2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Switched Second Order Sliding Mode Control with Partial Information

Mara Tanelli, Antonella Ferrara

Abstract— **This paper proposes a novel switched second order sliding mode (S-SOSM) control strategy in a partial information setting, *i.e.*, when only the sliding variable is accessible for measurements. The S-SOSM setting can deal with systems characterised by different levels of uncertainties associated with different regions of the state space and to accommodate different control objectives in the different regions by switching among appropriate SOSM controllers. The proposed approach stems from an *ad hoc* extension of a SOSM control algorithm which was introduced to cope with state-dependent uncertainties, and it is shown to ensure global finite-time convergence to the origin of the closed-loop system trajectory.**

I. Introduction and Motivation

In practical applications it is quite common to deal with dynamical systems with different degrees of uncertainty and/or different control objectives which are associated with different regions of the state space. Uncertainties can be linked to the system states because of state-dependent disturbances or different levels of confidence in the system model in different operating conditions. Furthermore, it is customary to have different control objectives according to the different regions of the state space visited by the closed-loop system. A typical example is the necessity of ensuring faster transients to quickly move towards the desired equilibrium point, while requiring finer tracking capabilities when close to the equilibrium itself. Devising switched algorithms has been recognised as an efficient way to achieve performance enhancement, as in general the benefits come with a quite limited increase in the controller complexity, and with easier tuning principles with respect to genuine adaptive solutions, see *e.g.*, [1], [2], [3]. Specifically, in [1] a state-feedback MPC algorithm for nonlinear systems has been proposed, where the state space was partitioned into different regions and the weighting matrices employed in the quadratic cost function were tuned in a different way in each region. The approach in [2], instead, was aimed at designing a switching supervisory unit to control nonlinear uncertain systems. Such a supervisor was employed to select the current controller among a set of pre-specified ones so as to ensure robust stability in the face of unknown disturbances. Finally, in [3] MIMO variable structure control systems were dealt with, proposing to employ a supervisor which selects a sliding mode controller among a family of possible ones with the aim of ensuring stability in presence of large plant uncertainties and of enabling improved transient performance.

M. Tanelli is with the Dipartimento di Elettronica e Informazione, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy. E-mail: tanelli@elet.polimi.it. A. Ferrara is with the Department of Computer Engineering and Systems Science, University of Pavia, Via Ferrata 1, 27100 Pavia, Italy. E-mail: antonella.ferrara@unipv.it.

Within this rationale, this paper proposes an extension of the results given in [4], where a formulation of second order sliding mode (SOSM) controllers was first presented. The idea is that of tuning a different SOSM control law for each region of the state space, adapting its parameters to the uncertainty levels and to the possibly different control objectives. Specifically, in this work the switched SOSM control problem is formulated and discussed in a *partial information* case, where only the sliding variable is accessible. the proposed S-SOSM control algorithm is inspired by the work in [5], where a SOSM algorithm to deal with state-dependent uncertainties and to ensure a global convergence of the closed-loop trajectories to the origin was provided. Specifically, we devise an *ad-hoc* modification of the algorithm in [5] conferring a suitable switched nature to the control law.

The stability and convergence properties of the resulting controller are analysed, proving finite-time convergence to the origin of the closed-loop trajectory.

II. Preliminaries

For what follows, it is worth recalling the structure and the basic features of the suboptimal SOSM controller (see *e.g.*, [6], [7]). For simplicity, we consider the so-called *auxiliary* system, which has the form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= f(z(t)) + g(z(t))v(t), \end{aligned} \quad (1)$$

where $z(t) = [z_1(t) \;\; z_2(t)]^T \in \mathbb{R}^2$ is the system state, $z_1(t)$ is the sliding variable, $v(t)$ is the control signal and $f(z(t))$ and $g(z(t))$ are uncertain, sufficiently smooth functions, satisfying all the conditions ensuring existence and uniqueness of the solution [8], together with the following bounds

$$0 < G_1 \leq g(z(t)) \leq G_2, \; |f(z(t))| \leq F. \quad (2)$$

The SOSM control problem is formulated as follows: given system (1), where $g(z(t))$ and $f(z(t))$ satisfy (2), design the control signal $v(t)$ so as to steer both $z_1(t)$ and $z_2(t)$ to zero in finite time. Under the assumption of being capable of detecting the extremal values z_{Max} of the signal $z_1(t)$, The suboptimal SOSM controller, see *e.g.*, [6], solves the problem using the auxiliary control law

$$v(t) = -\alpha V \,\mathrm{sign}\Big(z_1(t) - \beta z_{Max}\Big), \quad \beta = \frac{1}{2} \quad (3)$$
$$\alpha = \begin{cases} \alpha^* & \text{if} \quad [z_1(t) - \beta z_{Max}][z_{Max} - z_1(t)] > 0 \\ 1 & \text{else}, \end{cases}$$

