ω_a, Ω_b and Ω_c are affine in ξ. Hence, if the inequalities (31)-(33) are feasible for each $\xi \in \mathcal{V}(\mathcal{B}_\xi)$, then, by convexity, they are also satisfied for all $\xi \in \mathcal{B}_\xi$.

Since (from (31)) $Q_1 + Q_1' > 0$ and (by hypothesis) $Q_3 > 0$, it follows that matrix Π_0 defined in (26) is invertible. Hence if $\Lambda_4(\xi) < 0$, it follows that $\Lambda_3(\xi) < 0$. Hence, in view of (23) we conclude that (19) holds with $Q_2^{-1} = P > 0$. Hence, if $\mathcal{R} \subset \mathcal{S} \cap \mathcal{B}_\xi$ and considering $V(x) = x'Px$, it follows that $\dot{V} < 0$, which ensures that for all $\xi(0) \in \mathcal{R}$ the trajectory $\xi(t)$ belongs to $\mathcal{R}$ and approaches the origin as $t \to \infty$.

Now, consider the relations (32) and (33). Pre- and post multiplying (32) and (33) by Π_1^{-1}, lead to (12) and (15), respectively. It follows that the inclusion $\mathcal{R} \subset \mathcal{B}_\xi \cap \mathcal{S}$ is satisfied, which concludes the proof. □

Theorem 1 can be applied for computing the anti-windup gain while providing an estimate of the region of attraction of the system in (1). Often, we are also interested in computing an estimate $\mathcal{R}$ of the system region attraction as large as possible. To this end, the following optimization problem can be considered:

$$\max\ \mathrm{trace}(Q_2) : (31),(32),(33),\ \forall \xi \in \mathcal{V}(\mathcal{B}_\xi)\,. \tag{34}$$

since $\mathcal{R}$ is an ellipsoidal domain.

Remark 1. The region $\mathcal{B}_\xi$ corresponds to a region where the feasibility of the state dependent LMIs of the Theorem 1 should be verified. It is a priori fixed by the designer. In practice, it can be chosen as an hyper-rectangle, which allows a straightforward description as (11) and the vertices characterization. Of course, the assumption regarding the existence and uniqueness of the solutions in $\mathcal{B}_\xi$ must be respected.

Remark 2. Note that the maximization of $\mathrm{trace}(Q_2)$ is a criteria that leads to an implicit maximization of the size of $\mathcal{R}$. We stress that other classical size criteria of ellipsoidal sets such as volume maximization, minor axis maximization, minimization of trace of P and the maximization in certain directions (see, e.g., [16], [17], [25], [27]) can be also easily applied.

V. Numerical Example

Consider the nonlinear closed-loop system borrowed from [21]:

$$\begin{aligned} \dot{x}(t) &= (x^2(t)-1)x(t) + sat(v_c(t)) \\ y(t) &= x(t), \end{aligned} \tag{35}$$

and the controller

$$\begin{aligned} \dot{\eta}(t) &= -x(t) \\ v_c(t) &= \eta(t) - 2y(t). \end{aligned} \tag{36}$$

Consider $\mathcal{B}_\xi := \left\{\xi \in \mathbb{R}^2 : |\xi_1| \le \alpha_1, |\xi_2| \le \alpha_2\right\}$, where $\alpha_1 = 1.3$ and $\alpha_2 = 2.4$

Considering the DAR representation given in (7) with $z(t) = x^2$, we get for (35)-(36) the following:

$$\mathcal{A}_a(\xi) = \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix},\ \mathcal{A}_b(\xi) = \begin{bmatrix} x \\ 0 \end{bmatrix},\ \mathcal{A}_c(\xi) = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

$$\Omega_a(\xi) = \begin{bmatrix} x & 0 \end{bmatrix},\ \Omega_b(\xi) = -1\,,$$

$$\Omega_c(\xi) = 0\,,\ K = \begin{bmatrix} -2 & 1 \end{bmatrix}.$$

Based on optimization problem stated in (34), we have determined the estimate $\mathcal{R}_1$ of the region of attraction for $E_c = 0$. Figure 1 shows the estimate $\mathcal{R}_1$ obtained considering (34). In this case the matrix P is given by:

$$P = \begin{bmatrix} 0.9826 & -0.2750 \\ -0.2750 & 0.3211 \end{bmatrix}.$$

Applying the optimization problem (34) and considering $E_c \neq 0$, we obtain:

$$P = \begin{bmatrix} 0.8514 & -0.2547 \\ -0.2547 & 0.2498 \end{bmatrix} \text{ and } E_c = 5.2464\,.$$

Figure 2 shows the new estimate of the region of attraction for the above value of E_c which is denoted by $\mathcal{R}_2$. (solid line).