where V is the control gain, α is the so–called modulation factor, and z_{Max} is a piecewise constant function representing the value of the last singular point of $z_1(t)$ (*i.e.*, the most recent value z_{1_M} such that $z_2(t_M) = 0$). The closed-loop system trajectory converges onto the sliding manifold $z_1 = z_2 = 0$ in finite time provided that the control parameters α^* and V are chosen so as to satisfy the following constraints

$$\alpha^* \in (0, 1] \cap \left(0, \frac{3G_1}{G_2}\right) \qquad (4)$$
$$V > \max\left\{\frac{F}{\alpha^* G_1}, \frac{4F}{3G_1 - \alpha^* G_2}\right\}.$$

The control law (3) is such that the trajectories on the (z_1, z_2) plane are confined within limit parabolic arcs including the origin, and the absolute values of the coordinates of the trajectory intersections with the z_1 and z_2 axes decrease in time. Namely, as shown in [6], under conditions (4), one has $|z_1| \le |z_{Max}|$ and $|z_2| \le \sqrt{|z_{Max}|}$ and the convergence of z_{Max} to zero takes place in finite time. As a consequence, the origin of the state space, *i.e.*, $z_1 = z_2 = 0$, is reached in finite time since z_1 and z_2 are both bounded by $\max(|z_{Max}|, \sqrt{|z_{Max}|})$. This, in turn, implies that the control objective is attained.

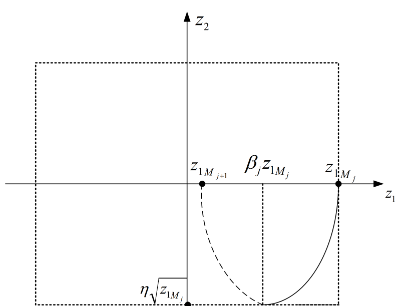

Fig. 1. Example of closed-loop trajectories induced by the SOSM algorithm in [5].

For what follows, it is interesting to remark that if the constant bounds on the system uncertainty $f(z(t))$ defined in (2) are assumed to hold only within a compact set contained in the state space $\mathbb{R}^2$, then the convergence properties which can be proved assume a local, or, better, a regional validity (see [5]). Thus, if one assumes instead that $f(z(t))$ is a class $\mathcal{K}$ function of z (see *e.g.*, [9]), *i.e.*, the uncertainty is state-dependent, then to achieve global convergence to zero of the system state it is necessary to devise an appropriate initialization phase, which ensures that the first extremal value is reached in finite time. To do this, in [5] it was shown that, assuming that a state-dependent bound of the form

$$|f(z(t))| \le \overline{F}(z), \qquad (5)$$

with $\overline{F}(z)$ being a known $\mathcal{K}$ function of z, holds, a control law of the type

$$v(t) = -\left(\overline{F}(z) + \kappa^2\right) \operatorname{sign}(z_1(t) - z_1(t_0)), \qquad (6)$$

with $\kappa > 0$, globally ensures that the first extremal point is reached in a finite time at $t = t_{M_1}$. Further, for all $t > t_{M_1}$, in order to ensure that between two successive extremal points a constant control amplitude can be chosen so that it can counteract the uncertain terms (which do not have *a priori* known constant upper bounds), one needs to employ a control strategy which makes use of a variable commutation point. This means that, instead of using $\beta = \frac{1}{2}$ in (3), a variable value of β is employed, the value of which is updated at all successive extremal points. The rationale behind this choice is that the commutation instant (and thus β) is chosen based on the fact that the state norm has exceeded a predefined upper-bound, so as to ensure that the control signal amplitude, tuned according to such a threshold on the uncertainty level, has enough authority to counteract it.

An example of the closed-loop trajectory achieved using the algorithm in [5] is shown in Figure 1. As can be seen, starting from a generic extremal point $z_{1_{M_j}}$, the state trajectory evolves along a parabolic arc until the sliding variables reaches the value $\beta_j z_{1_{M_j}}$, which determines the switching of the control gain sign. The value of β_j is updated at all subsequent extremal points in view of the current update on the bound on $f(z(t))$. It is worth pointing out that the control strategy in [5] is such that, once the sliding variables is equal to $\beta_j z_{1_{M_j}}$, it is ensured that its time derivative has not exceeded the value $\eta\sqrt{z_{1_{M_j}}}$, where η is a positive constant (see Figure 1). As a result, the closed-loop trajectory evolves within invariant sets of the type (see the rectangle in Figure 1)

$$\mathcal{I}_j \equiv \{(z_1, z_2) \in \mathbb{R}^2 : |z_1| \le z_{1_{M_j}}, |z_2| \le \eta\sqrt{z_{1_{M_j}}}\}. \qquad (7)$$

The existence of such invariant sets, which define upper bounds on $z_1(t)$ and $z_2(t)$, is in fact the key to provide constant bounds for $f(z(t))$ in each invariant set. An *ad hoc* extension of the algorithm will be devised in what follows to deal with the non-compactness of the state space regions. The core idea of the S-SOSM approach is that of tuning a

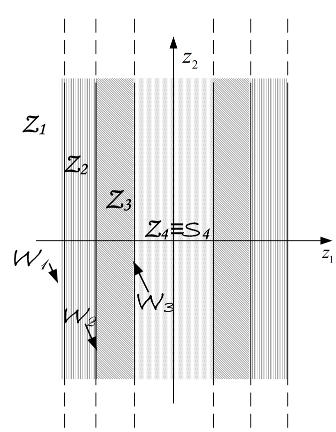

Fig. 2. An example of the state-space partitioning used in the proposed S-SOSM approach.

dedicated SOSM controller for each region of the state space, which is determined by different uncertainty levels and/or by possibly different control objectives.

In what follows, we will work under the following assumptions.

State-space partitioning

We assume that the state space $\mathcal{Z}$ of system (1) is partitioned into k regions, which are in fact *stripes*, $\mathcal{S}_i$, $i = 1, \dots, k$, all containing the origin, such that $\cup_i \mathcal{S}_i = \mathcal{Z}$ and with $\mathcal{S}_{i+1} \subset \mathcal{S}_i$. Further, we define as switching surfaces $\mathcal{W}_i = \partial \mathcal{S}_{i+1}$, $i = 1, \dots, k-1$ (see Figure 2). Finally, we assume that in each region $\mathcal{Z}_i = \mathcal{S}_i \cap \mathcal{S}_{i+1}$, $i = 1, \dots, k-1$, and in $\mathcal{Z}_k \equiv \mathcal{S}_k$, we may define different upper and lower bounds for the uncertainties, which will be specified in the following. Note that only one of these regions, namely the innermost one $\mathcal{Z}_k$, contains the origin (see again Figure 2). Specifically, it is assumed that the regions $\mathcal{S}_i$, $i = 2, \dots, k$ are defined as follows

$$\mathcal{S}_i := \{(z_1, z_2) : |z_1| \le \bar{z}_{1,i}, |z_2| \in (-\infty, +\infty)\}. \tag{8}$$

Uncertainty description

We consider the following bounds on the uncertain terms.

Case 1: Outermost Region $\mathcal{Z}_1$

In the outermost region $\mathcal{Z}_1$, the following bounds are given

$$\begin{aligned} 0 < G_{1,1} \le g(z(t)) \le G_{2,1} \\ |f(z(t)| \le \mathcal{F}_1(z), \end{aligned} \tag{9}$$

where $\mathcal{F}_1(z)$ is a known class $\mathcal{K}$ function of its argument.

Case 2: Regions $\mathcal{Z}_i$, $i = 2, \dots, k$

In the inner regions $\mathcal{Z}_i$, $i = 2, \dots, k$, the uncertainties are described as

$$\begin{aligned} 0 < \mathcal{G}_{1,i}(z) \le g(z(t)) \le \mathcal{G}_{2,i}(z) \\ |f(z(t)| \le \mathcal{F}_i(z), \end{aligned} \tag{10}$$

where

$$\begin{aligned} \mathcal{G}_{1,i}(z) &= G_{1,i}(z) + \overline{G}_{1,i} \\ \mathcal{G}_{2,i}(z) &= G_{2,i}(z) + \overline{G}_{2,i} \\ \mathcal{F}_i(z) &= F_i(z), \end{aligned} \tag{11}$$

where $G_{1i}(\cdot)$, $G_{2i}(\cdot)$ and $F_i(\cdot)$ are known class $\mathcal{K}$ functions and $\overline{G}_{1,i}$, $\overline{G}_{2,i}$ are known positive constants. Further, as in view of the proposed control algorithm it will be possible to bound the state of the system in the considered regions (see Equation (7)), a constant upper bound on the uncertain terms is assumed to be known, and $\forall\, i = 2, \cdots, k$, we can write

$$\begin{aligned} 0 < \overline{\mathcal{G}}_{1i} \le g(z(t)) \le \overline{\mathcal{G}}_{2i} \\ |f(z(t))| \le \overline{\mathcal{F}}_i. \end{aligned} \tag{12}$$

III. The S-SOSM Control Algorithm

This section is devoted to present the S-SOSM control algorithm in a partial information setting, *i.e.*, assuming that only the sliding variable is available for measurement. This assumption leads to a state space partitioning that (see Equation (8)) gives rise to regions $\mathcal{S}_i$ which are not limited anymore in the direction of z_2, as it was the case in [4] where rectangular regions were considered, thus losing the compactness property. Hence, in order to prove that the closed-loop trajectories still enjoy the convergence properties which have been shown to hold in the full information case treated in [4], one needs to devise a SOSM algorithm that forces such trajectories to evolve again across a finite sequence of invariant sets which contain the origin and shrink in size as time evolves, thereby causing the state to converge to the origin in finite time.