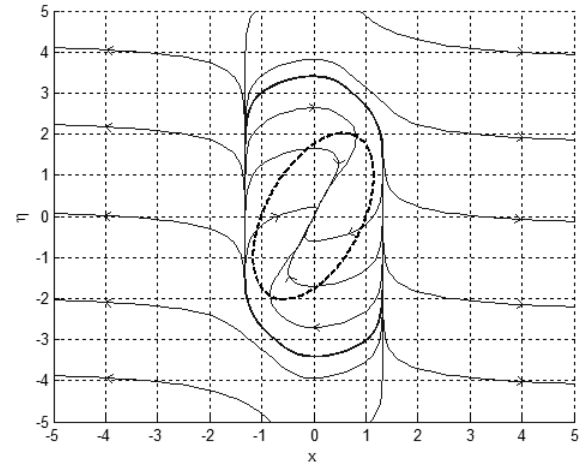

Fig. 1. Estimate of the region of attraction $\mathcal{R}_1$ without anti-windup.

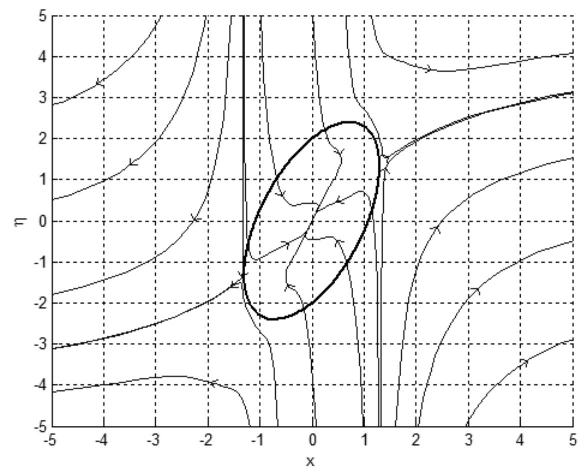

Fig. 2. Estimate of the region of attraction $\mathcal{R}_2$ with anti-windup.

Comparing Figures 1 and 2, we can note that the real region of attraction is greatly enlarged thanks to the additional anti-windup loop.

For comparison purposes, we present both estimates in Figure 3, where $\mathcal{R}_1$ is in dashed line and $\mathcal{R}_2$ is in solid line. The region where the control does not saturate is denoted by $\mathcal{R}_{ns}$.

Considering the initial condition $\xi(0) = [-0.6 \quad 1.16]'$, Figure 4 shows the trajectory of the output (y) and the control signal (v_c) in both cases (with and without the anti-windup strategy). In this case, note that, with anti-windup strategy (dashed-line), the control signal remains less time saturated and the transient performance is improved.

VI. Concluding Remarks

This paper has proposed an approach to compute anti-windup gains for a class of multivariable nonlinear systems subject to actuator saturation. The proposed design conditions relies on a differential algebraic representation of rational systems, which can model a broad class of nonlinear

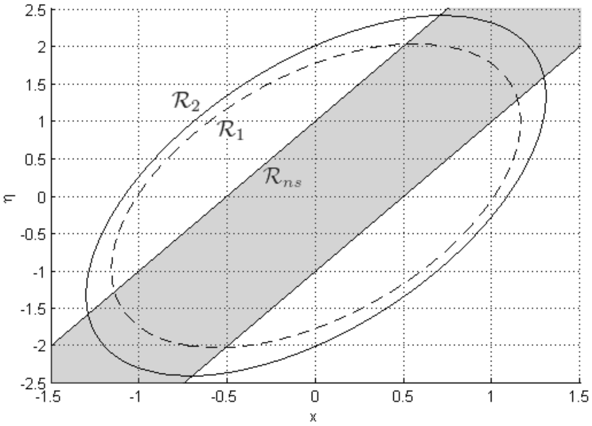

Fig. 3. Comparison of $\mathcal{R}_1$ and $\mathcal{R}_2$.

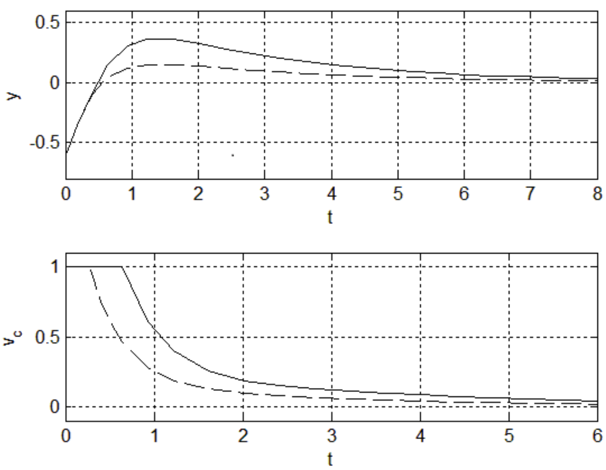

Fig. 4. Trajectory of the state x and the control signal.

systems. To deal with the saturation, we have considered a modified version of the generalized sector bound condition. From these elements, an LMI-based method has been devised to compute anti-windup gains aiming at the maximization of the estimates of the region of attraction of the closed-loop system. A numerical example has demonstrated the potentialities of the proposed approach.

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