To achieve this goal, the S-SOSM algorithm used within each region $\mathcal{S}_i$ will be based on an appropriate extension of the algorithm proposed in [5]. In our case, besides employing a variable β to adapt the controller parameters to the state-dependent uncertainty in the function $f(z)$, we have also to cope with different uncertainty levels in the function $g(z)$ and to properly manage the intersections between the state trajectory and the switching surfaces $\mathcal{W}_i$. These needs will lead to an *ad hoc* modification of the control law.

In particular, in the situation considered herein one needs to generate a sequence of compact regions that become positively invariant sets for the closed-loop trajectories and that are contained in the stripe-like regions $\mathcal{S}_i$ characterising the considered setting (see Equation (8)). To ensure that these invariant sets are all strictly contained in each other, thus contractive as we move to the origin, it will be needed to devise an adaptation rule also for the parameter η of the algorithm in [5] that defines the invariant sets induced by the controller (see Equation (7)). Furthermore, the intersection of the state trajectory with the switching surface $\mathcal{W}_i$ must affect the update of the sequence of extremal points $z_{1_{M_j}}$, which will thus contain also *non-canonical* extremal points, *i.e.*, extremal points with $z_2(t_{M_j}) \neq 0$. These *non-canonical* extremal points will be given by the intersections of the state trajectory with the boundaries of the regions $\mathcal{S}_i$, and need to be taken into account in order to ensure that a correct sequence of invariant sets is formed as the state evolves towards the origin.

Specifically, in [5] the sequence of such extremal points $\{z_{1_{M_j}}\}$ was shown to be contractive. Hence, for fixed η, one has that $\eta\sqrt{z_{1_{M_{j+1}}}} < \eta\sqrt{z_{1_{M_j}}}$. In our case, though, as we need to consider as extremal points also the abscissas of the intersections with the switching surfaces, it is necessary to act on η so as to ensure that

$$|z_{2_{M_{j+1}}}| < \eta_{j+1}\sqrt{z_{1_{M_{j+1}}}} < \eta_j\sqrt{z_{1_{M_j}}}, \tag{13}$$

which implies that the next invariant set of the type given in (7) with the adapted value of η is such that $\mathcal{I}_{j+1} \subset \mathcal{I}_j$ and that the sequence of the upper bounds for $z_2(t)$ is properly updated. Condition (13) must hold for both positive and negative amplitudes of the controller gain; hence, the state trajectories in both cases must be analysed.

Following a worst-case approach for the uncertainties, and assuming negative values of the gain $v(t) = -V_{M_j}$ and, say, $z(t) \in \mathcal{Z}_i$, $i = 2, \dots, k$ and $t_{M_j} < t \le t_{M_{j+1}}$, it yields

$$z_1(t) = z_1(t_{M_j}) - \frac{z_2(t)^2 - z_2(t_{M_j})^2}{2(\overline{\mathcal{F}}_i + \overline{\mathcal{G}}_{2,i} V_{M_j})}, \tag{14}$$

where $z_2(t_{M_j}) = 0$ if $z_1(t_{M_j})$ was a canonical extremal point, whereas $z_2(t_{M_j}) \neq 0$ if $z_1(t_{M_j}) = \bar{z}_{1,i+1}$, *i.e.*, if $z_1(t_{M_j})$ is a non-canonical extremal point due to an intersection with the switching surface $\mathcal{W}_{i+1}$ (see Equation (8)). Hence, from (14) one has

$$|z_{2_{M_{j+1}}}| = \sqrt{-2(\overline{\mathcal{F}}_i + \overline{\mathcal{G}}_{2,i} V_{M_j})\left[z_{1_{M_{j+1}}} - z_{1_{M_j}}\right]} + |z_2(t_{M_j})|. \tag{15}$$

With a similar reasoning, for the case $v(t) = V_{M_j}$ and, again, $z(t) \in \mathcal{Z}_i$, $i = 2, \ldots, k$ and $t_{M_j} < t \leq t_{M_{j+1}}$, it holds that

$$|z_{2_{M_{j+1}}}| = \sqrt{-2(\overline{\mathcal{F}}_i - \overline{\mathcal{G}}_{1,i} V_{M_j})\left[z_{1_{M_{j+1}}} - z_{1_{M_j}}\right]} + |z_2(t_{M_j})|. \tag{16}$$

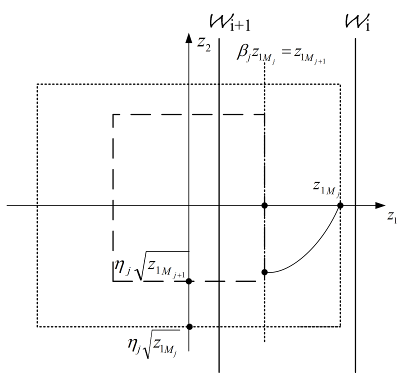

Fig. 3. Example of the closed-loop trajectories and of the update rules for the S-SOSM algorithm when (20) holds in Case 1.1.

We are now ready to introduce the proposed switched SOSM algorithm.

Algorithm 3.1: (Partial Information S-SOSM) Consider system (1), with the state space partitioned as in (8). Assume also that, for $z \in \mathcal{Z}_1$, $g(z(t))$ and $f(z(t))$ satisfy constraints (9), whereas for each $z \in \mathcal{Z}_i$, $i = 2, \ldots, k$, $g(z(t))$ and $f(z(t))$ satisfy constraints (12).

If $z \in \mathcal{Z}_1$, over the time interval to the first extremal point, *i.e.*, for $0 \leq t \leq t_{M_1}$, define the control signal as

$$v(t) = -\frac{1}{G_{1,1}}\left[\mathcal{F}_1(z(t)) + \nu\right]\operatorname{sign}(z_1(0)),\ t = 0 \tag{17}$$
$$v(t) = -\frac{1}{G_{1,1}}\left[\mathcal{F}_1(z(t)) + \nu\right]\operatorname{sign}\Big(z_1(t) - z_1(0)\Big),\ t \in (0, t_{M_1}]$$

with $\nu > 0$.

Over the time interval $t_{M_1} < t \leq t_{M_2}$ such that $z(t) \in \mathcal{Z}_i$, $i = 1, \ldots, k$, adopt the control law

$$v(t) = -V_{M_1}\operatorname{sign}\Big(z_1(t) - \beta_1 z_{1_{M_1}}\Big), \tag{18}$$

with

$$V_{M_1} = \frac{\pi}{G_{1,1}}\left[\overline{\mathcal{F}}_1 + \frac{1}{3}\eta^2\right], \quad \pi > 1 \tag{19}$$
$$\beta_1 = \max\left\{\frac{1}{2}, 1 - \frac{\eta^2}{2\left[\overline{\mathcal{F}}_1 + G_{2,1}V_{M_1}\right]}\right\}.$$

Over the generic time interval $t_{M_j} < t \leq t_{M_{j+1}}$, $j > 1$, if

$$\operatorname{sign}\Big(z_1(t) - \beta_j z_{1_{M_j}}\Big) > 0 \tag{20}$$

update the extremal point $z_{1_{M_{j+1}}}$ and set the value of the parameter η_{j+1} as follows:

- Case 1.1:
  If the sliding variable reaches its upper bound $\beta_j z_{1_{M_j}}$ while the state trajectory is within the open region $\mathcal{Z}_i$ (see Figure 3), *i.e.*, if
  $$z_1(t) \in \mathcal{Z}_i \setminus \mathcal{W}_i \text{ and } z_1(t) = \beta_j z_{1_{M_j}}, \tag{21}$$
  let (see again Figure 3)
  $$z_{1_{M_{j+1}}} = \beta_j z_{1_{M_j}},\ \eta_{j+1} = \eta_j. \tag{22}$$

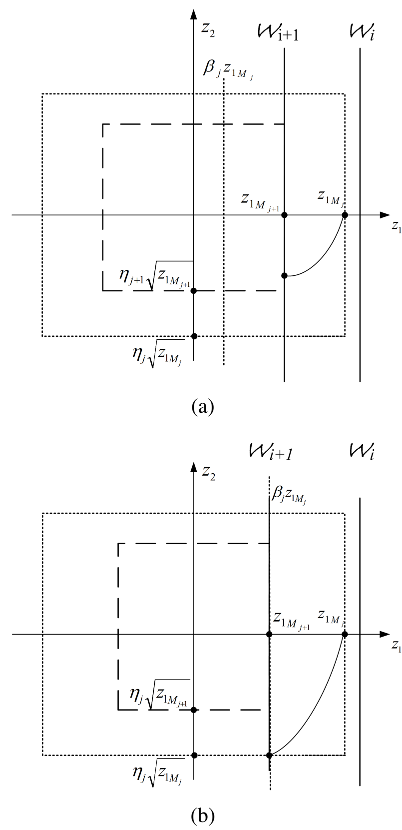

Fig. 4. Example of the closed-loop trajectories and of the update rules for the S-SOSM algorithm when (20) holds in Case 2.1a) (a) and Case 2.1b) (b).

- Case 2.1:
  If a switch occurs due to the intersection between the state trajectory and the switching surface $\mathcal{W}_{i+1}$, *i.e.*, if
  $$z_1(t) \cap \mathcal{W}_{i+1} \neq \emptyset \text{ and } z_1(t) \leq \beta_j z_{1_{M_j}}, \tag{23}$$
  two different situations may happen. Namely, if
  - Case 2.1a) (see Figure 4(a))
    $$z(t) \cap \mathcal{W}_{i+1} = \{\overline{z}_{1,i+1}, z_2\} \text{ with } z_2 < \eta_j\sqrt{z_{1_{M_j}}}, \tag{24}$$
    let (see again Figure 4(a))
    $$z_{1_{M_{j+1}}} = \overline{z}_{1,i+1},$$
    $$\eta_{j+1} \text{ such that} \tag{25}$$
    $$|z_{2_{M_{j+1}}}| < \eta_{j+1}\sqrt{z_{1_{M_{j+1}}}} < \eta_j\sqrt{z_{1_{M_j}}},$$

where $|z_{2_{M_{j+1}}}|$ is as in (15). On the other hand, if

– Case 2.1b) (see Figure 4(b))

$$z(t) \cap \mathcal{W}_{i+1} = \{\overline{z}_{1,i+1}, z_2\} \text{ with } z_2 = \eta_j \sqrt{z_{1_{M_j}}}, \tag{26}$$

let (see again Figure 4(b))

$$z_{1_{M_{j+1}}} = \overline{z}_{1,i+1},\ \eta_{j+1} = \eta_j. \tag{27}$$

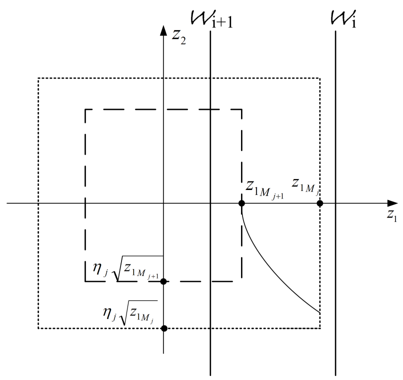

Fig. 5. Example of the closed-loop trajectories and of the update rules for the S-SOSM algorithm when (28) holds in Case 1.2.

If

$$\text{sign}\Big(z_1(t) - \beta_j z_{1_{M_j}}\Big) < 0 \tag{28}$$

update the extremal point $z_{1_{M_{j+1}}}$ and set the value of the parameter η_{j+1} as follows:

- Case 1.2:
  If the state trajectory is such that a canonical extremal point is encountered while within an open region $\mathcal{Z}_i$ (see Figure 5), *i.e.*, if

$$z_1(t) \in \mathcal{Z}_i \setminus \mathcal{W}_i \text{ and } z_2(t_{M_{j+1}}) = 0, \tag{29}$$

  let (see again Figure 5)

$$z_{1_{M_{j+1}}} = z_1(t_{M_{j+1}}),\ \eta_{j+1} = \eta_j. \tag{30}$$

  If a switch occurs due to the intersection between the state trajectory and the switching surface $\mathcal{W}_{i+1}$, *i.e.*, if

$$z_1(t) \cap \mathcal{W}_{i+1} \neq \emptyset \text{ and } z_2(t) \leq 0, \tag{31}$$

  two different situations may happen. Namely, if

– Case 2.2a) (see Figure 6(a))

$$z(t) \cap \mathcal{W}_{i+1} = \{\overline{z}_{1,i+1}, z_2\} \text{ with } z_2 < 0, \tag{32}$$

let (see again Figure 6(a))

$$\begin{aligned} z_{1_{M_{j+1}}} &= \overline{z}_{1,i+1}, \\ \eta_{j+1} &\text{ such that} \\ &|z_{2_{M_{j+1}}}| < \eta_{j+1}\sqrt{z_{1_{M_{j+1}}}} < \eta_j \sqrt{z_{1_{M_j}}}, \end{aligned} \tag{33}$$

where $|z_{2_{M_{j+1}}}|$ is as in (16). On the other hand, if

– Case 2.2b) (see Figure 6(b))

$$z(t) \cap \mathcal{W}_{i+1} = \{\overline{z}_{1,i+1}, z_2\} \text{ with } z_2 = 0, \tag{34}$$

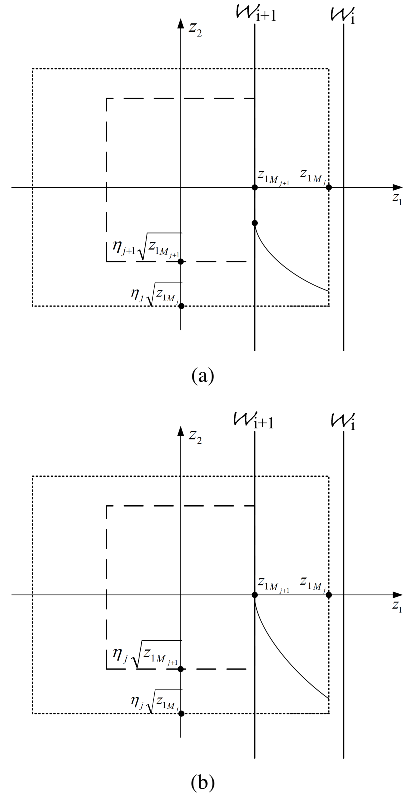

Fig. 6. Example of the closed-loop trajectories and of the update rules for the PM S-SOSM algorithm when (28) holds in Case 2.2a) (a) and Case 2.2b) (b).

let (see again Figure 6(b))

$$z_{1_{M_{j+1}}} = \overline{z}_{1,i+1},\ \eta_{j+1} = \eta_j. \tag{35}$$

Further, choose the control gain as

$$\begin{aligned} V_{M_{j+1}} &= \frac{\pi}{G_{1,i+1}} \left[\overline{\mathcal{F}}_i + \frac{1}{3}\eta_{j+1}^2 \right], \quad \pi > 1 \\ \beta_{j+1} &= \max\left\{ \frac{1}{2}, 1 - \frac{\eta_{j+1}^2}{2\left[\overline{\mathcal{F}}_i + G_{2,i+1}V_{M_{j+1}}\right]} \right\}, \end{aligned} \tag{36}$$

where $\overline{\mathcal{F}}_i = \overline{\mathcal{F}}_i(|z_1(t_{M_j})|, \eta_j\sqrt{|z_1(t_{M_j})|})$ is an upper bound on the function $\mathcal{F}_i(z)$ computed at any time instant $\{t_{M_j}\}$ in which the sequence of extremal points (canonical and non canonical) $\{z_{1_{M_j}}\}$ is updated (see also (7) and (12)).

Remark 3.1: Note that, with respect to the full measurement case where the core control law was based on the suboptimal SOSM approach, in this case we rely on the SOSM algorithm in [5], which in a sense is itself an extension of the suboptimal control law. However, the parameters which are now adapted have changed. In fact, we have a variable β_j (whereas in the full measurement case the value $\beta = 1/2$ was used), while we do not have anymore a variable α^*. In both cases, instead, there is an adaptation of the controller gain V_{M_j}. Note, however, that also in the case considered herein the resulting closed-loop trajectories, which still are parabolas, have a varying curvature, now due to the variability of β_j. It is topic of future research the investigation of the possible advantages that may derive from another modification of the control law in [5] to encompass a new adaptation parameter in the definition of the control gain V_{M_j} in the first of (36) which plays the role of α^* in the suboptimal SOSM control law. Of course, to do this while ensuring that the convergence properties are not lost, the expression of V_{M_j} in the first of (36) must be properly adjusted.

Remark 3.2: In the case where a negative amplitude of the controller gain is used, thus when (20) holds, in Cases 2.1a) and 2.1b) the closed-loop trajectories have a different behaviour with respect to the active invariant set. Specifically, in Case 2.1a) (see also Figure 6(a)) when the switch occurs the trajectory is already within the new (inner) invariant set belonging to the region $\mathcal{S}_j$ (see Equation (8)), denoted by $\mathcal{I}_{i+1,j}$. In Case 2.1b), instead, when the switch occurs the trajectory is outside the invariant set $\mathcal{I}_{i+1,j}$, which will be entered after the switch has occurred causing the use of a controller gain with positive amplitude.

Remark 3.3: It is worth mentioning that Case 2.2b), both for negative and positive gain amplitudes, represents the situation in which the trajectory intersects the switching surfaces at either the lower corner of the current invariant set or along the z_1 axis (see Figures 4(b) and 6(a)). The fact that the value of η_{j+1} is kept equal to η_j when the switch occurs is due to the fact that in this case the first and third terms of inequalities (15) and (16) are equal and thus the inequalities themselves are not meaningful. The contraction of the extremal point z_{M_j} is enough in these cases to ensure that the next invariant set which is created is a strict subset of the preceding one.

IV. Stability and Convergence Analysis

The proof of global finite-time convergence to the origin of the closed-loop trajectories follows the same conceptual development used in the full measurement case, see [4], [10]. We recall that the idea of the proof was the following. First of all, one shows that all the inner regions, which in the full measurement case are compact sets, contain a positively invariant set. This implies that, once such a set is entered, the state trajectory evolves within the region to which it belongs for all future times. Further, it is established that, independently of the region in which the initial condition is and of the specific point of the switching surface at which the switching occurs, the closed-loop trajectory does enter the next invariant set, so that it progressively moves towards the innermost region which contains the origin, and this happens in finite time. Further, in whichever region the initial condition is, the S-SOSM controller is shown to guarantee that the boundary of the adjacent inner region is reached in finite time, so that a switch actually occurs. Once these results were proved, the last step showed that, once the closed-loop trajectory entered the innermost region, the state is actually steered to the origin in finite time by the associated control law.

As such, the following facts can be proved.

Lemma 4.1: Consider the state space partitioning given in Assumption 1.b) and the regions defined in (8). Assume that the bounds (9) and (12) hold. Then, all the regions $\mathcal{S}_i$, $i = 1, \ldots, k$ of the state space $\mathcal{Z}$ of system (1), controlled with the S-SOSM Algorithm 3.1 contain a finite number of positive invariant sets of the type (7). For any admissible initial condition $z(0) \in \mathcal{Z}$, such invariant sets are progressively visited by the closed-loop trajectory $z(t)$, and the innermost invariant set $\mathcal{I}_{N,k}$ associated with the innermost region $\mathcal{S}_k$ is reached in finite time.

Based on the previous lemma, we can state the main result of this work.

Proposition 4.1: Consider the state space partitioning given in Assumption 1.b) and the regions defined in (8). Assume that the bounds (9) and (12) hold. Then, the origin of the state space $\mathcal{Z}$ of system (1) controlled with the S-SOSM Algorithm 3.1 is *globally finite-time attractive*, *i.e.*, for any admissible initial condition $z(0) \in \mathcal{Z}$, the closed-loop trajectory $z(t)$ is such that $(z_1(t), z_2(t)) \to \{0, 0\}$ in finite time.

V. Concluding Remarks

This paper presented a new SOSM algorithm which employs a switching rule based on which it is possible to adapt the controller parameters according to different levels of uncertainty and/or to different performance objectives associated with different regions of the state space. The proposed approach has been developed assuming that only the sliding variable can be accessed for measurements, and the origin of the closed-loop system was shown to be globally finite-time attractive.

References

[1] L. Magni, R. Scattolini, and M. Tanelli, "Switched model predictive control for performance enhancement," *International Journal of Control*, vol. 81, no. 12, pp. 1859–1869, 2008.

[2] D. Angeli and E. Mosca, "Lyapunov-based switching supervisory control of nonlinear uncertain systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 3, pp. 500–505, 2002.

[3] M. Corradini and G. Orlando, "Transient Improvement of Variable Structure Controlled Systems Via Multi-Model Switching Control," *Journal of Dynamic Systems, Measurement, and Control*, vol. 124, no. 2, pp. 321–326, 2002.

[4] M. Tanelli and A. Ferrara, "Switched second order sliding mode control," in *Proceedings of the IEEE Conference on Decision and Control, CDC 2010*, Atlanta, GA, 2010, pp. 5103–5108.

[5] G. Bartolini, A. Pisano, and E. Usai, "Global Stabilization for Nonlinear Uncertain Systems with Unmodeled Actuator Dynamics," *IEEE Transactions on Automatic Control*, vol. 46, no. 11, pp. 1826–1832, 2001.

[6] G. Bartolini, A. Ferrara, and E. Usai, "Chattering avoidance by second-order sliding mode control," *IEEE Transactions on Automatic Control*, vol. 43, no. 2, pp. 241–246, 1998.

[7] G. Bartolini, A. Ferrara, F. Levant, and E. Usai, "On second order sliding mode controllers," in *Lecture Notes in Control and Information Sciences*, K. D. Young and U. Özgüner, Eds., vol. 247. Berlin: Springer-Verlag, 1999, pp. 329–350.

[8] H. Khalil, *Nonlinear Systems*. Upper Saddle River, New Jersey: 2nd Edition, Prentice Hall, 1996.

[9] A. Isidori, *Nonlinear Control Systems*. London: 3rd Edition, Spriger, 1995.

[10] M. Tanelli and A. Ferrara, "Switched second order sliding mode control," Politecnico di Milano, Tech. Rep., 2010.